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Exploration of the phase structure of SU(N_c) lattice gauge theory with many Wilson fermions at strong coupling

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We explore aspects of the phase structure of $SU(2)$ and $SU(3)$ lattice gauge theories at strong coupling with many flavours N_f of Wilson fermions in the fundamental representation, including the relevance to recent searches for a conformal window. The pseudoscalar meson mass, the quark mass and other quantities are observed as functions of the hopping parameter, and we find deviations from the expected analytic dependence, at least for sufficiently large N_f. Implications of these effects for the phase structure and for the existence of a (first order) bulk phase and the Aoki phase are discussed in the case of $N_f/N_c \gg 1$.

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1. Introduction

In $SU(N_c)$ vector gauge theories, perturbation theory allows for a non-trivial infrared fixed point (IRFP)[1, 2]. When an IRFP exists in the very strong coupling region, investigation by lattice gauge theories is needed for going beyond perturbation theory. There have been a lot of recent lattice studies; see Refs. [3, 4] for reviews.

We are motivated by Ref. [5] in which a systematic lattice analysis over a wide range of the gauge coupling constant was performed. According to those authors, $N_f = 3$ in $SU(2)$ and $N_f = 7$ in $SU(3)$ belong to the conformal window. Their result was based on the prediction of the pion mass behaviour in the strong coupling limit, given in Ref. [6] as follows:

$$\cosh(m_\pi a) = 1 + \frac{(1-16\kappa^2)(1-4\kappa^2)}{8\kappa^2(1-6\kappa^2)}, \tag{1.1}$$

where $\kappa = \frac{1}{2m_0+8}$ and m_0 is the bare mass parameter of Wilson fermions. Eq. (1.1) implies that the pion becomes massless at $\kappa = 1/4$, independent of N_c and N_f. Therefore the authors of Ref. [5] investigated at $\beta = \frac{2N_c}{g^2} = 0$ in particular, which led to their prediction for the critical flavour number of the conformal window mentioned above.

However, with Wilson fermions the phase structure in the strong coupling limit is very complicated, as pointed out in Ref. [6], due to the existence of a parity-flavour broken phase (Aoki phase). In the Aoki phase $\kappa > \kappa_c = 1/4$ at $\beta = 0$, and on the phase boundary the charged pion is massless while the neutral pion is not[1].

Therefore, in Ref. [7] and this proceedings we check the particle spectroscopy in the strong coupling limit to determine whether Ref. [5] is correct, and we explore the phase structure of the Wilson fermion system with many flavours to determine whether the Aoki phase exists in the many flavour case.

2. Simulation details

We use N_f flavours of mass-degenerate Wilson fermions[7]. Dynamical configurations are generated by the standard Hybrid Monte-Carlo (HMC) algorithm with $N_{MD}\Delta\tau = 1$. Lattice sizes are $6^2\times12^2$, $8^2\times16^2$, $12^2\times24^2$ and $12^3\times24$ though not all of these sizes were used for every choice of N_f; this range of sizes allows us to look for finite size effects (for some N_f). In this exploration, we compute observables from $50\sim100$ configurations with $4\sim5$ intervals between trajectories after thermalizing. We monitor the following observables: m_π^2, m_ρ, the plaquette value($\langle\Box\rangle$), the axial Ward-Takahashi identity quark mass ($m_q^{\mathrm{AWI}} = \frac{\nabla_4\langle\sum_{\mathbf{x}}A_4(\mathbf{x},t)P(\mathbf{0},0)\rangle}{2\langle\sum_{\mathbf{x}}P(\mathbf{x},t)P(\mathbf{0},0)\rangle}$), the Polyakov loop, Creutz ratios, the chiral condensate (or the propagator norm), the lowest eigenvalues, and $\langle S(t)S(0)\rangle$ as a function of t. Due to page constraints in these proceedings we focus on a few observables most relevant to a discussion of the phase structure.

3. $N_c = 3$ case: Comparison with the data of Ref. [5]

Iwasaki et al.[5] concluded that the $N_f = 6$ case is in the confinement region because they found $N_{CG} > 10000$ in the Molecular-Dynamics (MD) evolution for the thermalizing process (they

[1]The order parameter of the Aoki phase is $\langle\bar\psi\gamma_5\tau_3\psi\rangle$.

did not attain the thermalization) and that the plaquette value was decreased toward zero, in contrast with the case of $N_f > 7$. Thus they interpreted these observations to be signals for a massless pion, in spite of no direct calculations of the pion mass and the quark mass. However, as plotted in the left panel of Fig. 1, we found results for the pion and quark masses as functions of $1/\kappa$ at $\beta = 0$ that differ from Ref. [5]: we did attain thermalization with $N_{CG} < 10000$ in MD step at $\kappa = 0.25$

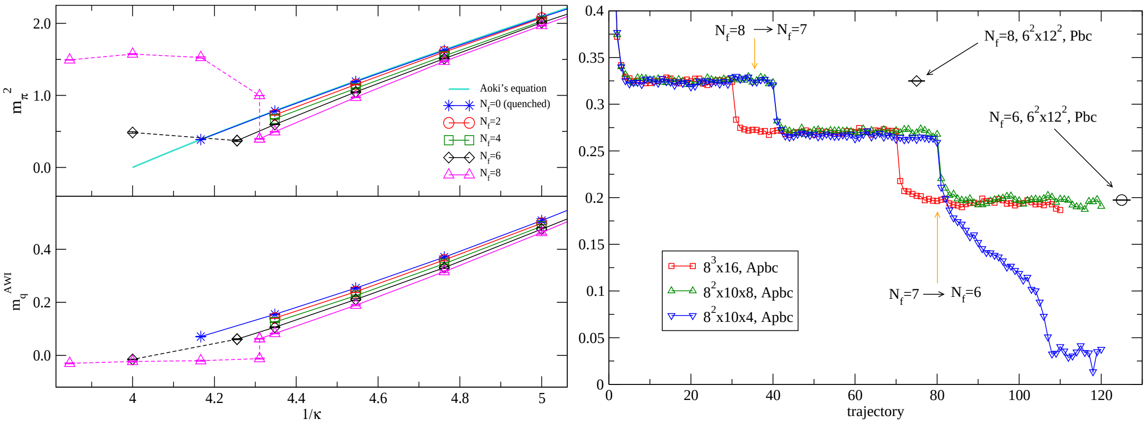

Figure 1: $SU(3)$ gauge theory with N_f flavours at $\beta = 0$. **Left panel:** m_π^2 and $m_q^{\rm AWI}$ as functions of $1/\kappa$. The analytic prediction is shown as a solid blue curve reaching $m_\pi^2 = 0$ at $1/\kappa = 4$. **Right panel:** Plaquette history in HMC and R-algorithm in various lattice setups.

for $N_f = 6$, and our pion mass is not zero as shown in the left panel of Fig. 1. In addition, our pion and quark masses depend on N_f such that only the quenched case ($N_f = 0$) obeys Eq. (1.1) leading to the conventional Aoki phase.

To understand the origin of this discrepancy between our data and theirs, we monitor the history of the plaquette value on various lattices with various setups. In addition to our own HMC simulations with periodic boundary conditions reported in Ref. [7], we now monitor the plaquette value by using MILC code[8]. The MILC code uses the R-algorithm with anti-periodic boundary conditions, which are the same algorithm and boundaries as were used by Iwasaki *et al.* in Ref. [5]. We generate the configurations with a cold start for $N_f = 8$, then switch to $N_f = 7$ from the last configuration of $N_f = 8$ and switch to $N_f = 6$ from that of $N_f = 7$, and we plot the plaquette value history in the right panel of Fig. 1. For larger N_t, the simulation is not affected by the different algorithms and boundary conditions. However, we found that the plaquette value at $N_t = 4$, where Iwasaki *et al.* obtained their conclusion, shows a large deviation.

Our observations at $N_t = 4$ are consistent with the results of Ref. [5], *i.e.* the small value of the plaquette and the pion mass. However, it seems clear that results with this small time extent is not representative of larger lattices; there seems to be some kind of the thermal phase transition. We conclude that our studies with larger N_t, which showed a massive pion, are more appropriate for zero-temperature discussions, while conclusions from Ref. [5] about the conformal window cannot be supported.

4. $N_c = 2$ case: Exploration of the phase structure

For $SU(2)$ gauge theory, Ref. [5] predicted that $N_f = 2$ is in the confinement region and that

$N_f = 3$ is in the conformal window for the same reason as discussed for $SU(3)$: $N_{CG} > 10000$. However, as we have already pointed out in the previous section, finding $N_{CG} > 10000$ during the thermalizing process is not reliable evidence for discovering a conformal window. Although we are unable to obtain direct results for $N_f = 2$ due to the very small value of the pion mass around $\kappa = 0.25$, we did obtain results for $N_f = 4, 6, 8, \cdots$ and we plot those in the upper left panel of Fig. 2 for various N_f flavours and in the upper right panel of Fig. 2 for $N_f = 0, 6$ and 12 especially to check finite size effects. In the upper panels of Fig. 2, there are a confinement phase and an

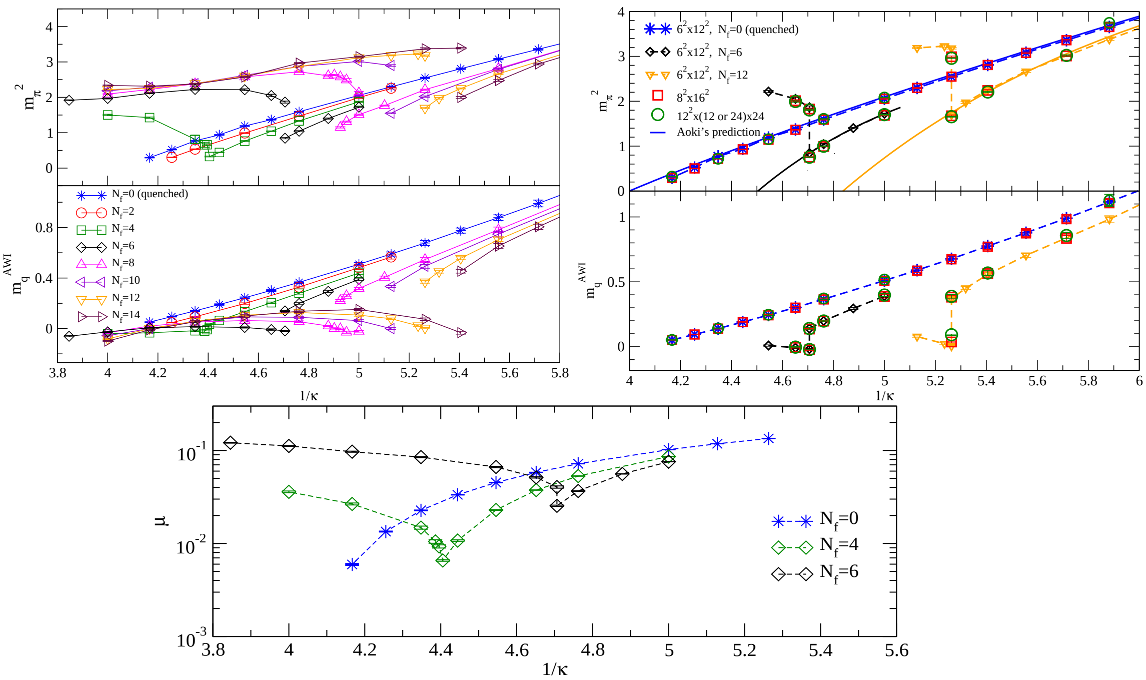

Figure 2: $SU(2)$ gauge theory with N_f flavours at $\beta = 0$ on a $6^2 \times 12^2$ lattice. **Upper left panel:** m_π^2 and $m_q^{\rm AWI}$ as functions of $1/\kappa$. **Upper right panel:** Volume independence for $N_f = 0, 6$ and 12. The solid blue curve represents the prediction of the equation. **Bottom panel:** $\mu = \sqrt{\lambda_0(H_W^2)}$, where $\lambda_0(H_W^2)$ is the lowest eigenvalue of H_W^2, as a function of $1/\kappa$.

unknown massive pion phase, and these two phases are separated by a transition at $\kappa = \kappa_d$[2].

Does an Aoki phase exist in the strong coupling region? The $SU(2)$ case is qualitatively the same as the $SU(3)$ case: the pion and the quark mass reveal the 2-state signal, the quenched case ($N_f = 0$) agrees with Eq. (1.1) but the dynamical case ($N_f > 0$) depends on N_f, and further there is no κ_c beyond which $m_\pi^2 = 0$. Namely our result is

$$0.25 = \kappa_c(N_f = 0) > \ldots > \kappa_c(N_f = 6) > \ldots > \kappa_c(N_f = 12) > \ldots, \tag{4.1}$$

if there exists κ_c in $N_f > 0$ case. Therefore it seems there is no standard Aoki phase due to non-existence of the massless pion over the whole region of κ.

Further we monitor the lowest eigenvalue in the bottom panel of Fig. 2, $\mu = \sqrt{\lambda_0(H_W^2)}$, where H_W is the hermitian Wilson-Dirac operator, in order to consider the phase structure of many Wilson

[2] This κ_d is not a precise value. Hysteresis is observed within a range of κ for $N_f \geq 6$ in $SU(2)$, and κ_d is a label for the location of that transition.

fermions. The lowest eigenvalue for $N_f > 0$ is not zero in both the confinement and the unknown-massive-pion phase, while the eigenvalue in $N_f = 0$ tends to go to zero as $\kappa \to 0.25$. According to the modified Banks-Casher relation[9], the order parameter of the Aoki phase is $\langle \bar{\psi}(x)\gamma_5\tau_3\psi(x)\rangle = 2\pi\rho_{H_W}(\lambda = 0)$ where $\rho_{H_W}(\lambda)$ is the eigenvalue distribution function of H_W. The data from small lattices suggest that the order parameter of the Aoki phase is zero, and this is consistent with the absence of a standard Aoki phase in the case with many Wilson fermions. However, it would be necessary to investigate on the larger lattices and to calculate the spectrum of low-lying eigenvalues before making a strong conclusion, and this is left to future studies.

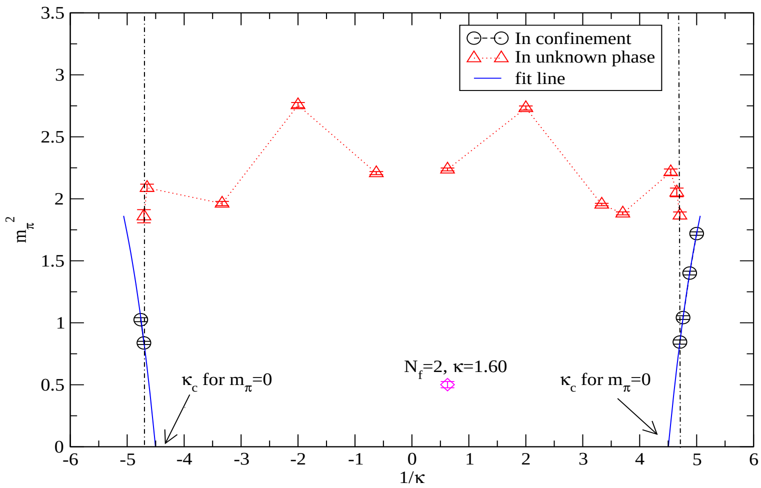

Figure 3: m_π^2 as a function of not only positive but negative κ for $N_f = 6$ at $\beta = 0$ on $6^2 \times 12^2$.

The outline of the phase structure in our research is shown in Fig. 3. The existence of a massive pion phase (the red triangles in the figure) is apparent, but without that information one might try to obtain κ_c by extrapolating to $m_\pi^2 = 0$. In that way one would find that there is the gap of κ_c ($\kappa_c^+ \neq \kappa_c^-$) for $m_\pi^2 = 0$ between the positive and the negative region. According to the traditional interpretation, this is the Aoki phase containing a massless charged pion. However, the massive pion phase (red triangles) indicates that, at least with the many Wilson fermions ($N_f/N_c \gg 1$), another phase has appeared instead of a standard Aoki phase.

Therefore, in Ref. [7] and this proceedings, we don't have a clear signal for the existence of an Aoki phase and the Sharpe–Singleton first order scenario[10], but instead we have obtained evidence for a different phase at strong coupling.

5. Summary and Discussion

We explored $SU(2)$ and $SU(3)$ gauge theories with many Wilson fermions and reported result in Ref. [7] and additional results in these proceedings. In contrast to Ref. [5], we find that (i) The pion mass in the confinement region depends on N_f. (ii) As defined by extrapolation to $m_\pi^2 = 0$, κ_c depends on N_f and shifts from 0.25. (iii) No massless pion exists for any value of κ. Thus the extrapolated κ_c is not a very useful quantity, and extrapolation to $m_\pi = 0$ is not valid when searching for an Aoki phase. (iv) There is the 2-state signal and hysteresis around $\kappa = \kappa_d$, which depends on N_f. (v) The $N_f = 6$ case in $SU(3)$ gauge theory with Wilson fermions is not in the confinement region because the pion mass at $\kappa_c = 0.25$ is not zero. Simulations at $N_t = 4$ are not representative of larger N_t values, so we do not validate the claim[5] that the critical flavour of the conformal window is 7 ($N_f^* \neq 7$). (vi) There is a standard Aoki phase in the quenched case ($N_f = 0$), but the situation is markedly different in the large flavour case ($N_f/N_c \gg 1$).

Now we make an effort to identify the unknown massive pion phase. The Polyakov loop is

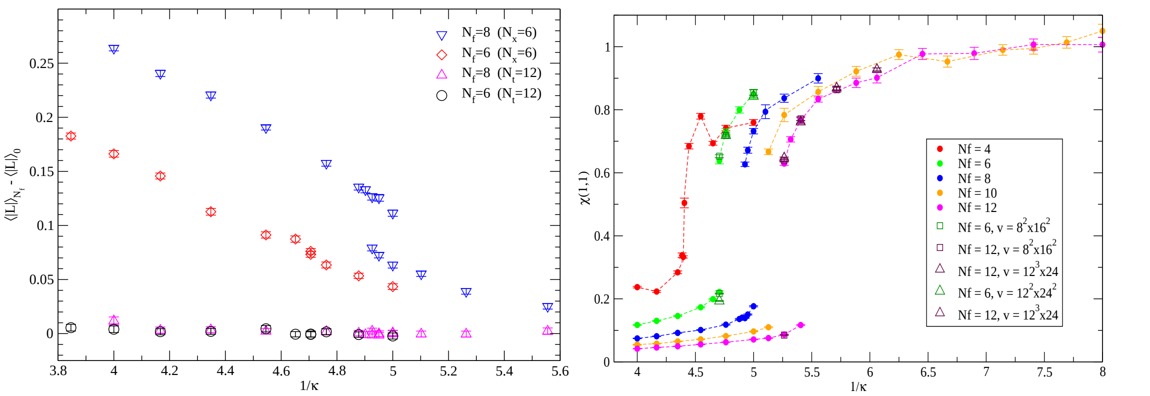

Figure 4: Left panel: Polyakov loop normalized by the quenched case, $\langle |L| \rangle_{N_f} - \langle |L| \rangle_{N_f=0}$, as a function of $1/\kappa$. Here we take the absolute value of the Polyakov loop. **Right panel:** Creutz ratio, $\chi(1,1)$, with 10 APE smearings as a function of $1/\kappa$.

plotted in the left panel of Fig. 4. Along the small extent ($N_x = 6$ on $6^2 \times 12^2$), the Polyakov loop shows finite temperature behaviour with a 2-state signal and hysteresis. However, the Polyakov loop along the large extent ($N_t = 12$ on $6^2 \times 12^2$) doesn't show any transitions and is consistent with the quenched case ($N_f = 0$). This means there is no finite temperature transition on large lattices, so the unknown phase is not the result of a confinement-deconfinement phase transition.

The right panel of Fig. 4 is a Creutz ratio, $\chi(1,1)$. At $\beta = 0$ the static quark is confined (modulo string breaking), so the Creutz ratio at short distances is near unity in lattice units. We compute $\chi(1,1)$ for $\kappa > \kappa_d$, and find that it becomes small in contrast with its value in the confinement phase. This relative difference might imply a transition from confinement to weak coupling.

Although we monitored various quantities to identify the phase, unfortunately at present we cannot determine the true nature of this unknown massive-pion phase. Still, this phase is the confirmation of a (first order) bulk phase transition at the strong coupling, as pointed out in Refs. [11, 12] by observing the high- and the low-plaquette phase and the chiral condensate with the first order transition. Our result is the first observation of that longer list of quantities (the pion mass, the quark mass and so on) and the fact that the standard Aoki phase seems to be invisible due to this bulk phase transition at strong coupling. Furthermore, it is found that this bulk phase transition even appears in the strong coupling limit ($\beta = 0$) with Wilson fermions. This implies that the (first order) bulk phase transition is caused by fermionic dynamics only, not by the gauge dynamics, namely this phase is independent of the type of lattice gauge actions.

Although it's difficult to determine which phase is realized, it will be possible to determine whether the standard Aoki phase has appeared or not, by observing the order parameter of the Aoki phase, $\langle \bar{\psi}\gamma_5\tau_3\psi \rangle$. Refs. [13, 14] carried out a calculation of the order parameter for $N_f = 2$

in $SU(3)$ ($N_f/N_c < 1$) by using the twisted mass Wilson fermions. Let us remember again the bottom panel of Fig. 2. When this figure is turned upside-down, a result similar to the data of the order parameter might be obtained in $SU(2)$, due to the very rough relation $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle \sim \frac{1}{\mu}$ on finite lattices. If this is true, $SU(2)$ twisted mass fermions might show a bulk phase transition in $N_f = 2$, instead of Aoki phase. Therefore we are calculating the order parameter in $N_f = 2$ and $N_f \geq 6$ in $SU(2)$ gauge theories ($N_f/N_c \geq 1$) with twisted mass Wilson fermions. It should be possible to clarify the phase structure of a system with (many) Wilson fermions in the near future.

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