Shock wave in the Friedmann-Robertson-Walker space-time

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According to ’t Hooft shock waves in the Minkowski space-time, as well as in (A)dS can be used to describe ultra relativistic particles collisions. In this talk the generalization of this construction for the ultrarelativistic particles in the Friedman-Robertson-Walker space-time is presented.
1. Introduction

The shock gravitation waves can be obtained by Lorentz transformations in the ultrarelativistic limit of the stationary solutions of classical gravitation theory [1]. At 1987 G. ’t Hooft [2] proposed to use the Aichelburg-Sexl shock wave [1] as ultrarelativistic particle moving in Minkowski space-time. The problem of shock waves construction in different metrics is related with consideration of ultrarelativistic particles collision models in these metrics. Shock waves in (anti) de Sitter ((A)dS) space-time have been constructed in [3], [4] and used to describe ultra relativistic particles collisions [5], [6] in these backgrounds. Subtleties of obtaining shock waves in dS space are discussed in [7].

The goal of the present paper is to get shock waves in the Friedmann-Robertson-Walker (FRW) metric

\[ ds_0^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \] (1.1)

Our starting point is the McVittie metric [8]

\[ ds^2 = \frac{(1-\mu)^2}{(1+\mu)^2} dt^2 + a^2(t) (1+\mu)^4 (dx^2 + dy^2 + dz^2), \] where \( \mu = \frac{m}{2a(t)r}, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad m \text{ is a parameter} \] (1.2)

by choosing \( a(t) = e^{Ht} \) we can get the Schwarzschild-de Sitter metric.

For McVittie metric (1.2) with \( a(t) = t^n \) the Hubble parameter \( H \equiv \frac{\dot{a}}{a} = \frac{n}{t} \) [9]. The Ricci scalar \( R = 12H^2 + \frac{1+\mu}{1-\mu}H \). If \( \mu = 1 \), then we have singularity, which can be interpreted such as cosmological Big Bang singularity. Note, that if \( t \to \infty \), the metric (1.2) reduces to the Schwarzschild black hole of mass \( m \) [10].

In the present paper we construct the shock wave in FRW space-time with \( a(t) = t^n \). In Section 2 we consider relations between coordinates of five-dimensional (5D) Minkowski space-time on non-stationary hyperboloid and FRW space-time. In Section 3 we apply the small mass approximation to the McVittie metric with \( a(t) = t^n \). In Section 4 we make boost of the McVittie metric in the small mass approximation. The boosted metric has a divergence and in Section 5 we consider this divergence in the generalized function setting. The shock wave metric is presented in Section 6. In Section 7 we shortly summarize our results.

2. Coordinates relations

Let us remind, that in the five-dimensional (5D) Minkowski space-time

\[ dS_{5M}^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2, \] (2.1)

the stationary hyperboloid

\[ -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = A^2 = \text{const} \] (2.2)
is 4D de Sitter space-time.

This relation can be generalized for FRW case. The explicit coordinate transformations, which show the equivalence between a four-dimensional spatially flat cosmology and an appropriate sub-manifold in the flat 5D Minkowski space-time, have been presented in [11].

In the present Section we consider the analog to the de Sitter metric (2.1) with non-stationary hyperboloid condition

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = b^2(t)$$

(2.3)

and FRW metric (1.1) with

$$a(t) = t^n.$$  

(2.4)

We suppose that coordinates related by the expressions:

$$Z_0 = \frac{1}{2}a(t) - \frac{1}{2}b^2(t) + \frac{1}{2}a(t)(x^2 + y^2 + z^2),$$

(2.5)

$$Z_4 = \frac{1}{2}a(t) + \frac{1}{2}b^2(t) - \frac{1}{2}a(t)(x^2 + y^2 + z^2),$$

$$Z_1 = a(t)x, \quad Z_2 = a(t)y, \quad Z_3 = a(t)z.$$

Substituting (2.5) to (2.1) we get the following condition to $a(t)$ and $b(t)$:

$$-\left( \frac{da(t)}{dt} \frac{b(t)}{a(t)} \right)^2 + 2\frac{da(t)}{dt} \frac{db(t)}{dt} \frac{b(t)}{a(t)} + 1 = 0.$$  

(2.6)

For considering case (2.4), from (2.6) we obtain $b^2(t)$:

$$b^2(t) = \frac{t^2}{n(n-2)}.$$  

(2.7)

3. McVittie metric in the small mass approximation

Let us consider the small mass approximation of (1.2) with (2.4). Let $m$ is a small parameter. To zero and first orders in $m^2$ the coefficients of (1.2) are as follows:

$$\frac{(1-\mu)^2}{(1+\mu)^2} \approx 1 - 4\mu, \quad (1+\mu)^4 \approx 1 + 4\mu.$$  

(3.1)

Thus we can consider

$$ds^2 = -(1 - 4\mu)dt^2 + a^2(t)(1 + 4\mu)(dx^2 + dy^2 + dz^2) \quad \text{or}$$

$$ds^2 = ds_0^2 + 4\mu(ds_0^2 + 2dt^2).$$  

(3.2)

Sustainable $da(t) = nt^{n-1}dt$ and (2.7), $dt^2$ can be represented by the form:

$$dt^2 = \frac{(da(t))^2}{n^2(n(n-2)b^2(t))^{n-1}}.$$  

(3.4)
5. Regularization

According to (2.5) we have the following expressions for $dt^2$ and $\mu$:

$$ dt^2 = \frac{(d(Z_0 + Z_1))^2}{n^2(n(n-2)(-Z_0^2 + Z_1^2 + Z_4^2))^{n-1}}, \quad \mu = \frac{m}{2\sqrt{Z_i^2}}, \quad i = 1, 3. \quad (3.5) $$

Thus, the metric (3.3) can be considered in the plane coordinates (2.5):

$$ ds^2 = dS_{SM}^2 + \frac{2m}{\sqrt{Z_i^2}} \left( dS_{SM}^2 + \frac{2d(Z_0 + Z_4)^2}{n^2(n(n-2)(-Z_0^2 + Z_1^2 + Z_4^2))^{n-1}} \right) \quad (3.6) $$

4. Boost

Consider the Lorentz transform in 5D Minkowski space-time:

$$ Z_0 = \gamma(\bar{Z}_0 + v\bar{Z}_1), \quad Z_1 = \gamma(\bar{Z}_1 + v\bar{Z}_0), \quad \gamma = \frac{1}{\sqrt{1-v^2}}. \quad (4.1) $$

The hyperboloid condition (2.3) is invariant under the Lorentz transformations

$$ -Z_0^2 + Z_1^2 + Z_4^2 = -\bar{Z}_0^2 + \bar{Z}_1^2 + \bar{Z}_4^2 \quad \text{and} \quad b^2(t) = \bar{b}^2(\bar{t}). \quad (4.2) $$

Let us apply the Lorentz transform (4.1) to metric (3.6):

$$ ds^2 = dS_{SM}^2 + \frac{2\bar{m}}{\sqrt{\gamma^2(\bar{Z}_0 + \bar{Z}_1)^2 + \bar{Z}_2^2 + \bar{Z}_3^2}} \left( dS_{SM}^2 + \frac{2d(\bar{Z}_0 + \bar{Z}_4)^2}{n^2(n(n-2)(-\bar{Z}_0^2 + \bar{Z}_1^2 + \bar{Z}_4^2))^{n-1}} \right), \quad (4.3) $$

where $d\bar{S}_{SM} = -d\bar{Z}_0^2 + d\bar{Z}_1^2 + d\bar{Z}_4^2, \bar{m} = m\gamma$.

The ultrarelativistic limit $\gamma \to \infty, v \to 1$ to (4.3) is

$$ ds^2 \bigg|_{v \to 1} \to dS_{SM}^2 + \frac{4\bar{m}\gamma}{\sqrt{\gamma^2(\bar{Z}_0 + \bar{Z}_1)^2 + \bar{Z}_2^2 + \bar{Z}_3^2}} \left( \frac{d(\bar{Z}_0 + \bar{Z}_4)^2}{n^2(n(n-2)(-\bar{Z}_0^2 + \bar{Z}_1^2 + \bar{Z}_4^2))^{n-1}} \right). \quad (4.4) $$

In the next section we consider the following limit

$$ \lim_{\gamma \to \infty} \frac{\bar{m}\gamma d(\bar{Z}_0 + \bar{Z}_1)^2}{\sqrt{\gamma^2(\bar{Z}_0 + \bar{Z}_1)^2 + \bar{Z}_2^2 + \bar{Z}_3^2}} \quad (4.5) $$

in the generalized function meaning.

5. Regularization

For consideration of the limit $\lim_{\gamma \to \infty} \int_{-\infty}^{\infty} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}}$, we can use the following trick:

$$ \int_{-\infty}^{\infty} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}} = \int_{-\infty}^{0} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}} + \int_{0}^{\infty} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}} + \int_{-1}^{1} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}}. $$

$$ \lim_{\gamma \to \infty} \int_{-\infty}^{\infty} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}} \to \int_{-\infty}^{0} \frac{f(U) dU}{|U|} + \int_{0}^{\infty} \frac{f(U) dU}{|U|} + \lim_{\gamma \to \infty} \int_{-1}^{1} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}}. $$
Examine the last integral. It is evidently that
\[
\int_{-1}^{1} \frac{\gamma f(U) dU}{\sqrt{\gamma^2 U^2 + X^2}} = \int_{-1}^{1} \frac{\gamma f(U) - f(0) dU}{\sqrt{\gamma^2 U^2 + X^2}} + \int_{-1}^{1} \frac{\gamma f(0) dU}{\sqrt{\gamma^2 U^2 + X^2}}.
\] (5.1)

We can just write
\[
\lim_{\gamma \to \infty} \int_{-1}^{1} \frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} [f(U) - f(0)] dU \to \int_{-1}^{1} \frac{f(U) - f(0)}{|U|} dU.
\] (5.2)

Direct calculation gives that
\[
\int_{-1}^{1} \frac{\gamma f(0) dU}{\sqrt{\gamma^2 U^2 + X^2}} = \ln \frac{\gamma^2 + \gamma \sqrt{\gamma^2 + X^2}}{-\gamma^2 + \gamma \sqrt{\gamma^2 + X^2}} f(0).
\] (5.3)

Using the Taylor series for large \( \gamma \) in logarithm argument
\[
\ln \frac{\gamma^2 + \gamma \sqrt{\gamma^2 + X^2}}{-\gamma^2 + \gamma \sqrt{\gamma^2 + X^2}} \to \ln \frac{4\gamma^2}{X^2},
\] (5.4)

we can write
\[
\lim_{\gamma \to \infty} \left[ \int_{-\infty}^{\infty} \frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} [f(U) dU - f(0) \ln \gamma^2] \right] = -f(0) \ln \frac{X^2}{4} + \int_{-\infty}^{\infty} \left( \frac{1}{|U|} \right)_{\text{reg}} f(U) dU + O\left(\frac{1}{\gamma^2}\right)
\]

where
\[
\int_{-\infty}^{\infty} \left( \frac{1}{|U|} \right)_{\text{reg}} f(U) dU = \int_{-1}^{1} \frac{f(U) - f(0)}{|U|} dU + \int_{-\infty}^{1} \frac{1}{|U|} f(U) dU + \int_{1}^{\infty} \frac{1}{|U|} f(U) dU
\] (5.5)

or
\[
\lim_{\gamma \to \infty} \left[ \frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} - \delta(U) \ln \gamma^2 \right] = -\delta(U) \ln \frac{X^2}{4} + \left( \frac{1}{|U|} \right)_{\text{reg}}
\] (5.6)

6. Shock wave

After the regularization we have the gravitational waves metric
\[
d s_f^2 = d S_{5M}^2 + F(\bar{U}) \delta(\bar{U}) d(\bar{U})^2, \quad \bar{m} = \bar{m} \ln \gamma^2, \quad \bar{U} = \bar{U}_0 + \bar{U}_1,
\] (6.1)

where
\[
F(\bar{U}) = \frac{4\bar{m}}{n^2(n(n-2)\bar{b}^2(i))n-1}, \quad \bar{b}^2(i) = -\bar{U} \bar{V} + Z_2^2 + Z_3^2 + Z_4^2, \quad \bar{V} = Z_0 - Z_1.
\] (6.2)
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Conclusively, the shape of shock wave in 5D Minkowski space-time on non-stationary hyperboloid is

\[ F(0) = \frac{4\tilde{m}}{n^2(n(n-2)(\tilde{Z}_2^2 + \tilde{Z}_3^2 + \tilde{Z}_4^2))^{n-1}}. \] (6.3)

Consider metric (6.1) in cosmological coordinates. Suppose that initially, before boost, we had some symmetry \( Z_1 = Z_2 = Z_3. \)

Applying the Lorenz transform (4.1) to \( Z_1, Z_2, Z_3 \) we obtain

\[ \gamma(\tilde{Z}_1 + v\tilde{Z}_0) = \tilde{Z}_2 = \tilde{Z}_3. \] (6.4)

From here we get

\[ \tilde{b}^2(\tilde{t}) = \tilde{a}^2(\tilde{t})\tilde{l} \quad \text{or} \quad \tilde{a}^2 = \frac{\tilde{b}^2(\tilde{t})}{\tilde{l}}, \] (6.5)

where \( \tilde{b}^2(\tilde{t}) = b^2(t) = \frac{t^2}{n(n-2)} \) and \( \tilde{l} \) can has the following forms

\[ \tilde{l} = \tilde{x} + \frac{v}{2}(1 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2) - \frac{\tilde{y} + \tilde{z}}{2\gamma}, \] (6.6)

\[ \tilde{l} = \tilde{x} + \frac{v}{2}(1 + \tilde{x}^2 + 2\tilde{y}^2) - \frac{\tilde{y}}{2\gamma}, \] (6.7)

\[ \tilde{l} = \tilde{x} + \frac{v}{2}(1 + \tilde{x}^2 + 2\tilde{z}^2) - \frac{\tilde{z}}{\gamma}. \] (6.8)

Accordingly, we obtain that \( \tilde{U} = \frac{\pm t}{\sqrt{n(n-2)\tilde{l}}} \left( \tilde{x} + \frac{1 + \tilde{r}^2 - \tilde{l}}{2} \right), \tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2. \)

Thus, the wave shape in cosmological coordinates has the form:

\[ F(0) = \frac{4\tilde{m}}{n^2\tilde{l}^2(n-1)}. \] (6.9)

7. Summarizing

It is proposed to use the boosted McVittie metric such as model of ultrarelativistic particle in the Friedmann-Robertson-Walker space-time with \( a(t) = t^n. \) The shock wave corresponding ultrarelativistic particle in the Friedmann-Robertson-Walker space-time is constructed.

The authors thank to Andrey Bagrov for discussions.

I.A. is supported in part by grants of Russian Ministry of Education and Science Nsh–8265.2010.1 and by RFBR grant 08-01-00798.

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