Lattice Quantum Gravity - an Update

Jan Ambjørn

The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø,
Denmark
E-mail: ambjorn@nbi.dk

Jerzy Jurkiewicz

Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland
E-mail: jurkiewicz@th.if.uj.edu.pl

Renate Loll[*]

Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The
Netherlands.
E-mail: r.loll@uu.nl

We advocate lattice methods as the tool of choice to constructively define a background-independent theory of Lorentzian quantum gravity and explore its physical properties in the Planckian regime. The formulation that arguably has most furthered our understanding of quantum gravity (and of various pitfalls present in the nonperturbative sector) uses dynamical triangulations to regularize the nonperturbative path integral over geometries. Its Lorentzian version in terms of Causal Dynamical Triangulations (CDT) – in addition to having a definite quantum signature on short scales – has been shown to reproduce important features of the classical theory on large scales. This article recaps the most important developments in CDT of the last few years for the physically relevant case of four spacetime dimensions, and describes its status quo at present.

---

[*]Speaker.

1. Quantum gravity and the lattice

Gravity remains the only fundamental interaction which we have not yet been able to formulate and understand as a full quantum theory. Even basic issues remain open, for example, whether gravity at the most fundamental level will be truly unified with the weak, strong and electromagnetic interactions, or have more of a separate status, in the spirit of the classical theory. Research in quantum gravity is driven by a number of simple, but profound questions: What are the quantum origins of space, time and our universe? What is the microstructure of spacetime, and can it *explain* macroscopic gravitational interactions and perhaps even the universe's observed large-scale structure? Are "space", "time" and "causality" fundamental or emergent concepts in a setting where spacetime geometry is allowed to undergo large quantum fluctuations?

Apart from its appeal as a white spot on the map of our understanding of fundamental high-energy physics, the specific reason why this topic is of inherent interest to the lattice community is the apparent need in quantum gravity for nonperturbative methods to model and understand the relevant Planck-scale physics. Lattice and Monte Carlo techniques, adapted to systems of dynamical geometry (such as gravity), provide powerful tools for addressing such issues. For low-dimensional systems of quantum geometry the validity and usefulness of such methods has been demonstrated long ago, and reviewed at previous lattice conferences under headings like "lattice gravity and random surfaces" [1]. Similar techniques can be applied to fully-fledged four-dimensional quantum gravity, but the situation here is less clear-cut, which is not surprising in view of our limited understanding of this theory. As will be described in what follows, attempts are under way to *define* quantum gravity as the scaling limit of a specific statistical system of dynamical geometry. For the physically relevant case of four spacetime dimensions, the only way we can currently study the existence and properties of this nonperturbative limit is via lattice methods. In other words, despite the fact that quantum gravity is at a much earlier stage of theory building, compared with a theory like QCD, numerical methods – in conjunction with analytical and theoretical modelling – can be used in a profitable way to explore what this theory may be. This also implies that the more foundational aspects of theory development are currently at least on a par with purely simulation-technical aspects, like improving efficiency or increasing the lattice size.

At this stage, the only points of reference and comparison for lattice quantum gravity are alternative and (likewise incomplete) nonperturbative formulations in the continuum. In addition, because of the requirement of covariance, there are also considerable challenges in defining and evaluating observable quantities, which can be used to characterize the physical properties of the theory. The specific candidate theory of quantum gravity described below arises from a confluence of ideas from general relativity (in particular, gravity-specific properties like dynamical geometry and background independence), high-energy physics (in particular, the use of path integral and renormalization group methods), and, equally crucially, lattice field theory. This approach of "*Quantum Gravity from Causal Dynamical Triangulations (CDT)*" was last reported on during plenary talks at Lattice 2000 and 2001 [2], when the formulation was still in its infancy, and far from deriving results in the physically interesting case of four dimensions. The remainder of this presentation constitutes a brief progress report on the many interesting developments that have taken place since then, focussing on four-dimensional results. More extensive recent reviews of the field can be found in [3].

2. Causal Dynamical Triangulations 101

Viewed from a larger perspective, CDT quantum gravity is an outright conservative approach in the sense of relying exclusively on standard quantum field-theoretic tools and principles, applied to the situation where spacetime is not regarded as fixed, but itself part of the nonperturbative dynamics. It builds on techniques which have been well tested in the study of systems of random surfaces and Euclidean[1] models of quantum gravity, and does not invoke or presently require "exotic" ingredients like strings, loops, branes, extra dimensions or new symmetries. By contrast, it is an approach with few free parameters, whose outcomes are by construction robust. This means that if it can be shown to lead to a viable theory of quantum gravity, the theory will be reasonably unique. On the other hand, if in the future it produces results which are inconsistent (for example, because its classical limit is in contradiction with Einstein's general relativity), it will be difficult to fix this by twiddling with the parameters of the model.

Since dynamically triangulated models of quantum gravity are amenable to numerical methods, which can and do produce numbers and results, the above considerations are not merely of a theoretical nature, as is illustrated by the fate of the Euclidean precursor of CDT. This candidate theory of four-dimensional quantum gravity generated considerable excitement in the early 1990s before it was understood gradually that it suffers from fatal degeneracies, which prevent the emergence of macroscopic, classical spacetimes of dimension four. Numerical simulations were crucial in bringing about this result (for a summary of these developments, as well as a complete bibliography, see [4]). This illustrates that the presence of explicit computational consistency checks, combined with a small number of free parameters and a high degree of universality (independence of the continuum theory of the details of the lattice discretization) means that in practice quantum gravity theories from dynamical triangulations can be falsified. Despite being a hallmark of any good physical theory, falsifiability has become somewhat of a rarity in more speculative areas of high-energy theory, including quantum gravity. What we would like to emphasize here is the importance – in the absence of any direct probes of Planck-scale physics – of "computational experiments" in providing criteria for the viability of candidate theories for quantum gravity.

In technical terms, quantum gravity from causal dynamical triangulations is a nonperturbative implementation of the gravitational path integral. It has already passed several nontrivial tests and has produced unprecedented results, as will be described below. In the process, it also has highlighted a number of unexpected features (and pitfalls) due to the nonperturbative nature of the construction, which permits large quantum fluctuations on small scales.

The idea of constructing a nonperturbative gravitational path integral which captures Lorentzian, causal properties of the spacetimes to be summed over goes back to a paper from 1998 [5], where it was also demonstrated by explicit, analytic computation that the idea works in two dimensions and produces a result distinct from previous Euclidean models of 2d quantum gravity. The first results for the physical, four-dimensional theory were published in 2004 [6].

As a warm-up, consider the path integral for a nonrelativistic particle of mass *m* in one dimen-

---

[1]Instead of space*times* with Lorentzian signature, Euclidean gravity works with purely spatial geometries, which do not have a notion of time or causality. Euclidean gravity was a popular starting point for cosmological path integrals back in the 1970s and '80s and (usually for reasons of simplification) continues to be used in some path integral formulations of full gravity.
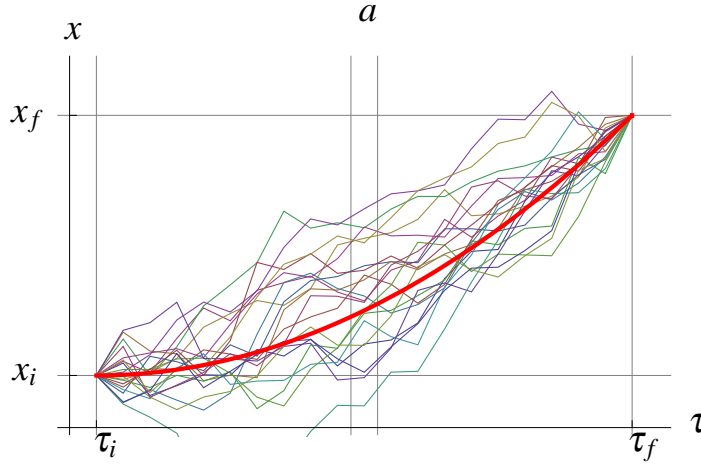
Figure 1: Sample paths or "histories" $x(\tau)$ from a regularized version of the path integral of the nonrelativistic particle in imaginary time $\tau := -it$, generated by a Monte Carlo simulation. The total time interval $T = \tau_f - \tau_i$ has been subdivided into time steps of length a, and the trajectories are piecewise linear. The average path $\langle x(\tau) \rangle$ is indicated by the central fat line.

sion, moving in a harmonic oscillator potential, and subject to fixed boundary conditions $x_i = x(t_i)$ and $x_f = x(t_f)$ at some initial and final times t_i and t_f. The corresponding path integral describes the transition amplitude from x_i to x_f as a superposition of amplitudes exp $iS[x(t)]$ of all possible particle trajectories with the given boundary conditions, where $S[x(t)]$ is the classical action of the entire path $x(t)$. The superposition gives rise to an "average path", the expectation value $\langle x(t) \rangle$ in the given ensemble (c.f. Fig. 1, which shows a sample of paths in Euclidean, imaginary time $\tau := -it$, generated by a Monte Carlo simulation). The typical size of the deviation $\delta x(\tau)$ of a general history $x(\tau) = \langle x(\tau) \rangle + \delta x(\tau)$ from this expectation value can be computed explicitly, yielding

$$\langle \delta x(\tau)^2 \rangle = \frac{\hbar}{2m\omega} \frac{\cosh \omega T - \cosh \omega (T - 2\tau)}{\sinh \omega T}, \tag{2.1}$$

where ω is the oscillator's frequency, $T = \tau_f - \tau_i$ is the total length of the time interval considered, and the time τ runs from $\tau_i = 0$ to T. Anticipating a similar quantum superposition in gravity, where each "path" will represent a curved spacetime, we will be interested both in the "average universe" and deviations from it. In this case, the scale of quantum fluctuations of some linear distance x is expected to be $\langle |\delta x| \rangle \propto \sqrt{\hbar G}$ instead of the $\langle |\delta x| \rangle \propto \sqrt{\hbar/m\omega}$ of the particle case, with G denoting Newton's constant.

Quantum gravity from causal dynamical triangulations is a nonperturbative and background-independent realization of the formal *gravitational path integral* (a.k.a. the "sum over histories") on a differential manifold M,

$$Z(G, \Lambda) = \int_{\mathscr{G}(M) = \frac{\text{Lor}(M)}{\text{Diff}(M)}} \mathscr{D}[g_{\mu\nu}] \, e^{iS^{\text{EH}}[g_{\mu\nu}]}, \tag{2.2}$$

where S^{EH} is the four-dimensional Einstein-Hilbert action, Λ the cosmological constant, and the path integral is to be taken over all spacetimes $[g_{\mu\nu}] \in \mathscr{G}(M)$ (Lorentzian metrics $g_{\mu\nu}$ modulo

diffeomorphisms), with specified boundary conditions. In other words, each path is now a four-dimensional, curved spacetime geometry $[g_{\mu\nu}]$, which can be thought of as a three-dimensional, spatial geometry developing in time. The weight associated with each $[g] \in \mathscr{G}(M)$ is given by the Einstein-Hilbert action

$$S^{\text{EH}}[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-\det g}\, (R[g, \partial g, \partial^2 g] - 2\Lambda). \tag{2.3}$$

To evaluate this quantum field-theoretic path integral, one proceeds in close analogy with the path-integral quantization of the nonrelativistic particle described above. The latter is defined as the continuum limit of a regularized sum over paths, where the contributing "virtual" paths are taken from an ensemble of piecewise straight paths, with the time interval a for each step going to zero in the limit. The method of CDT turns the corresponding gravitational path integral (2.2) into a well-defined regularized and finite expression, which can be evaluated and whose continuum limit can be studied systematically [7]. The CDT prescription consists in representing the space $\mathscr{G}(M)$ of all Lorentzian spacetimes in terms of a set of triangulated, piecewise flat (ie. piecewise Minkowskian) manifolds[2].

The idea of approximating curved spacetimes by much simpler, triangulated objects was introduced in the classical theory of General Relativity by Regge [8], and first applied in the quantum context in a seminal paper by Roček and Williams [9]. Note that the objectives of the classical and quantum theories differ significantly: in the former, one usually wants to approximate a given, classical solution to the Einstein equation locally as well as possible. By contrast, when using such geometries in the path integral, one wants to approximate the space of *all* geometries. It should be pointed out that just like in the particle case, where the path integral in the continuum limit is dominated by nowhere differentiable paths, typical geometries contributing to the gravitational path integral also turn out to be highly nonclassical.

The geometry of the triangulated manifolds is almost everwhere flat and therefore trivial, and can carry curvature in a delta function-like manner only at its two-dimensional subsimplices (the triangles), where three or more four-simplices meet. This regularization in terms of dynamical lattices implies a vast truncation of the number of degrees of freedom, from the local field tensor $g_{\mu\nu}(x)$ to a discrete set of edge lengths for the four-simplices, plus the information of which pairs of simplices are glued together pairwise.

For the purposes of causal dynamical triangulations, the simplicial approximation $\mathscr{G}_{a,N}$ of $\mathscr{G}$ contains all simplicial manifolds T obtained from gluing together at most N four-dimensional, triangular building blocks of typical edge length a, with a again playing the role of a UV cut-off (see Fig. 2). What makes the construction *causal* is the fact that the gluing of the Minkowskian four-simplices respects a global notion of (proper) time, akin to the requirement of global hyperbolicity usually imposed in classical gravity. The regularized gravitational path integral in CDT is then given by

$$Z_{a,N}^{\text{CDT}} = \sum_{\substack{\text{triangulated causal} \\ \text{spacetimes}\, T \in \mathscr{G}_{a,N}}} \frac{1}{C_T} e^{iS^{\text{Regge}}[T]}, \tag{2.4}$$

---

[2]Unlike in the particle case, there is no embedding space; all geometric spacetime data are defined intrinsically, just like in the classical theory.
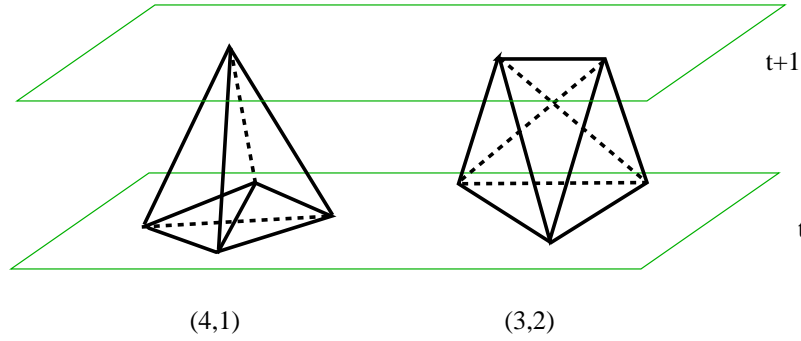
Figure 2: The two fundamental building blocks of CDT are four-simplices with flat, Minkowskian interior. They are spanned by spacelike edges, which lie entirely within spatial slices of constant proper time t, and timelike edges, which interpolate between adjacent slices of integer time. A building block of type (m,n) has m of its vertices in slice t, and n in slice $t+1$.

where S^{Regge} is the Regge version of the Einstein-Hilbert action associated with the simplicial spacetime T, and C_T denotes the order of its automorphism group (see [10, 11] or the recent reviews [12] for an explicit expression of S^{Regge} as well as other construction details). The discrete volume N acts as an infrared cutoff. We still need to consider a suitable continuum or scaling limit

$$Z^{\text{CDT}} := \lim_{\substack{N \to \infty \\ a \to 0}} Z_{a,N}^{\text{CDT}} \tag{2.5}$$

of (2.4), while renormalizing the original bare coupling constants of the model, in order to arrive at a theory of quantum gravity.[3] The two limits in (2.5) are usually tied together by nominally keeping fixed a physical four-volume $V_4 := a^4 N$. In order to make the evaluation of Z amenable to Monte Carlo simulations, one still needs to convert the sum over complex amplitudes to a sum over real Boltzmann weights. Despite the fact that no suitable Wick rotation is known for arbitrary curved metrics, such a prescription fortunately does exist for the causal triangulations under consideration [7]. As is familiar from lattice field theory, one then takes $a \to 0$, such that the individual discrete building blocks shrink to zero. This should be contrasted with some other approaches to quantum gravity, which postulate the existence of fundamental discreteness at the Planck scale, and consequently identify the lattice spacing a with the Planck length ℓ_{Pl}. In this case one never takes a continuum limit $a \to 0$, which has the disadvantage that the quantum dynamics at the Planck scale is *not* universal and has a large degree of arbitrariness. In CDT applications, since the limit $a \to 0$ can in practice never be reached on a finite lattice, one must make sure that a is always much smaller than the scale at which one is trying to extract physical results.

Let us summarize the key features of the construction scheme thus introduced. Unlike what is possible in the continuum theory, the path integral (2.4) is defined directly on the physical configuration space of *geometries*. It is nonperturbative in the sense of including geometries which are "far away" from any classical solutions, and it is background-independent in the sense of performing the sum "democratically", without distinguishing any given geometry (say, as a preferred background). However, these attractive properties of the regularized path integral are only useful because *we are able to evaluate Z^{CDT} quantitatively*, with an essential role being played by Monte

---

[3]Note that the existence of a physically meaningful limit is not automatic, but something that needs to be shown.
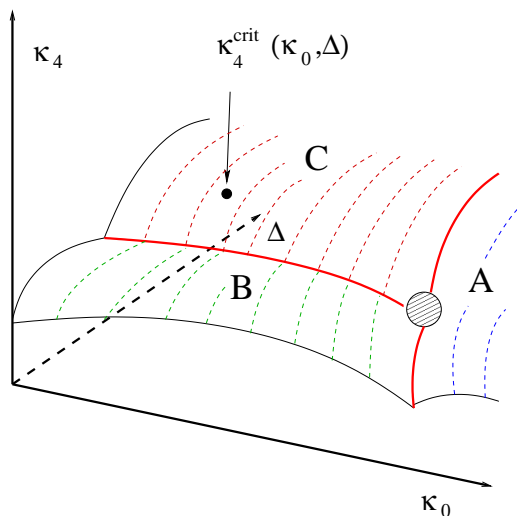
Figure 3: The phase diagram of Lorentzian quantum gravity from CDT, with κ_0 and κ_4 denoting the bare inverse Newton's constant and (up to an additive shift) the bare cosmological constant. After fine-tuning to the subspace where the cosmological constant is critical (tantamount to performing the infinite-volume limit), there are three phases: A and B (the Lorentzian analogues of the degenerate branched-polymer and crumpled phases of the Euclidean approach), and a *new* phase C, where an extended, four-dimensional universe emerges. The parameter Δ in CDT parametrizes a finite relative scaling between space- and time-like distances which is naturally present in the Lorentzian case.

Carlo simulations. These, together with the associated finite-size scaling techniques [13], have enabled us to extract information about the nonperturbative, strongly coupled quantum dynamics of the system, which is currently not accessible by analytical methods, neither in this nor any other approach to quantum gravity. This mirrors the role played by lattice simulations in determining the nonperturbative behaviour of QCD (although one should keep in mind that the latter is a theory we already know *much* more about than quantum gravity).

As far as we are aware, CDT is the only nonperturbative approach to quantum gravity which has been able to dynamically generate its own, physically realistic background from nothing but quantum fluctuations. More than that, because of the minimalist set-up and the methodology used (quantum field theory and critical phenomena), the results obtained are robust in the sense of being largely independent of the details of the chosen regularization procedure and containing few free parameters. As we already pointed out in Sec. 2 above, it is therefore one of the rare instances of a candidate theory of quantum gravity which can potentially be falsified. Our investigations of both the quantum properties and the classical limit of this candidate theory are at this stage not sufficiently complete to provide conclusive evidence that we have found *the* correct theory of quantum gravity. However, results until now have been unprecedented and very encouraging, and have thrown up a number of nonperturbative surprises, some of which we will summarize next.

3. Key findings of CDT – the phase diagram

One important lesson learned for nonperturbative gravitational path integrals from CDT quantum gravity is that the ad-hoc prescription of integrating over curved Euclidean *spaces* of metric

signature (++++) instead of the physically correct curved Lorentzian space*times* of metric signature (−+++) generally leads to inequivalent and (in $d = 4$) incorrect results. Euclidean quantum gravity, as advocated by Hawking and others [14], adopts the Euclidean version of the path integral mainly for the technical reason of being able to use real weights $\exp(−S^{\mathrm{eu}})$ instead of the complex amplitudes $\exp(iS^{\mathrm{lor}})$ in its evaluation. The same is done in perturbative quantum field theory on flat Minkowski space, where one can rely on the existence of a well-defined Wick rotation to relate correlation functions in either signature, an option that is *not* available in continuum gravity beyond perturbation theory on a Minkowski background.

CDT quantum gravity has given us the first explicit example of a nonperturbative gravitational path integral (in a toy model of two-dimensional gravity [5]) which is exactly soluble and leads to distinct and inequivalent results, depending on whether the sum over histories is taken over Euclidean spaces or Lorentzian spacetimes. (More precisely, the latter are Euclidean spaces which are obtained by Wick rotation – which *does* exist for the class of simplicial spacetimes under consideration – from Lorentzian spacetimes). Only those histories are summed over which possess a global time slicing with respect to which no spatial topology changes are allowed to occur. After Wick rotation, this set constitutes a strict subset of all Euclidean (triangulated) spaces. Note that general Euclidean spaces possess no natural notion of time or causality and in this sense branching in all directions is always present.

A crucial insight of CDT quantum gravity is that a similar result holds also in four dimensions. The geometric degeneracy of the phases (in the sense of statistical systems) found in Euclidean dynamical triangulations and the resulting absence of a good classical limit [15, 16] can again in part be traced to the proliferation of branching "baby universes". As demonstrated by the CDT results in [6, 10], the requirement of microcausality (absence of causality-violating points) of the individual path integral histories leads to a different phase structure, compared with the previous Euclidean approach. The breakthrough result of Lorentzian CDT is that its phase diagram now possesses a third and *qualitatively new* phase, in which the universe on large scales is extended and four-dimensional (Fig. 3), exactly as required by classical General Relativity! As indicated on the figure, to obtain an infinite-volume limit the bare cosmological constant κ_4 has to be fine-tuned to the critical surface from *above*, since $\kappa_4 > \kappa_4^{\mathrm{crit}}$ characterizes the region where the (Euclidean) partition function Z^{CDT} exists and is finite.

On the critical surface, phases A and B can be understood as Lorentzian analogues of the two degenerate phases of the Euclidean models, and do not appear interesting from a continuum point of view [10]. The new and physically interesting phase – more on which below – is phase C. What is curious about the phase structure of four-dimensional CDT quantum gravity is its resemblance with that of Hořava-Lifshitz gravity [17], which has been spelled out further in [18, 19]. It gives rise to the intriguing conjecture that there may be a universal phase diagram governing systems of higher-dimensional, dynamical geometry, and accomodating a variety of gravity theories, some of which may be anisotropic in space and time. Another question that arises is that of the order of the phase transitions between the three phases, indicated by the red lines in Fig. 3. Their determination is numerically challenging, and a preliminary investigation of the A-C transition in [20] turned out inconclusive. Some of these problems have now been overcome and new results on both the A-C and the B-C transition will appear in due course [21].
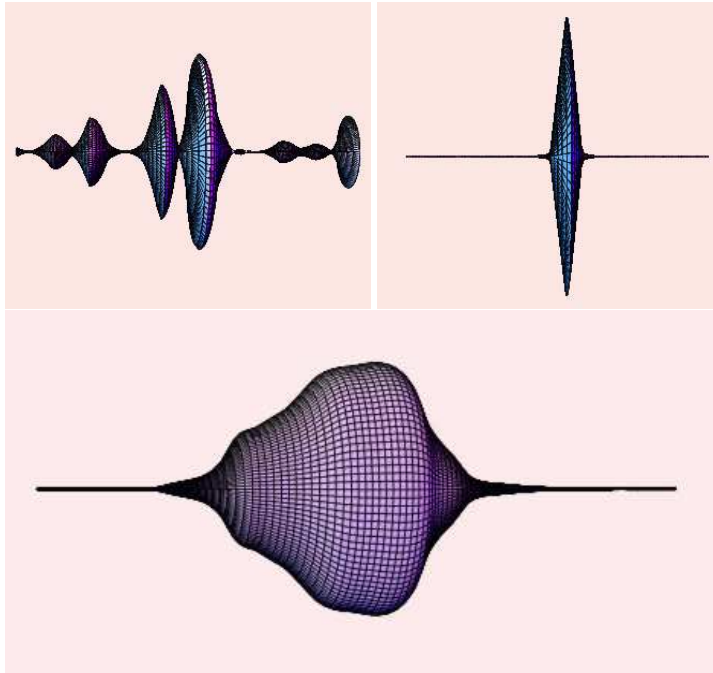
Figure 4: The volume profiles of typical path integral configurations in phases A, B and C (no averaging over histories involved). Phase C (bottom figure) is the one where extended four-dimensional geometries emerge. The figures show *only* the curves $V_3(\tau)$ – made into bodies of revolution about the horizontal time axis – and no local fluctuations of geometry.

4. Key findings of CDT – the dynamical emergence of spacetime as we know it

What is the nature of the extended spacetime found in phase C of CDT quantum gravity, and what quantitative criteria do we apply to distinguish between the three phases? Examining individual path integral histories will only be of limited use, since in the limit $a \rightarrow 0$ their geometry will become highly singular, similar to that of the nowhere differentiable paths which constitute the carrier space of the path integral of the nonrelativistic particle in the continuum limit [22]. What we must do instead is to define and measure geometric *quantum observables*, evaluate their expectation values on the ensemble of geometries and draw conclusions about the behaviour of the "quantum geometry" generated by the computer simulations (that is, the ground state of minimal Euclidean action).

One such observable is given by the overall shape of the universe, more precisely, the three-volume $V_3(\tau)$ as a function of proper time τ. Already by comparing Monte Carlo "snapshots" of typical shapes, one observes completely different qualitative behaviours in the three phases (Fig. 4). Remarkably, inside phase C the microscopic building blocks superposed in the nonperturbative path integral arrange themselves into an extended quantum spacetime whose macroscopic shape is that of the well-known *de Sitter universe* [23, 11]. This amounts to a highly nontrivial test of the classical limit, about which it is notoriously difficult to make any definite statements in most models of nonperturbative quantum gravity. The dynamical mechanism by which this happens is not understood in detail, however, it is clear that "entropy" (in other words, the measure of the path
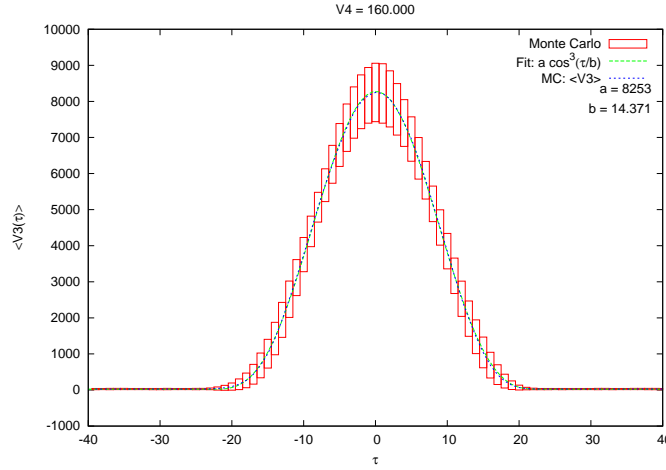
Figure 5: The average shape $\langle V_3(\tau)\rangle$ of the CDT quantum universe in phase C, fitted to that of Euclidean de Sitter space (the "round four-sphere") with rescaled proper time, $\langle V_3(\tau)\rangle = a\cos^3(\tau/b)$. Measurements taken for a universe of four-volume $V_4 = 160.000$ and time extension $T = 80$. The fit of the Monte Carlo data to the theoretical curve for the given values of a and b is impressive. The vertical boxes quantify the typical scale of quantum fluctuations scale around $\langle V_3(\tau)\rangle$.

integral, or the number of times a given Boltzmann factor $\exp(-S)$ is realized) plays a crucial role in producing the outcome. It means that the nature of this semiclassical limit is *truly nonperturbative*, in the sense that the tentative continuum limit of the path integral is found in a region of the bare coupling constant space where the entropy of the various geometric configurations contributes at the same order of magnitude as the exponential of the action. As we have pointed out in [20], this is reminiscent of certain phenomena in condensed matter physics, like the Kosterlitz-Thouless transition in the XY model.

The manner in which we have identified (Euclidean) de Sitter space from the computer data is by looking at the expectation value of the volume profile $V_3(t)$. From the line element of Lorentzian de Sitter space in proper-time coordinates,

$$ds^2 = -dt^2 + c^2\cosh^2\left(\frac{t}{c}\right)\, d\Omega_{(3)}^2, \tag{4.1}$$

with $d\Omega_{(3)}^2$ denoting the line element of the unit three-sphere, one can immediately read off the classical volume profile

$$V_3(t) = 2\pi^2(c\cosh\frac{t}{c})^3, \quad c = const, \tag{4.2}$$

which for $t > 0$ gives rise to the familiar, exponentially expanding universe, thought to give an accurate description of our own universe at late times, when matter can be neglected compared with the repulsive force due to the positive cosmological constant. Because the CDT simulations for technical reasons have to be performed in the Euclidean regime, we must compare the expectation value of the shape with those of the analytically continued expression of (4.2), with respect to the Euclidean time $\tau := -it$. After normalizing the overall four-volume and adjusting computer proper
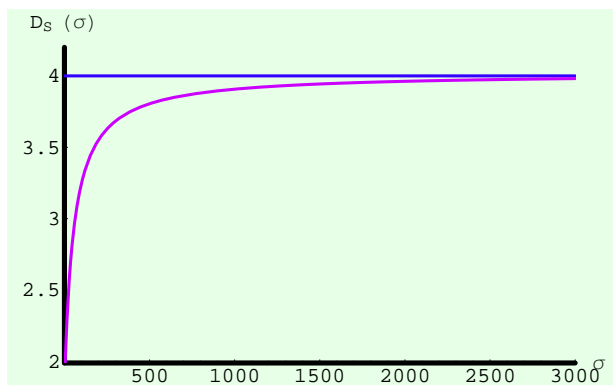
Figure 6: The spectral dimension $D_S(\sigma)$ of the CDT-generated quantum universe (lower curve, error bars not included), contrasted with the corresponding curve for a classical spacetime, which for sufficiently short distances is simply given by the constant function $D_S(\sigma) = 4$.

time by a constant to match continuum proper time, the average volume profile obtained is depicted in Fig. 5.

A few more things are noteworthy about this result. First, despite the fact that the discrete CDT construction treats space and time differently, at least on large scales the full isotropy is restored by the ground state of the theory for precisely one choice of identifying proper time in the continuum. Second, the computer simulations necessarily have to be performed for finite, compact spacetimes, which also means that a specific choice has to be made for the spacetime topology. For simplicity, to avoid having to specify boundary conditions, it is usually chosen to be $S^1 \times S^3$, with time compactified[4] and spatial slices which are topological three-spheres. What is reassuring is the fact that the bias this choice could in principle have introduced is "corrected" by the system, which clearly is driven dynamically to the topology of a four-sphere (or as close to it as is permitted by the kinematical constraint imposed on the three-volume, which is not allowed to vanish at any time). Lastly, we have also analyzed the quantum fluctuations around the de Sitter background; they match to good accuracy a continuum saddlepoint calculation in minisuperspace [11], which is one more indication that we are indeed on the right track.

5. Key findings of CDT – getting a handle on Planckian physics

Having presented some of the evidence that CDT quantum gravity does possess the correct classical limit, let us now turn to the *new* physics we are ultimately after, namely, what happens to gravity and the structure of spacetime at or near the Planck scale. One way of probing the short-scale quantum structure of the universe is by setting up a *diffusion process* on the ensemble of spacetimes, and studying an associated quantum observable. For a classical manifold, it is well known that the speed with which an initially localized diffusion process spreads depends on the dimension of the space. Conversely, given a space *M* of unknown properties, it can be assigned a so-called *spectral dimension D_S* by studying the leading-order behaviour of the average return probability $\mathcal{R}_V(\sigma)$ (of random diffusion paths on *M* starting and ending at the same point *x*) as a

---

[4]the period is chosen much larger than the time extension of the universe and does no influence the result

function of the diffusion time σ,

$$\mathscr{R}_V(\sigma) := \frac{1}{V(M)} \int_M d^d x\, P(x,x;\sigma) \propto \frac{1}{\sigma^{D_S/2}}, \quad \sigma \le V^{2/D_S}, \tag{5.1}$$

where $V(M)$ is the volume of M, and $P(x,y;\sigma)$ the solution to the heat equation on M. Diffusion processes can be defined on very general spaces, for example, on fractals, which are partially characterized by their spectral dimension (usually not an integer, see [24]). Relevant for the application to quantum gravity is that the expectation value $\langle \mathscr{R}_V(\sigma) \rangle$ can be measured on the ensemble of CDT geometries, giving us the spectral dimension of the dynamically generated quantum universe, with the astonishing result that $D_S(\sigma)$ depends on the linear scale $\sqrt{\sigma}$ probed [25]! The measurements from CDT quantum gravity, extrapolated to all values of σ, lead to the lower curve in Fig. 6, with asymptotic values $D_S(0) = 1.82 \pm 0.25$, signalling highly nonclassical behaviour near the Planck scale, and $D_S(\infty) = 4.02 \pm 0.1$, which is compatible with the expected classical behaviour. We conclude that the quantum geometry dynamically generated by CDT is definitely not a classical manifold on short scales.

What is even more remarkable is the fact that the same kind of short-scale "dynamical dimensional reduction" has been found recently in a couple of different quantum field-theoretic approaches to quantum gravity, namely, a nonperturbative renormalization group flow analysis of gravity [26] and the novel Hořava-Lifshitz quantum gravity already mentioned earlier [27]. Whether there is a common underlying reason for this remarkable coincidence – which might tell us something deeper about the nature of quantum gravity – remains to be understood. Within the CDT framework, further indications for nonclassicality at Planckian distances come from measurements of geometric structures in spatial slices $\tau = const$. [10], including a measurement of their Hausdorff and spectral dimensions, and of shell decompositions of both space and spacetime [28].

6. Quantum gravity - quo vadis?

For a long time now, there has been plenty of abstract reasoning on the nature of nonperturbative quantum gravity, that is, what the theory should look like and what kind of properties it should have *if only* we knew what it was. On the one hand, it is of course natural to appeal to general principles in the absence of any experimental or observational guidance on how to construct the theory. On the other hand, our so-called intuition – mostly coming from studying classical gravity and quantum fields on a fixed background – may seriously mislead us when speculating about the nature of spacetime at the Planck scale. What lattice quantum gravity (in the form of dynamical triangulations or causal dynamical triangulations) provides us with is an "experimental lab", a calculational framework to study systems of fluctuating geometry *quantitatively* in a nonperturbative regime. In dimension two, where comparisons with analytical models are available, this leads to sensible results. In dimension four, it is currently the only way to extract nonperturbative information about these systems. In particular, it has uncovered several completely unexpected, but presumably generic features, for example, the fact that the signature of the geometry can make a crucial difference, the fact that a superposition of d-dimensional geometries is not necessarily d-dimensional, indeed, that such superpositions are usually so degenerate that they possess no classical limit at all, and the fact that the conformal divergence of the Euclidean path integral can be cured by "entropic contributions".

CDT's toolbox has enabled us to uncover these nonperturbative properties and at the same time make quantitative statements about covariant properties of quantum geometry, including its dimension and volume profile. In principle the framework is also able to test nonperturbative predictions from other fundamental theories containing gravity, if and when they will be made, subject only to the usual numerical limitations of the lattice. Clearly, much remains to be done, but the results already obtained underline the power and utility of lattice methods, also in situations where spacetime itself is dynamical.

References

[1] M.J. Bowick, *Random surfaces and lattice gravity*, *Nucl. Phys. Proc. Suppl.* **63** (1998) 77-88 [hep-lat/9710005]; G. Thorleifsson, *Lattice gravity and random surfaces*, *Nucl. Phys. Proc. Suppl.* **73** (1999) 133-145 [hep-lat/9809131]; A. Krzywicki, *Random manifolds and quantum gravity*, *Nucl. Phys. Proc. Suppl.* **83** (2000) 126-130 [hep-lat/9907012].

[2] R. Loll, *Discrete Lorentzian quantum gravity*, *Nucl. Phys. Proc. Suppl.* **94** (2001) 96-107 [hep-th/0011194]; J. Ambjørn, *Strings, quantum gravity and noncommutative geometry on the lattice*, *Nucl. Phys. Proc. Suppl.* **106** (2002) 62-70 [hep-lat/0201012].

[3] J. Ambjørn, J. Jurkiewicz and R. Loll, *Quantum gravity as sum over spacetimes Lect. Notes Phys.* **807** (2010) 59-124 [arXiv:0906.3947, gr-qc]; J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *CDT - an entropic theory of quantum gravity* [arXiv:1007.2560, hep-th].

[4] R. Loll, *Discrete approaches to quantum gravity in four-dimensions*, *Living Rev. Rel.* **1** (1998) 13 [gr-qc/9805049].

[5] J. Ambjørn and R. Loll, *Non-perturbative Lorentzian quantum gravity, causality and topology change*, *Nucl. Phys. B* **536** (1998) 407-434 [hep-th/9805108].

[6] J. Ambjørn, J. Jurkiewicz and R. Loll, *Emergence of a 4D world from causal quantum gravity*, *Phys. Rev. Lett.* **93** (2004) 131301 [hep-th/0404156].

[7] J. Ambjørn, J. Jurkiewicz and R. Loll, *A nonperturbative Lorentzian path integral for gravity*, *Phys. Rev. Lett.* **85** (2000) 924-927 [hep-th/0002050]; *Dynamically triangulating Lorentzian quantum gravity*, *Nucl. Phys. B* **610** (2001) 347-382 [hep-th/0105267].

[8] T. Regge, *General relativity without coordinates*, *Nuovo Cim. A* **19** (1961) 558-571.

[9] M. Rocek and R.M. Williams, *Quantum Regge calculus*, *Phys. Lett. B* **104** (1981) 31-37.

[10] J. Ambjørn, J. Jurkiewicz and R. Loll, *Reconstructing the universe*, *Phys. Rev. D* **72** (2005) 064014 [hep-th/0505154].

[11] J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *The nonperturbative quantum de Sitter universe*, *Phys. Rev. D* **78** (2008) 063544 [0807.4481, hep-th].

[12] J. Ambjørn, J. Jurkiewicz and R. Loll, *Quantum gravity as sum over spacetimes*, *Lect. Notes Phys.* **807** (2010) 59-124 [0906.3947, gr-qc]; J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *CDT - an entropic theory of quantum gravity* [1007.2560, hep-th].

[13] M.E.J. Newman and G.T. Barkema, *Monte Carlo methods in statistical physics*, Clarendon Press, Oxford (1999).

[14] G.W. Gibbons and S.W. Hawking (eds.), *Euclidean quantum gravity*, World Scientific, Singapore (1993).

[15] P. Bialas, Z. Burda, A. Krzywicki and B. Petersson, *Focusing on the fixed point of 4d simplicial gravity*, *Nucl. Phys. B* **472** (1996) 293-308 [hep-lat/9601024]; P. Bialas, Z. Burda, B. Petersson and J. Tabaczek, *Appearance of mother universe and singular vertices in random geometries*, *Nucl. Phys. B* **495** (1997) 463-476 [hep-lat/9608030].

[16] B.V. de Bakker, *Further evidence that the transition of 4D dynamical triangulation is 1st order*, *Phys. Lett.* **B** 389 (1996) 238-242 [hep-lat/9603024].

[17] P. Hořava, *Quantum gravity at a Lifshitz point*, *Phys. Rev. D* **79** (2009) 084008 [0901.3775, hep-th].

[18] J. Ambjørn, A. Görlich, S. Jordan, J. Jurkiewicz and R. Loll, *CDT meets Horava-Lifshitz gravity*, *Phys. Lett. B* **690** (2010) 413-419 [1002.3298, hep-th].

[19] P. Hořava, *General covariance in gravity at a Lifshitz point* [1101.1081, hep-th].

[20] J. Ambjørn, A. Görlich, J. Jurkiewicz, R. Loll, J. Gizbert-Studnicki, T. Trzesniewski, *The semiclassical limit of causal dynamical triangulations*, *Nucl. Phys. B* **849** (2011) 144-165 [1102.3929, hep-th].

[21] J. Ambjørn, S. Jordan and R. Loll, to be published.

[22] M. Reed and B. Simon, *Methods of modern mathematical physics, vol. 2*, Academic Press (1975).

[23] J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *Planckian birth of the quantum de Sitter universe*, *Phys. Rev. Lett.* **100** (2008) 091304 [0712.2485, hep-th].

[24] D. ben-Avraham and S. Havlin, *Diffusion and reactions in fractals and disordered systems*, Cambridge University Press (2000).

[25] J. Ambjørn, J. Jurkiewicz and R. Loll, *Spectral dimension of the universe*, *Phys. Rev. Lett.* **95** (2005) 171301 [hep-th/0505113].

[26] O. Lauscher and M. Reuter, *Fractal spacetime structure in asymptotically safe gravity*, *JHEP* **0510** (2005) 050 [hep-th/0508202].

[27] P. Hořava, *Spectral dimension of the universe in quantum gravity at a Lifshitz point*, *Phys. Rev. Lett.* **102** (2009) 161301 [0902.3657, hep-th].

[28] J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, *Geometry of the quantum universe*, *Phys. Lett. B* **690** (2010) 420-426 [1001.4581, hep-th].