Summation of threshold singularities in QCD

Yu. D. Chernichenko
International Center for Advanced Studies, Gomel State Technical University, Belarus
E-mail: `chern@gstu.gomel.by`

O. P. Solovtsova[*]
International Center for Advanced Studies, Gomel State Technical University, Belarus
E-mail: `solvtsova@gstu.gomel.by`

We present a new relativistic Coulomb-like threshold resumming factor, L-factor, in quantum chromodynamics for arbitrary orbital angular momentum $\ell \geq 0$. The analysis is performed within the framework of completely covariant of quasipotential approach in quantum field theory formulated in the relativistic configurational representation for the case of two particles of unequal masses. A new model expression for the Drell ratio $R(s)$ is suggested to summarize the threshold singularities into a main potential contribution.

The XIXth International Workshop on High Energy Physics and Quantum Field Theory
8-15 September 2010
Golitsyno, Moscow, Russia

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[*]Speaker.

1. Introduction

In comparing theoretical results with experimental data it is important to use the simplest single-argument functions which allow one to check direct consequences of the theory without using model assumptions. The cross-section for e^+e^- annihilation into hadrons or its ratio to the leptonic cross-section, the Drell ratio $R(s)$, which have a straightforward connection with experimentally measured quantities can play the role of these the simplest single-argument functions. The $R(s)$-function, have a resonance structure that is difficult to discribe, at the present stage of a theory, without model considerations. Moreover, the basic method of calculations in quantum field theory, becomes ill-defined due to the threshold singularities of the form $(\alpha/v)^n$, where $v = \sqrt{1 - 4m^2/s}$ is a quark velocity, and m is a quark mass. Consequently, the real expansion parameter in the threshold region is α/v. Obviously, it becomes to be singular, when the velocity $v \to 0$. Thus, at the description of quark-antiquark systems close to threshold we can not cut off the perturbative series even if the expansion parameter, the QCD coupling constant α_S, is small [1]. The problem is well known from QED [2]. To obtain meaningful result these threshold singularities of the form $(\alpha/v)^n$ have to be summarized. In the nonrelativistic of case for the Coulomb interaction

$$V(r) = -\frac{\alpha}{r} \tag{1.1}$$

this resummation is realized the known S-factor Gamov–Sommerfeld–Sakharov [3]–[5]

$$S_{\mathrm{nr}} = \frac{X_{\mathrm{nr}}}{1 - \exp(-X_{\mathrm{nr}})}, \qquad X_{\mathrm{nr}} = \frac{\pi\alpha}{v_{\mathrm{nr}}}, \tag{1.2}$$

which is related to the wave function of the continuous spectrum at the origin by $|\psi(0)|^2$. Here $2v_{\mathrm{nr}}$ is the relative velocity of two nonrelativistic particles. The corresponding nonrelativistic expression can also be obtained for higher ℓ states (see, for instance, [6], [7]).

In the relativistic theory the nonrelativistic approximation needs to be modified. For the first time the relativistic modification of the S-factor (1.2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in [8] (see also [9]) and it consisted in the change $v_{\mathrm{nr}} \to v$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$. Just the same form of the S-factor for the interaction of two particles of equal masses was later suggested in [10]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [11]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [12]. This factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation.

The new method to relativistic generalization of the S-factor in the case of two particles of equal masses was developed by Milton and Solovtsov in [13]. Their the method is based on the relativistic quasipotential (RQP) approach proposed by Logunov and Tavkhelidze [14] in the form suggested by Kadyshevsky [15]. At present the RQP approach continues to remain one of the methods of the study of component objects (see, for instance, [16], [17]). In the method developed in [13], the possibility of transformation of quasipotential (QP) equation from momentum space into relativistic configurational representation in the case of two particles of equal masses (see [18]) has been used also. Moreover, it is important the potential (1.1) that used by them possesses

the QCD-like behaviour (see [19]). Thus, the application of the quasipotential approach in QCD developed by them, gives the following expression for the relativistic S-factor:

$$S(\chi) = \frac{X(\chi)}{1 - \exp[-X(\chi)]}, \qquad X(\chi) = \frac{\pi\alpha}{\sinh\chi}, \qquad (1.3)$$

where χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$, by $2m\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (1.3) can be expressed in terms of v as $X(\chi) = \pi\alpha\sqrt{1-v^2}/v$. The method proposed by them in [13] has been generalized in [20] successfully to get the following expression for the relativistic L-factor ($\ell \geq 0$) in the case of two particles of equal masses:

$$L(\chi) = \prod_{n=1}^{\ell} \left[1 + \left(\frac{\alpha}{2n\sinh\chi}\right)^2 \right] \frac{X(\chi)}{1 - \exp[-X(\chi)]}, \qquad (1.4)$$

where the function $X(\chi)$ is determined in (1.3). Applications of the factor (1.3) for describing some hadronic processes can be found in [21]–[23]. Recently, the relativistic S-factor (1.3) has been applied in [24] to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron.

We note that the solution containing arbitrary functions of r with period i, the so-called the i-periodic constants, with the same potential was investigated in [25]. However, the using of such solution is suitable for the spectral problems only. Other forms of the QP equation with the Coulomb potential were considered in [26].

In this talk we discuss the generalization of the previous study started in [13]. The basis of our consideration is the completely covariant RQP approach in quantum field theory (see [27], [28]) formulated in the relativistic configuration representation for the case of interaction between two relativistic particles that have unequal masses.

2. Relativistic threshold L-factor

The resumming threshold factors appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe–Salpeter (BS) amplitude of two charged particles $\chi_{\mathrm{BS}}(x)$ at $x = 0$ (see [29]). The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (1.1), leads to formula (1.2) with a substitution $\alpha \to 4\alpha_{\mathrm{s}}/3$ for QCD. The possibility of using the RQP approach for our task is based on the fact that the relationship of the BS amplitude at $x = 0$ with the RQP wave function in the relativistic configuration representation, $\psi_q(\boldsymbol{\rho})$, is $\chi_{\mathrm{BS}}(x = 0) = \psi_q(\boldsymbol{\rho})|_{\rho=i}$.

The basis of our consideration is the completely covariant QP equation for the RQP partial wave function $\varphi_\ell(\rho, \chi)$ connected with the wave function of interacting particles, $\psi_q(\boldsymbol{\rho})$, through the expansion on a Legendre function of the first kind $P_\mu^\nu(z)$. This equation constructed in [30] for the case of two relativistic particles of unequal masses given by (in the following we will use the

system of units $c = \hbar = 1$)

$$
\frac{2}{\pi} \int\limits_0^\infty d\chi' \frac{(\sinh\chi')^{2\ell+2}(-1)^{\ell+1}}{\rho^{(\ell+1)}} \left(2\cosh\chi - 2\cosh\chi'\right) \left(\frac{d}{d\cosh\chi'}\right)^\ell \left(\frac{\sin(\rho\chi')}{\sinh\chi'}\right) \times
$$
$$
\times \left(\frac{d}{d\cosh\chi'}\right)^\ell \frac{1}{\sinh\chi'} \int\limits_0^\infty d\rho' \frac{\rho'\sin(\rho'\chi')}{(-\rho')^{(\ell+1)}} \varphi_\ell(\rho',\chi) = \frac{2\mu}{m'} \frac{V(\rho;E_q)\,\varphi_\ell(\rho,\chi)}{\rho}, \tag{2.1}
$$

where $m' = \sqrt{m_1 m_2}$ is mass of an effective relativistic particle, emerging instead of the system of two particles and carrying the total c. m. energy of the interacting particles, $\sqrt{s}$, proportional to its the energy $m'E_q$ (see [27], [28]) and $\mu = m_1 m_2 /(m_1 + m_2)$ is the usual reduced mass; χ is the rapidity which is related to $E_q = \sqrt{1+\mathbf{q}^2}$ as $E_q = \cosh\chi$, and the function $(-\rho)^{(\ell+1)} = i^{\ell+1}\Gamma(\ell+1+i\rho)/\Gamma(i\rho)$ is the generalized power [18] where $\Gamma(z)$ is the gamma-function.

Generalizing the method developed in [13] (see also [30]–[32]), we will seek a solution of RQP equation (2.1) with the potential (1.1) in the form

$$
\varphi_\ell(\rho,\chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta} R_\ell(\zeta,\chi), \tag{2.2}
$$

where the ζ-integration is performed in the complex plane over a contour with end points α_- and α_+ (see [30] for details). In the case when the interaction vanishes, $\alpha \to 0$, the solution $\varphi_\ell(\rho,\chi)$ should reproduce the known free wave function:

$$
\lim_{\alpha\to 0} \varphi_\ell(\rho,\chi) = \rho\, p_\ell(\rho,\cosh\chi) \xrightarrow[\rho\to\infty]{} \frac{\sin(\rho\chi - \pi\ell/2)}{\sinh\chi}, \tag{2.3}
$$

where the function $p_\ell(\rho,\cosh\chi)$ is the solution of Eq. (2.1) in the case when the interaction is switched off, $V(\rho;E_q) \equiv 0$.

The L-factor in the nonrelativistic case is defined by derivative of the order ℓ of the the wave function at $r = 0$. In the relativistic case, instead of the derivative, one has to use its finite difference analog [18]:

$$
\Delta^* = \frac{1}{i} \left[\exp\left(i\frac{\partial}{\partial\rho}\right) - 1\right]. \tag{2.4}
$$

Thus, the relativistic L-factor is connected with the RQP partial wave function $\varphi_\ell(\rho,\chi)$ as follows

$$
L_{\text{uneq}}(\chi) = \lim_{\rho\to i} \left| \frac{\Gamma(2\ell+2)}{(2\sinh\chi)^\ell \Gamma^2(\ell+1)} (\Delta^*)^\ell \left[\frac{\varphi_\ell(\rho,\chi)}{\rho}\right] \right|^2. \tag{2.5}
$$

The resulting solution Eq. (2.1) with the potential (1.1) at arbitrary $\ell \geq 0$ which does not contain the i-periodic constant, represents in terms of hypergeometrical function by

$$
\varphi_\ell(\rho,\chi) = N_\ell(\chi)(-\rho)^{(\ell+1)} e^{i\rho\chi + iA\chi + i\pi(\ell+1)} \times
$$
$$
\times F\left(\ell+1-iA, \ell+1-i\rho; 2\ell+2; 1-e^{-2\chi}\right), \quad A = \frac{\alpha\mu}{m'\sinh\chi}, \tag{2.6}
$$

where the normalization constant $N_\ell(\chi)$ can be obtained from the condition (2.3).

By using Eqs. (2.3) and (2.4)–(2.6), we finally find the following expression for the relativistic L-factor in the case of two particles of unequal masses:

$$L_{\text{uneq}}(\chi) = \prod_{n=1}^{\ell} \left[1 + \left(\frac{\alpha\mu}{m'n\sinh\chi} \right)^2 \right] S_{\text{uneq}}(\chi). \tag{2.7}$$

Here

$$S_{\text{uneq}}(\chi) = \lim_{\rho\to i} \left| \frac{\varphi_0(\rho,\chi)}{\rho} \right|^2 = \frac{X_{\text{uneq}}(\chi)}{1 - \exp\left[-X_{\text{uneq}}(\chi) \right]}, \quad X_{\text{uneq}}(\chi') = \frac{2\pi\alpha\mu}{m'\sinh\chi'}, \tag{2.8}$$

is the relativistic S-factor in the case of two particles of unequal masses [30], and the rapidity χ is related to the total c. m. energy, $\sqrt{s}$, as $(m'^2/\mu)\cosh\chi = \sqrt{s}$. The functions $\sinh\chi$ and $X_{\text{uneq}}(\chi)$ in Eqs. (2.7) and (2.8) can be expressed in terms of the "velocity" u

$$u = \sqrt{1 - \frac{4m'^2}{s - (m_1 - m_2)^2}}, \tag{2.9}$$

in the form $\sinh\chi = 2\mu u/(m'\sqrt{1-u^2})$, $X_{\text{uneq}}(\chi) = \pi\alpha\sqrt{1-u^2}/u$. The square of relative 3-momentum $\mathbf{k}'$ for an effective relativistic particle, having mass m', the total c. m. energy of interacting particles, $\sqrt{s}$, and emerging instead of the system of two particles with their the relative relativistic velocity $\mathbf{v}$, gives by the expression (see [27], [28])

$$\sqrt{s} = \frac{m'}{\mu}\sqrt{m'^2 + \mathbf{k}'^2}, \quad \frac{\mathbf{k}'^2}{2\mu} = \mu\left(\frac{1}{\sqrt{1 - |\mathbf{v}|^2}} - 1 \right). \tag{2.10}$$

Thence, taking into consideration the determination (2.9) and expressions in Eq. (2.10), we find

$$|\mathbf{v}| = \frac{2u}{1+u^2}, \quad \mathbf{k}'^2 = \mu^2(u'_{\text{rel}})^2, \tag{2.11}$$

where

$$u'_{\text{rel}} = \frac{2u}{\sqrt{1-u^2}} \tag{2.12}$$

is the relative velocity of an effective relativistic particle with mass m'.

Thus, in terms of relative velocity of an effective relativistic particle (2.12), the S-factor (2.8) and L-factor (2.7) are given by expressions

$$S_{\text{uneq}}(u'_{\text{rel}}) = \frac{X_{\text{uneq}}(u'_{\text{rel}})}{1 - \exp\left[-X_{\text{uneq}}(u'_{\text{rel}}) \right]}, \quad X_{\text{uneq}}(u'_{\text{rel}}) = \frac{2\pi\alpha}{u'_{\text{rel}}}, \tag{2.13}$$

$$L_{\text{uneq}}(u'_{\text{rel}}) = \prod_{n=1}^{\ell} \left[1 + \left(\frac{\alpha}{nu'_{\text{rel}}} \right)^2 \right] S_{\text{uneq}}(u'_{\text{rel}}). \tag{2.14}$$

The S-factor in Eq. (2.13) only formally has the same form, as the nonrelativistic S-factor (1.2). However, the S-factor in Eq. (2.13) has an obviously relativistic nature since as the argument r (the module of radius-vector $\mathbf{r}$) in the Coulomb potential (1.1) and the relative velocity of an effective relativistic particle (2.12), according to Eq. (2.11), possesses this property as well.
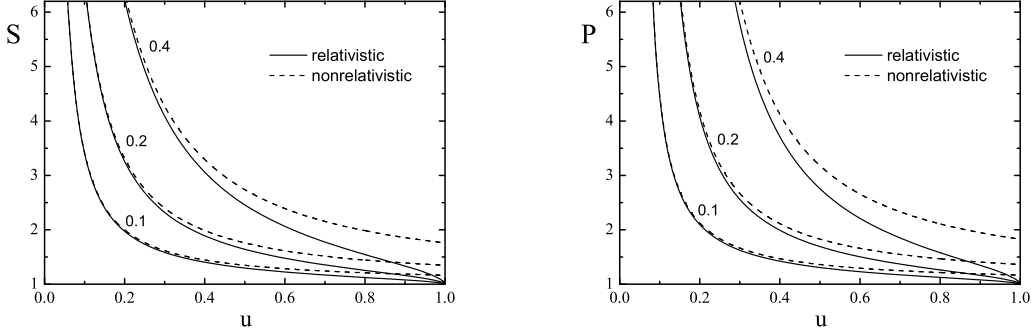
Figure 1: Behavior of the *S*- and *P*-factors at different values of the parameter α (the numbers at the curves).

The relativistic threshold factors (2.13) and (2.14) has the following important properties. In the nonrelativistic limit, $u \ll 1$, they reproduces the well-known nonrelativistic result. In the relativistic limit, $u \to 1$, the factors (2.13) and (2.14) go to unity. In the case of equal masses they coincides with *S*-factor (1.3) and *L*-factor (1.4). In the ultrarelativistic limit, as it was argued in [33], [34], the bound state spectrum vanishes since mass of an effective relativistic particle $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory where an additional dimensional parameter appears. Thus, within a potential model the relativistic factors (2.13) and (2.14) reproduces both the known nonrelativistic and the expected ultrarelativistic limits.

To illustrate the differences between the new relativistic *S*- and *P*-factors in Eqs. (2.13) and (2.14) ($\ell = 1$) with their the nonrelativistic analoges in more detail, in Fig. (1) we plot the behavior of these factors as functions of u at different values of the parameter α (the numbers at the curves).

From this figures one can see that in the region of nonrelativistic values of u, $u \leq 0.2$, where their the influence are big, the differences between the new relativistic *S*- and *P*-factors and their the nonrelativistic analoges are practically absent. However, when α increases, the nonrelativistic expressions gives a less suitable result in the region of large values u, in particular, as $u \to 1$.

Thus, the above analysis demonstrates that the relativistic *S*- and *L*-factors in Eqs. (2.13) and (2.14), as would be expected, coincides in form with their the nonrelativistic analoges. However, the relative velocity of an effective relativistic particle (2.12) emerging instead of the system of two particles, now plays role of the parameter of velocity, but not the relativistic relative velocity of interacting particles, **v**.

The principal contribution to the function $R(s)$ for the vector current with the *S*-factor can be written as

$$R(s) \to R_V^{(0)}(s) = \left[1 - \frac{(m_1 - m_2)^2}{s} \right]^2 \left[\frac{u(3 - u^2)}{2} + \frac{(m_1 - m_2)^2}{2s} u^3 \right] S(u, \alpha), \qquad (2.15)$$

where the total c. m. energy of interacting particles, $\sqrt{s}$, according to Eq. (2.9), can be expressed in terms of the "velocity" u as $s = [(m_1 + m_2)^2 - (m_1 - m_2)^2 u^2]/(1 - u^2)$. The corresponding expression without the *S*-factor can be found in paper [35]. By using this formula, we study the influence of the *S*-factor to the function $R_V^{(0)}$. For Fig. (2) is shown dependence of the behaviour of the value $R_V^{(0)}$ with the new *S*-factor (2.13) as a function of dimensionless variable $\sqrt{s}/(m_1 + m_2)$ at different values of the parameter α (the numbers at the curves). This figure shows that the influence
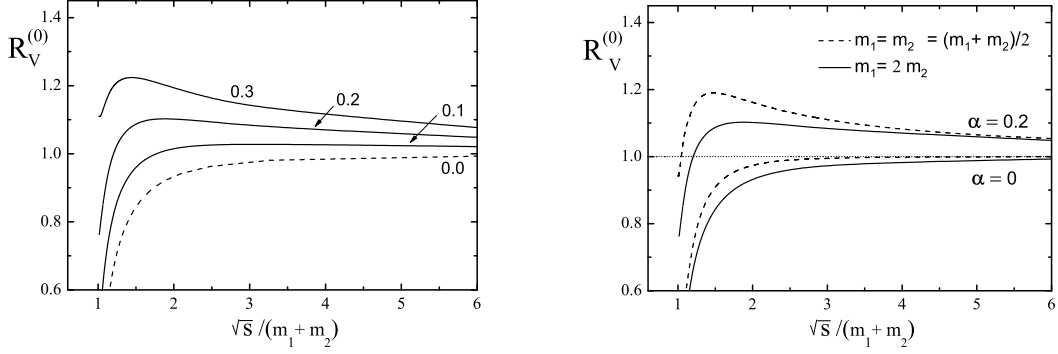
Figure 2: Behavior of the function $R_V^{(0)}$.

of the new S-factor (2.13) is much stronger in the threshold region and with growing energy $\sqrt{s}$ weakens, and all curves approach unity.

3. Conclusion

Thus, the new relativistic Coulomb-like threshold resumming L-factor (2.14) in QCD for arbitrary orbital angular momentum $\ell \geq 0$ was obtained. For this aim the relativistic quasipotential equation in relativistic configuration representation [27] with the Coulomb potential of interaction of two relativistic particles of unequal masses was used. The new L-factor reproduce both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential. As the L-factor (2.14) was obtained within the framework of completely covariant method, one can expect that this factor takes into account more adequately relativistic nature of interaction. We have suggested new expression for $R(s)$ in which threshold singularities are summarized by a potential contribution. It was demonstrated that the new relativistic S-factor has the influence on the behavior of the function $R(s)$.

Acknowledgments

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