

Pion form factor in time-like region in a framework of relativistic composite quark model.

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The analytical properties of the pion form factor in the complex plane of momentum transfers are discussed in the Poincaré-invariant quark model. It is shown, that these analytical properties are similar to ones following from basic principles of quantum field theory. The strong dependence of pion form factor on the type of the pion wave functions is obtained in the time-like momentum transfers region. The conditions of resonance behavior of the form factor in this region are formulated in our model. The simple examples of wave functions giving the resonance behavior of the form factor are constructed.

*The XIXth International Workshop on High Energy Physics and Quantum Field Theory, QFTHEP2010
September 08-15, 2010, Golitsyno, Moscow, Russia*

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1. Introduction.

At present time the electromagnetic structure of mesons is the object of intensive studying. In particular, a large number of the impressing measurements of pion space-like [1, 2], and pion and kaon time-like form factors [3, 4] have been performed in recent years. The theoretical interest to these objects is connected with the limits of applicability of the nonperturbative composite quark model and with the determination of the region of the perturbative QCD.

Among the most successful modern approaches to the description of the pion electromagnetic form factor in terms of quark degrees of freedom it is necessary to note ones based on ADS/CFT duality [5, 6] and the Poincaré-invariant quantum mechanics (PIQM) [8, 9, 10]. The both of approaches gives a good description of form factor in space-like region. But ADS/CFT approach does not give the description of finite width of ρ -resonance in time-like region, and analytical properties of form factor do not coincide with ones following from general principles of quantum field theory (QFT). Therefore is of interest to study the time like behavior and analytical properties of form factor in PIQM approach.

2. Pion form factor in the instant form of PIQM.

PIQM is a relativistic quantum theory with finite number of degrees of freedom (the general review of PIQM see in [7]). In this theory mesons are described as bound state of constituent quark and anti-quark. In our work we use the one of possible formulations of PIQM - the instant form PIQM (see, e.g. [8]). In the case of two point constituents our approach gives the following expression for the pion form factor in space-like region [8]:

$$F_\pi(t) = \frac{1}{4} \int_{\Omega_{ss'}} ds ds' \frac{\varphi(s)\varphi(s')}{\sqrt{(s-4M^2)(s'-4M^2)}} \frac{(-t)(s+s'-t)}{\lambda^{3/2}(s,t,s')}, \quad (2.1)$$

here the region of integration is $\Omega_{ss'} = \{(s, s') | s \in [4M^2, +\infty); s' \in [s_1(s, t), s_2(s, t)]\}$,

$$s_{1,2}(s, t) = s + t - \frac{ts}{2M^2} \pm \frac{1}{2M^2} \sqrt{(-t)(4M^2 - t)s(s - 4M^2)}, \quad (2.2)$$

$\lambda(s, t, s') = s^2 + t^2 + s'^2 - 2(ss' + st + s't)$, M is constituent mass.

The instant-form wave function $\varphi(s)$ is connected with the wave function depending on the absolute value of three-momentum $u(k)$:

$$\varphi(s) = \sqrt[4]{s} \frac{\sqrt{s-4M^2}}{2} u\left(\frac{\sqrt{s-4M^2}}{2}\right).$$

The function $u(k)$ is normalized by a condition:

$$\int_{-\infty}^{\infty} k^2 u^2(k) dk = 1. \quad (2.3)$$

The deal of our work is to analyze the analytical properties and behavior of model expression (2.1) in space-like ($t < 0$) and time-like ($t > 0$) region of transferred momentum, and dependence of this behavior on the kind of wave functions $u(k)$.

3. Analytical properties of form factor expression.

The expression (2.1) is difficult for analysis, because of complicated form of limits of integration and integrand. To simplify this expression we will use properties of symmetry of the integration region and integrand. It is easy to see, that line $s' = s$ is the axes of symmetry of $\Omega_{ss'}$ at $t < 0$ and the integrand is invariant under the permutation of variables s and s' . Hence, integration in the (2.1) can be carried out over the half of the region $\Omega_{ss'}$. The new region of integration is:

$$\Omega_{1ss'} = \Omega_{ss'} \cap \{(s, s') | s \in [0, \infty), s' \in [0, s]\} .$$

Let us introduce a new variables:

$$\begin{aligned} x &= s + s' - t , \\ y &= \sqrt{4ss'} . \end{aligned} \quad (3.1)$$

In this variables, the region $\Omega_{1ss'}$ maps to the region in plane (x, y) :

$$\Omega_{xy} = \{(x, y) | x \in [x_{min}(t), +\infty), y \in [y_1(x, t), y_2(x, t)]\} .$$

The expression of form factor in the new variables has the form:

$$F_\pi(t) = \frac{(-t)}{8\sqrt{2}} \int_{x_{min}(t)}^{+\infty} \int_{y_1(x, t)}^{y_2(x, t)} \psi(x, y, t) \frac{xy^{3/2} dy dx}{\sqrt{(t+x)^2 - y^2} (x^2 - y^2)^{3/2}} , \quad (3.2)$$

here $x_{min}(t) = (4M^2 - t) + 2M\sqrt{4M^2 - t}$, $y_1(x, t) = x\sqrt{\frac{4M^2}{4M^2 - t}}$ and $y_2(x, t) = x + t$, $\psi(x, y, t)$ is a product of wave functions $u(k)$ in new variables:

$$u\left(\frac{\sqrt{s - 4M^2}}{2}\right) u\left(\frac{\sqrt{s' - 4M^2}}{2}\right) = \psi(s + s' - t, \sqrt{4ss'}, t) .$$

The analytical properties of expression (3.2) are described by the theorem:

Theorem. Integral representation (3.2) defines an analytical function in a plane t with a cut from $4M^2$ to infinity, if the following conditions are satisfied:

1. The path of integration in x -plane γ_x is begun in point $x_{min}(t)$ and goes to infinity in the half-plane $\Re x > 0$.
2. In y -plane the path of integration γ_y exists between points $y_1(x, t)$ and $y_2(x, t)$ with condition:

$$\forall x \in \gamma_x : \max_{y \in \gamma_y} |\psi(x, y, t)| < \infty .$$

3. If $|x| \rightarrow \infty$, so

$$\max_{y \in \gamma_y} |\psi(x, y, t)| \sim o\left(\frac{1}{\sqrt{|x|}}\right) .$$

4. $\forall \tau > 4M^2 : \psi(x^*, y^*, \tau - i0) = \psi^*(x, y, \tau + i0)$.

The proof of the theorem is based on the proof of absolute convergence of the integral (3.2). Let us remark, that existing singular point of the integrand $y_2(x,t) = x+t$ no affect on absolute convergence.

The existence of the cut in the t -plane is proved with help of properties of integration limits in (3.2). If $t = \tau \pm i0$ ($\tau < 4M^2$), so $x_{min}(t)$, $y_1(x,t)$ and $y_2(x,t)$ are real, and values of $F_\pi(\tau \pm i0)$ are real also. If $\tau > 4M^2$, so limits of integration and value of integral became complex, and values of limits on the upper lip of the cut in t -plane are complex conjugate with ones on the lower lip. Hence, the values of $F_\pi(\tau \pm i0)$ are not equal on the upper and lower lip. From condition 4 the relation follows:

$$F_\pi(\tau - i0) = (F_\pi(\tau + i0))^* . \quad (3.3)$$

4. Wave functions in sense of PIQM in space- and time-like regions.

In our model, information about interaction between quarks in the pion is contained in wave functions $u(k)$. We will use phenomenological approach to obtain the possible wave functions. Wave functions, which are using in this type of models, can be divided in two large classes:

1. Wave functions which has no singularities¹ in k -plane. The example of such wave function and corresponding function ψ from (3.2) is Gaussian wave function:

$$u(k) = N_{HO}(b)e^{\frac{-k^2}{2b^2}} , \quad \psi(x,y,t) = N_{HO}^2(b)e^{\frac{M^2}{b^2}} e^{-\frac{x+t}{8b^2}} . \quad (4.1)$$

2. Wave functions which has power type singularities in plane k . The example is:

$$u(k) = N_{PL}(b) \left(1 + \frac{k^2}{b^2}\right)^{-n} , \quad \psi(x,y,t) = \frac{N_{PL}^2(b)b(4b^2)^{2n}}{\left(\frac{y^2}{4} + \alpha(x+t) + \alpha^2\right)^n} , \quad (4.2)$$

here $\alpha = 4(b^2 - M^2)$.

The normalization constants are defined by condition (2.3), and parameter b is describing the confinement scale in k -space.

The numerical calculations with different wave functions shows, that wide class of the wave functions, constructed with taking into account the confinement, allows to fit the experimental points in space-like region (see, e.g. [11]). The results of calculation of expression (3.2) with ψ -functions (4.1) and (4.2) at space-like and time-like t are shown on figure 1.

On this figure, we can see a good agreement between expression (3.2) and Breit-Wigner approximation of experimental data in space-like region:

$$F_{\pi BW}(t) = \frac{4\kappa^2}{4\kappa^2 - t - 2i\kappa\Gamma} , \quad (4.3)$$

here parameters of approximation $\kappa = 0.375$ GeV and $\Gamma = 0.1$ GeV, are given in the work [5].

As figure 1(a) shows, the time-like behavior of form factor is more strongly dependent on wave function, than space-like behavior. The Gaussian wave function with any values of parameters,

¹In our terms, the point z_s is *singularity* of a function $f(z)$ if $\lim_{z \rightarrow z_s} f(z) = \infty$.

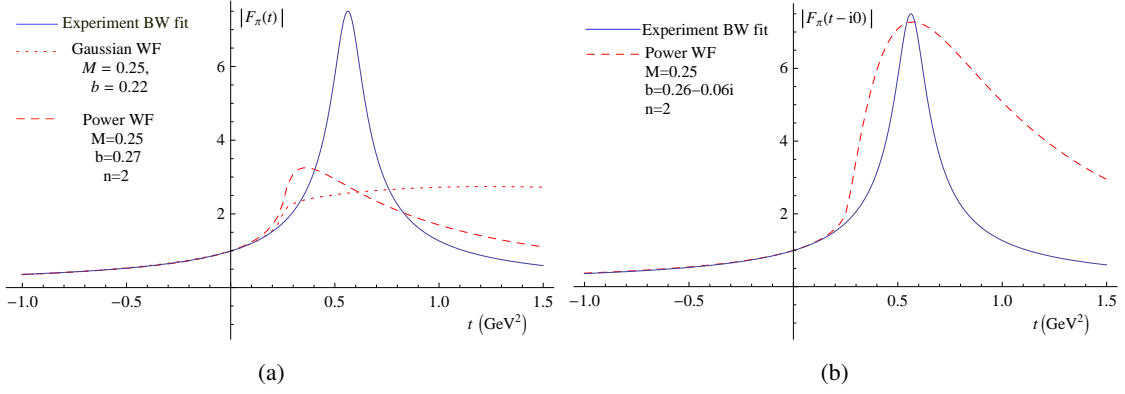


Figure 1: Results of numerical calculations. (a) Solid line - Breit-Wigner approximation of experimental data, dotted line - wavefunction (4.1) and dashed line - wavefunction (4.2) with real b parameter. (b) Solid line - Breit-Wigner approximation of experimental data, dashed line - wavefunction (4.2) with complex b parameter.

can not describe the time-like behavior, but, as figure 1(b) shows, that the wave function (4.2) demonstrates a qualitative agreement with experiment, if parameter b is complex. Unfortunately, usage of complex values of b leads to the contradiction with condition 4 of the theorem. In this case, form factor became complex in space-like region, and relation (2.3) is broken. Considering the strong dependence of time-like behavior of expression (2.2) from the wave function, we can supply the problem of construction of wave function, which describes the time-like and space-like behavior of form factor.

5. Conditions of existence of the time-like resonance.

The origin of a resonance behaviour of expression (3.2) in case of power-like wave function (4.2) is due to the contribution of singularities of a wave function to the value of integral. To construct a model wave function, giving the resonance in the pion form factor in time-like region we need some criterion, connecting the wave function singularities with position of a peak of form factor. To derive this criterion, we will introduce a new variable:

$$t = 4M^2 - z^2. \quad (5.1)$$

In the z -plane form factor becomes a one-valued function. The transformation (5.1) maps the space-like region of t into the positive ray of real z - axes and the upper and lower lips of the cut into the positive and negative directions of imaginary z -axes respectively. The peak of absolute value of form factor in time-like region is caused by the singularities of $F_\pi(z)$ in the left half of z plane. The singularity of function $F_\pi(z)$ in the point $z_s = z_s'' + iz_s'$ ($z_s'' < 0$) leads to maximum of $|F_\pi(t)|$, which is located near the point $t_{max} = z_s''^2 + 4M^2$. Hence, we need a criterion, which connects the position of singularities of ψ function in y plane with singularities of form factor in plane z .

To obtain this criterion, one can note, that for divergence of integral (3.2) in some point z_s , we will place the singularity of function ψ to the end point of integration path in plane y . The motions of the end points $y_{1,2}(x, t)$ are dependent on integration path in x -plane, therefore we will place the

singularity to some point, which is independent on integration paths in planes of x and y . When taken into account the expressions for $y_{1,2}(x, t)$ and $x_{min}(t)$ in (3.2), one can find, that this point is:

$$y_s(z_s) = y_1(x_{min}(4M^2 - z_s^2), 4M^2 - z_s^2) = y_2(x_{min}(4M^2 - z_s^2), 4M^2 - z_s^2) = 2M(z_s + 2M),$$

In terms of function ψ , the criterion for the singularity of $F_\pi(z)$ in point z_s has the form:

$$\psi(x_{min}(4M^2 - z_s^2), 2M(z_s + 2M), 4M^2 - z_s^2) = \infty. \quad (5.2)$$

It is easy, to rewrite this criterion in terms of function $u(k)$:

$$u(\sqrt{K}) = \infty, \quad (5.3)$$

here $K = \frac{M}{4}(z_s - 2M)$.

6. Construction of model wave functions. Numerical results.

Now we will construct the sample model wavefunction, based on the function (4.2) with using of criterion (5.3). Our model wave function will satisfy the condition 4 of the theorem, and provide the description of experiment in space-like region. The simplest rational function, which satisfies condition 4, and allows to introduce the singularities in plane z , has the form:

$$u_s(k) = \frac{N}{[(k^2 - K)(k^2 - K^*)]^n}.$$

The asymptotic behaviour of this function in the neighborhood of the point \sqrt{K} is $(k - \sqrt{K})^{-n}$, but asymptotic, when $k \rightarrow \infty$ is k^{-4n} . Such a dramatic difference, leads to a mismatch of behavior of form factor in space-like and time-like regions. Thus, we will add a growing factor to the numerator of a function:

$$u_{model}(k) = N \left[\frac{k^2 + a^2}{(k^2 - K)(k^2 - K^*)} \right]^n. \quad (6.1)$$

In order to ensure coincidence with the experiment in space-like region, we must impose the condition of the asymptotic matching of new wave function with the old one:

$$\lim_{k \rightarrow \infty} \frac{1}{N_{PL}(b_0)} \left(1 + \frac{k^2}{b_0^2} \right)^n u_{model}(k) = 1. \quad (6.2)$$

We obtain the following equation for the constant a , when taken into account the condition (6.2) and the normalization condition (2.3):

$$\int_{-\infty}^{+\infty} \left[\frac{k^2 + a^2}{(k^2 - K)(k^2 - K^*)} \right]^{2n} k^2 dk = b_0^{3-4n} B(2n - 3/2, 3/2). \quad (6.3)$$

Condition (6.2) also fixes the real part of z_s , so that $K = \frac{1}{4}(iMz'_s - 2b_0^2)$, and $N = (b_0^{3-4n} B(2n - 3/2, 3/2))^{-1/2}$.

The numerical calculations with new wave function are more complicated problem, than with old one, because we introduced an additional singularities into wave function. Hence, it necessary

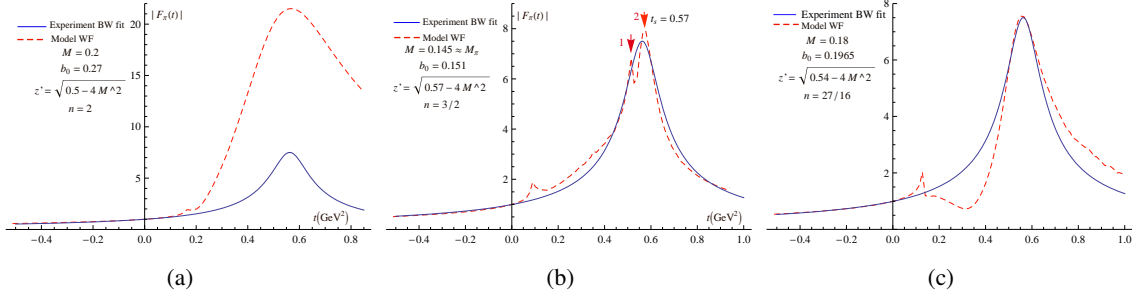


Figure 2: Results of numerical calculations with model wavefunction. Solid line - Breit-Wigner approximation of experimental data, dashed line - model WF with $n = 2$ and correct bypassing of $y_{\psi_1}^+$. (b) Dashed line - model WF with $n = 3/2$ and correct bypassing of $y_{\psi_1}^+$. (c) Dashed line - model WF with $n = 27/16$ and without bypassing of $y_{\psi_1}^+$.

to choose the path of integration in y -plane, which bypasses correctly this new singularities. The continuity of form factor in point $t = 4M^2$ is the condition for choosing of this contour. The function ψ corresponded with wave function (5.1) has four singular points:

$$y_{\psi_1}^{\pm}(x, t) = \pm \sqrt{-4(\alpha_1(x+t) + \alpha_1^2)},$$

$$y_{\psi_2}^{\pm}(x, t) = \pm \sqrt{-4(\alpha_1^*(x+t) + (\alpha_1^*)^2)}.$$

here $\alpha_1 = -4(K + M^2)$. The correct choice of the contour is the path which is begun in a point $y_1(x, t)$, bypasses the point $y_{\psi_1}^+$ from the outer side, and ends in the point $y_2(x, t)$.

The results of numerical calculations with model wave function are presented on a figure 2. If parameter n is integer, we can provide a correct analytical continuation of form factor to time-like region, by choosing of integration path. Unfortunately, as figure 2(a) shows, the integer value of n is not able to describe the time-like behavior of form factor. Hence, we need $1 < n < 2$, but in this case, we can't choose the correct integration path because of the appearance of additional cuts in plane y . In this case the integration must be carried out not on the complex plane y , but on the Riemannian surface of the integrand. This is a difficult computational problem.

The results of calculations with and without bypassing of additional singularity are shown on figures 2(b) and 2(c). This calculations are made without taking into account the additional cuts, as a result, there is a break of continuity of $F_{\pi}(t)$ in a point $t = 4M^2$ on this figures. This results can be regarded as indicative, because the contribution of non continuity of integrand is small. The resonance in point 2 on the figure 2(b) is corresponding to the criterion, and resonance in point 1 is due to bypass the singularity $y_{\psi_1}^+$, if we don't bypass the singularity, this resonance disappear, as shown on figure 2(c).

This results are leading to the conclusion, that our model wave function can describe only the small vicinity of the resonance.

7. Discussion.

As shown above, the analytical properties of our model expression for the pion form factor are the similar to ones following from basic principles of QFT. This similarity allows us to hope,

that the correct choice of constituents wave function will describe correctly the time-like behavior of form factor. This hope is supported by the fact, that space-like behavior of form factor can be described by large class of wave functions, practically independent from time-like behavior.

Thus, the time-like behavior of the pion form factor, carries much more information about interaction between constituents, than space-like one. Reconstruction of model wave function, which will describe this behavior is nontrivial problem. In this paper, an effective enough attempt was made to create the instrument for such reconstruction - the criterion of time-like resonance behavior.

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