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Some exact solutions in vielbane gravity

Edward Rakhmetov[1]
Skobeltsyn Institute of Nuclear Physics, M.V. Lomonosov Moscow State University
1(2), Leninskie gory, GSP-1, Moscow 119991, Russian Federation
E-mail: `dsparel@mail.ru`

Sergey Keyzerov
Skobeltsyn Institute of Nuclear Physics, M.V. Lomonosov Moscow State University
1(2), Leninskie gory, GSP-1, Moscow 119991, Russian Federation
E-mail: `errar@www-hep.sinp.msu.ru`

We present a short introduction in vielbane (Möller) gravity theory and study a Schwarzschild solution and self-consistent solutions for Kaluza-Klein theories with spontaneous compactification, which can be obtained in this interesting generalization of General Relativity.

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[1] Speaker

1.Introduction

In this paper we describe an interesting generalization of General Relativity, suggested by C. Möller [1]. Save this little introduction and brief review Möller gravity, we are going to talk about exact solutions, which can be obtained in this theory. Namely, Schwarzschild solution and self-consistent solutions for Kaluza-Klein theories with spontaneous compactification.

We are trying to find self-consistent solutions of kind $M_4{}^xS^n$. Where M_4 is 4-dimensional Minkovsky space and S^n is n-dimensional sphere. Such solutions are useful to obtain after dimensional reduction effective 4-dimentional theories with CP-violation, which might be very interesting in particle and high energy physics[2,3]. But unfortunately, such solutions aren't known to be in General Relativity (except the case n=1), so that we must consider other theories.

The next motivation is an attempt to explain modern astrophysics data (such as rotation curves of stars in spiral galaxies) by means of gravity modification on large scales, without involving "Dark Matter" concept [4]. Very roughly speaking, the effects, which we might explain with dark matter, are strongly pronounced for spiral galaxies, and, quite the contrary, faintly pronounced for elliptical and dwarf spherical galaxies [5]. If so, we must find theory, in which Kerr-Solution, is vastly modified in comparison with Kerr-Solution in General Relativity. And, on the contrary, Schwarzschild solution, which described dwarf spherical galaxies, need only slight modification, or even doesn't need ones. Vielbein gravity is good for this purpose, because it has this kind of solutions.

And last, but not least: we are looking for a theory with faintly Lorentz symmetry violation. In Möller gravity theory, we have more restriction on frame vectors, then in General Relativity, because of additional asymmetric part of motion equations. Thus we have a hope to reveal Lorentz symmetry violation in some cases, when frame vectors give us a preferential direction in space-time because of this restriction. In other words, we can obtain faintly Lorentz symmetry violation; because of the action of the theory is not invariant with regard to rotations of frame vectors in some cases. But in this paper we are not going to discuss this case.

2.Vielbein (Möller) gravity theory

It was C. Möller, who first offered this theory in 1978 [1]. Möller gravity theory is a metric theory, in which metric tensor

$$g_{\mu\nu} \equiv \sum_{\alpha=0}^{3} \delta(\alpha\alpha) g(\alpha)_\mu g(\alpha)_\nu \equiv g(\alpha)_\mu g(\alpha)_\nu\,, \quad \delta(\alpha\beta) \equiv diag\{-1,1,1,1\} \qquad (2.1)$$

is constructed from orthonormal tetrad (vielbein):

$$g(\alpha)_\mu g(\beta)^\mu \equiv \delta(\alpha\beta) \qquad (2.2)$$

Here indexes in brackets are frame (vielbein) indexes, which are altered from 1 to 4, and summation is considered over repeated indexes. Stress tensor (denoted as $f(\alpha)_{\mu\nu}$) for this vector fields is, as usual:

$$f(\alpha)_{\mu\nu} \equiv \partial_{[\mu} g(\alpha)_{\nu]} \qquad (2.3)$$

Where antisymmetrization over indexes in square brackets is considered. Coordinate indexes can be turned into vielbein indexes as it present below:

$$C_{\mu\nu}(\alpha)g(\alpha)_{\lambda} \equiv C_{\mu\nu\lambda} \quad (2.4)$$

We can obtain Riche tensor from stress tensor:

$$R(\alpha\beta) = \frac{1}{2}\partial(\alpha) f(\mu\mu\beta) + \frac{1}{2}\partial(\beta) f(\mu\mu\alpha) + \frac{1}{2}\partial(\mu) f(\beta\alpha\mu)$$
$$+\frac{1}{2}\partial(\mu) f(\alpha\beta\mu) - \frac{1}{2} f(\alpha\beta\mu) f(\nu\nu\mu) - \frac{1}{2} f(\beta\alpha\mu) f(\nu\nu\mu) + \quad (2.5)$$
$$-\frac{1}{2} f(\mu\nu\alpha) f(\nu\mu\beta) - \frac{1}{2} f(\mu\nu\alpha) f(\mu\nu\beta) + \frac{1}{4} f(\alpha\mu\nu) f(\beta\mu\nu)$$

Where $\partial(\alpha) \equiv g(\alpha)^{\mu}\partial_{\mu}$ (2.6)

By means of convolution we can produce from stress tensor 3 different scalars:

$$L_1 \equiv f_{\alpha\beta\gamma} f^{\alpha\beta\gamma} \; L_2 \equiv f_{\alpha\beta\gamma} f^{\beta\alpha\gamma} \; L_3 \equiv f^{\alpha}{}_{\alpha\gamma} f_{\beta}{}^{\beta\gamma} \quad (2.7)$$

Then, in the general case, the simplest quadratic in partial derivative action is:

$$S = \int_X \left(k_0 + k_1 L_1 + k_2 L_2 + k_3 L_3\right)\sqrt{g}\,dx \quad (2.8)$$

Where k_0, k_1, k_2, k_3 are arbitrary dimensional constants.

Using (2.5) we can obtain more usual expression for Einstein-Möller action (accurate within full divergence):

$$S = \int_X \left(k_0 + k_3 R + k_1' L_1 + k_2' L_2\right)\sqrt{g}\,dx \quad (2.9)$$

We can not write for action terms of kind third and fourth, if we use metric tensor itself without fielbeine formalism. Thus we have generalization of any metric gravity.
As we can see, Möller gravity theory coincides with General Relativity, if k_1', k_2' is equal to 0.

Denote action variation over $g(\mu)_{\alpha}$ as $X(\mu)^{\alpha} \equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g(\mu)_{\alpha}}$, (2.11)

we can write symmetric and asymmetric part of motion equations $X(\mu)^{\alpha} \equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g(\mu)_{\alpha}} = 0$

separately. Symmetric part:

$$X^{(\alpha\beta)} = -4k_3\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\left(R+\Lambda\right)\right) + 2\left(2k_1' + k_2'\right)\nabla_{\mu} f^{(\alpha\beta)\mu} - \left(2k_1' - k_2'\right) f^{[\mu\nu]\alpha} f_{[\mu\nu]}{}^{\beta} +$$
$$+\left(2k_1' + k_2'\right) f^{\mu\nu(\alpha} f^{\beta)}{}_{\mu\nu} - 2k_2' f^{\alpha}{}_{\mu\nu} f^{\beta\mu\nu} + 2g^{\alpha\beta}\left(k_1' f^{\mu\nu\lambda} f_{\mu\nu\lambda} + k_2' f^{\mu\nu\lambda} f_{\nu\mu\lambda}\right) = 0 \quad (2.11)$$

Asymmetric part:

$$X^{[\alpha\beta]} = 2\left(2k_1' - k_2'\right)\nabla_{\mu} f^{[\alpha\beta]\mu} - 4k_2'\nabla_{\mu} f^{\mu\alpha\beta} - \left(2k_1' - 3k_2'\right) f^{\mu\nu[\alpha} f^{\beta]}{}_{\mu\nu} = 0 \quad (2.12)$$

if k_1', k_2' is equal to 0, asymmetric part vanish, and symmetric part gives us General Relativity.
As we can see in Möller gravity theory, we have more restriction on frame vectors, then in General Relativity, because of additional asymmetric part of motion equations.

3.Schwarzschild solution in Möller gravity

In this paragraph small latin letters are altered from 1 to 3. Let us write well-known Schwarzschild metric in the following way:

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\alpha(r)}\left\{dr^2 + r^2\left(d\varphi^2 + \sin^2\varphi d\chi^2\right)\right\} \tag{3.1}$$

Ansatz for metric tensor:

$$g_{\alpha\beta} = \begin{pmatrix} -e^{2\gamma} & 0 \\ 0 & e^{2\alpha}\bar{g}_{pq} \end{pmatrix} \tag{3.2}$$

Ansatz for the vielbein:

$$g(0)_0 = e^{\gamma} \qquad g(a)_q = e^{\alpha}\bar{g}(a)_q \qquad g(0)_q = 0 \qquad g(a)_0 = 0 \tag{3.3}$$

$$g(0)^0 = e^{-\gamma} \qquad g(a)^q = e^{-\alpha}\bar{g}(a)^q$$

Where $g(a)^q$ is orthonormal set for 3-dimensional flat space.

$$\bar{g}(1)_q = \begin{pmatrix} \cos\varphi \\ -r\sin\varphi \\ 0 \end{pmatrix} \qquad \bar{g}(2)_q = \begin{pmatrix} -\sin\varphi\cos\chi \\ -r\cos\varphi\cos\chi \\ r\sin\varphi\sin\chi \end{pmatrix} \qquad \bar{g}(3)_q = \begin{pmatrix} -\sin\varphi\sin\chi \\ -r\cos\varphi\sin\chi \\ -r\sin\varphi\cos\chi \end{pmatrix} \tag{3.4}$$

After some boring permutations we can obtain 3 equations for two parameters:

$$2\alpha_{,rr} + \alpha_{,r}^2 + 4r^{-1}\alpha_{,r} = \frac{\Lambda}{2}e^{2\alpha} + \kappa\left(2\alpha_{,r}^2 - 2\gamma_{,rr} - \gamma_{,r}^2 - 2\alpha_{,r}\gamma_{,r} - 4r^{-1}\gamma_{,r}\right) \tag{3.5}$$

$$\alpha_{,r}^2 + 2r^{-1}\alpha_{,r} + 2\gamma_{,r}\alpha_{,r} + 2r^{-1}\gamma_{,r} = \frac{\Lambda}{2}e^{2\alpha} - \kappa\left(2\alpha_{,r}^2 + \gamma_{,r}^2 + 4r^{-1}\alpha_{,r}\right) \tag{3.6}$$

$$\alpha_{,rr} + r^{-1}\alpha_{,r} + r^{-1}\gamma_{,r} + \gamma_{,rr} + \gamma_{,r}^2 = \frac{1}{2}e^{2\alpha}\Lambda + \kappa\left(-2\alpha_{,rr} + \gamma_{,r}^2 - 2\alpha_{,r}\gamma_{,r} - 2r^{-1}\alpha_{,r}\right) \tag{3.7}$$

Where $\kappa \equiv \dfrac{2k_1' + k_2'}{2k_3}$ (3.8)

If so, whether system is overdetermined, or these equations are not independent. It can be shown, that system has solutions only if κ =0. But if so, we have the same system as in General Relativity! In other words, Möller theory has the same Schwarzschild solution, as it appears in General Relativity. Peculiar moment is that, then this solution appears not only in case, when the constants of the theory are small, as was shown by Möller [1], but in case of arbitrary constants too, when the relation $2k_1' + k_2' = 0$ is valid. If this relation is broken, there is no spherical symmetric Schwarzschild-like solutions in Möller gravity theory.

4.Self-consistent solution of kind $M_4{}^x S^3$ with spontaneous compactification

In this paragraph large latin letters are altered from 0 to 7, except 4, small latin letters are altered from 0 to 3 and small greece letters are altered from 5 to 7. As it mentioned above, manifold is $M_4{}^x S^3$. Ansatz for the vielbein is shown below:

$g(\alpha)_\mu(x^A) = h(\alpha)_\mu(x^\lambda)$ depends only from 4-dimentional coordinates (4.1)

$$g(\alpha)_q = 0 \tag{4.2}$$

$$g(a)_\mu = 0 \tag{4.3}$$

$g(a)_q(x^A) = r\bar{g}(a-4)_q(x^l)$ depends only from additional coordinates (4.4)

Where $h(\alpha)_\mu$ is a vielbein on Minkovsky space M_4 and $\bar{g}(n)_q$ is a vielbein on 3-dimensional sphere S^3. Then there are two nontrivial motion equations:

$$R[h](\alpha\beta) - \frac{1}{2}\left(R[h] + 6r^{-2} - \Lambda\right)\delta(\alpha\beta) = \frac{1}{4k_3}Y[h](\alpha\beta) \quad (4.7)$$

$$2r^{-2}\delta(pq) - \frac{1}{2}\left(R[h] + 6r^{-2} - \Lambda\right)\delta(pq) = 4\kappa r^{-2}\delta(pq) - 6\frac{k_2'}{k_3}r^{-2}\delta(pq) \quad (4.6)$$

Where $Y^{\alpha\beta} \equiv X^{(\alpha\beta)} + 4k_3\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\left(R + \Lambda\right)\right)$ (4.8)

For Minkovsky space $R[h](\alpha\beta) = 0$, so that we have from (4.6)-(4.7) conditions for constants k_1', k_2' and compactification radius r^{-2}

$$k_2' = -\frac{1}{3}k_3 \qquad k_1' = -\frac{1}{6}k_3 \qquad r^{-2} = \frac{1}{6}\Lambda \quad (4.9)$$

If constants are like (4.9), we have for metric tensor a searching solution of kind $M_4{}^x S^3$.
That is to say, with such constants, 4-dimantional dynamics allows solutions like plane Minkovsky space, and the other dimensions are compactified into 3-dimensional sphere.

5.Conclusions

In vielbane gravity Schwarzschild solution appears not only in case, when additional constants of the Einstein-Möller action are small, as was shown by Möller [1], but in case of arbitrary constants too, when some relations for these constants are valid. . If this relations are broken, there is no spherical symmetric Schwarzschild-like solutions in Möller gravity theory.

4-dimantional dynamics allows solutions like plane Minkovsky space with a large spectre of theory parameters, when additional dimensions are spontaneously compactifed in to 3-dimentional sphere.

The action of the theory is not invariant with regard to rotations of frame vectors. It can result in the theory with Lorentz symmetry violation, when some relations for these constants are valid, but in this paper we are not describe such cases.

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