PROCEEDINGS OF SCIENCE

Haag's theorem in $SO(1,k)$ invariant quantum field theory

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Generalized Haag's theorem has been proven in $SO(1,k)$ invariant quantum field theory. Apart from the above mentioned $k+1$ variables there can be arbitrary number of additional coordinates including noncommutative ones in the theory. New consequences of generalized Haag's theorem are obtained.
It has been proven that the equality of four-point Wightman functions in two theories leads to the equality of elastic scattering amplitudes and thus the total cross-sections in these theories.
In space-space noncommutative quantum field theory in four-dimensional case it has been proved that if in one of the theories under consideration S-matrix is equal to unity, then in another theory S-matrix is unity as well.

The XIXth International Workshop on High Energy Physics and Quantum Field Theory
8-15 September 2010
Golitsyno, Moscow, Russia

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PoS(QFTHEP2010)080

1. Introduction

In this report we consider Haag's theorem – one of the most important results of axiomatic approach in quantum field theory (see [1, 2]). In the usual Hamiltonian formalism it is assumed that asymptotic fields are related with interacting fields by unitary transformation.
Haag's theorem shows that in accordance with Lorentz invariance of the theory the interacting fields are also trivial which means that corresponding S-matrix is equal to unity. So the usual interaction representation can not exist in the Lorentz invariant theory. In four dimensional case in accordance with the generalized Haag's theorem four first Wightman functions coincide in two related by the unitary transformation theories.

Let us recall that n-point Wightman functions $W(x_1,\dots,x_n)$ are $\langle \Psi_0, \varphi(x_1)\dots\varphi(x_n)\Psi_0\rangle$, where Ψ_0 is a vacuum vector. Actually in accordance with translation invariance Wightman functions are functions of variables $\xi_i = x_i - x_{i+1}$, $i = 1,\dots,n-1$. At first Haag's theorem is proved in $SO(1,3)$ invariant theory in four dimensional case.

In last years noncommutative generalization of QFT - NC QFT - attracts interest of physicists as in some cases NC QFT is the low energy limit of string theory (see [3, 4, 5]). Haag's theorem in NC QFT was considered in [6, 7].

NC QFT is defined by the Heisenberg-like commutation relations between coordinates

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{1.1}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.
It is very important that NC QFT can be formulated in commutative space if the usual product between operators (precisely between corresponding test functions) is substituted by the star (Moyal-type) product.

Now the theories in spaces of arbitrary dimensions are widely considered. Thus it is interesting to consider Haag's theorem in the general case of $k+1$ commutative variables (time and k spatial coordinates) and arbitrary number m of noncommutative coordinates. This extension of Haag's theorem is a goal of our work.

2. Generalized Haag's theorem in four dimensional case

In axiomatic QFT fields $\varphi(f)$ smeared on all four coordinates are unbounded operators in the state vectors space. In the derivation of Haag's theorem it is necessary to assume that fields smeared on the spatial coordinates are well defined operators.

Let us recall generalized Haag's theorem in four dimensional case.

Theorem. *Let $\varphi_1(f,t)$ and $\varphi_2(f,t)$ belong to two sets of irreducible operators in Hilbert spaces H_1 and H_2 correspondingly. We assume that both theories are Poincare invariant, that is*

$$U_j(a,\Lambda)\,\varphi_j(x)U_j^{-1}(a,\Lambda) = \varphi_j(\Lambda x + a), \tag{2.1}$$

$$U_j(a,\Lambda)\Psi_{0j} = \Psi_{0j}, \qquad j = 1,2. \tag{2.2}$$

Let us suppose also that there exists unitary transformation V relating fields and vacuum states as well in two theories at any t:

$$\varphi_2(f,t) = V\varphi_1(f,t)V^{-1}, \tag{2.3}$$

$$c\Psi_{02} = V\Psi_{01}, \tag{2.4}$$

where c is a complex number with module one.

Let as suppose that vacuum states in two theories are invariant under the same unitary transformation.

If in two theories there are not states with negative energies then four first Wightman functions coincide in two theories.

It is important to note that, given rather common assumptions, condition (2.4) is a direct consequence of eq. (2.3) (see **Statement** at the end of the next section).

Let us give the idea of the proof.

The invariance of the vacuum states implies that in accordance with conditions (2.1) and (2.2) Wightman functions coincide at equal times.

$$(\Psi_{01}, \varphi_1(t,x_1), \ldots, \varphi_1(t,x_n)\Psi_{01}) = (\Psi_{02}, \varphi_2(t,x_1), \ldots, \varphi_2(t,x_n)\Psi_{02}) \tag{2.5}$$

Let us recall that Wightman functions are analytical functions in tubes T_n^- : $\nu \in T_n^-$ if $\nu = \xi - i\eta$, ξ is arbitrary, $\eta \in V^+$, which means that $\eta_0^2 - \vec{\eta}^2 > 0$.

It can be shown that the equality of Wightman functions at equal times and their analyticity lead to equality of four first Wightman functions in two theories related by unitary transformation at all points.

Let us point out that noncommutative coordinates belong to the boundary of analyticity of Wightman functions. As in the derivation of Haag's theorem only transformations of coordinates which belong to the domain of analyticity are essential, we omit the dependence of vectors on additional variables.

3. Extension of generalized Haag's theorem

Let us obtain the extension of Haag's theorem on the $SO(1,k)$ invariant theory.

As at $n > k$ vectors $\xi_i = (0, \vec{\xi}_i)$ are linearly dependent, then vectors related to them with Lorentz transformation are linearly dependent too and thus can not form the open set. Thus they can not determine Wightman functions.

Let us show that if $n \leq k$ then Wightman functions coincide in two theories under consideration.

Indeed, as the vectors $\vec{\xi}_i$ are arbitrary we can choose vector $\xi_2 = (0, \vec{\xi}_2)$ in such a way that it be orthogonal to ξ_1. Continuing this procedure we obtain the set of vectors ξ_i in such a way that they all be orthogonal one to another.

As $\xi_i \perp \xi_j$ if $i \neq j$, then also $\alpha\xi_i \perp \beta\xi_j$, $\alpha, \beta \in \mathbb{R}$ are arbitrary. If $\tilde{\xi}_i = L\xi_i$, where L is a real Lorentz transformation, then $\tilde{\xi}_i \perp \tilde{\xi}_j$, $i \neq j$ and also $\alpha\tilde{\xi}_i \perp \beta\tilde{\xi}_j$. So these points form the open subset and thus fully determine Wightman functions owing to their analiticity.

As in two theories related by unitary transformation at equal times first $k+1$ Wightman functions coincide on the open set, then these functions coincide in all points.

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Statement. *Condition (2.4) holds if vacuum vectors Ψ_0^i are translation invariant normalized states with respect to shifts $U_i(a)$ along one of the coordinates, satisfying $SO(1,k)$ symmetry.*

Proof. Indeed, it is easy to note that operator $U_1^{-1}(a)V^{-1}U_2(a)V$ commutes with operators $\tilde{\varphi}_1(t,\vec{x})$ and, thanks to irreducibility of the operator set, is proportional to identity operator. In limit $a=0$ we obtain that

$$U_1^{-1}(a)V^{-1}U_2(a)V = \mathbb{I}. \tag{3.1}$$

From eq. (3.1) it follows that if

$$U_1(a)\Psi_0^1 = \Psi_0^1, \tag{3.2}$$

then

$$U_2(a)V\Psi_0^1 = V\Psi_0^1, \tag{3.3}$$

that is condition (2.4) is satisfied. □

4. Consequences of Haag's theorem

Let us obtain consequences of Haag's theorem in $SO(1,k)$ invariant theory $k \geq 3$. In accordance with reduction formula

$$\begin{aligned}\langle p_1' \ldots p_l'^{out} \mid p_1 \ldots p_m^{in}\rangle = \\ & i^{l+m}\int dy_1 \ldots dy_l\, dx_1 \ldots dx_m\, f^*_{p_1'}(y_1) \times \ldots \\ & \times f^\star_{p_l'}(y_l)\vec{K}_{y_1}\ldots\vec{K}_{y_l}\Big(\Psi_0, T\left(\varphi(y_1)\ldots\varphi(x_m)\right)\Psi_0\Big)\times \\ & \times \vec{K}_{x_1}\ldots\vec{K}_{x_m} f_{p_1}(x_1)\ldots f_{p_m}(x_m),\end{aligned} \tag{4.1}$$

where $K_x = \left(\partial_\mu\partial^\mu\right)_x + m^2$ is Klein-Gordon operator and $f_p(x) = \frac{e^{-ipx}}{(2\pi)^{3/2}}$ is a corresponding wave function.

Let us consider $2 \Rightarrow k-1$ processes. We see that amplitudes of these processes coincide in two theories.

From the equality

$$W_1(x_1,\ldots,x_4) = W_2(x_1,\ldots,x_4)$$

it follows that

$$< p_3, p_4|p_1, p_2 >_1 = < p_3, p_4|p_1, p_2 >_2 \tag{4.2}$$

for any p_i. Having applied this equality for the forward elastic scattering amplitudes, we obtain that, according to the optical theorem, the total cross-sections for the fields $\varphi_1(x)$ and $\varphi_2(x)$ coincide. If now the S-matrix for the field $\varphi_1(x)$ is unity, then it is also unity for field $\varphi_2(x)$.

Let us consider Haag's theorem in the $SO(1,1)$ invariant field theory. In accordance with previous result equality of only two-point Wightman functions takes place. Let us prove that if one of considered theories is trivial, that is the corresponding S-matrix is equal to unity, then another is trivial too.

Let us point out that in the $SO(1,1)$ invariant theory it is sufficient that the spectral condition, which implies non existence of tachyons, is valid only in respect to commutative coordinates. Also it is sufficient that translation invariance is valid only in respect to the commutative coordinates. The equality of two-point Wightman functions in the two theories leads to the following conclusion: if local commutativity condition in respect to commutative coordinates is fulfilled and the current in one of the theories is equal to zero, then another current is zero as well.

Indeed as $W_1(x_1,x_2) = W_2(x_1,x_2)$, then also

$$< \Psi_0^1, j_{\bar{f}}^1 j_f^1 \Psi_0^1 >=< \Psi_0^2, j_{\bar{f}}^2 j_f^2 \Psi_0^2 >= 0, \tag{4.3}$$

where

$$j_f^i = (\Box + m^2)\,\varphi_f^i.$$

If, for example, $j_f^1 = 0$, then in the positive metric case

$$j_f^2 \Psi_0^2 = 0. \tag{4.4}$$

From the last formula and local commutativity condition it follows that [2]

$$j_f^2 \equiv 0.$$

Our statement is proved.

5. Conclusions

Generalized Haag's theorem has been proven in $SO(1,k)$ invariant quantum field theory. Apart from the above mentioned $k+1$ variables there can be arbitrary number of additional coordinates including noncommutative ones in the theory.

In $SO(1,k)$ invariant theory new consequences of generalized Haag's theorem are obtained. It has been proven that the equality of four-point Wightman functions in two theories leads to the equality of elastic scattering amplitudes and thus to the equality of the total cross-sections in these theories. Also it has been shown that at $k > 3$ the equality of $(k+1)$ - point Wightman functions in two theories leads to the equality of scattering amplitudes of some inelastic processes.

In $SO(1,1)$ invariant theory it has been proved that if in one of the theories under consideration *S*-matrix is equal to unity, then in another theory *S*-matrix is unity as well.

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