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QCD Rotator with Light Quarks up to NNL Order

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We consider 2-flavour QCD with light quark masses in a small spatial box, where the low-lying excitations are that of an $O(4)$ rotator. This problem can be treated in chiral perturbation theory (δ-regime). Up to NNL order the final result depends on the low-energy constants F, Λ_1, Λ_2 and B. Comparing these results with numerical simulations in QCD should help to determine the low-energy constants to good precision.

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1. Introduction

At low energies QCD is dominated by Goldstone boson physics and can be described by chiral perturbation theory (ChPT) [1, 2, 3]. Effective field theories contain an infinite number of operators. By introducing a suitable power counting scheme, only a finite number of operators enter up to a given order in the calculation. The associated low-energy constants (LECs) have to be determined by experiments or from numerical lattice QCD simulations.

ChPT can also be applied if the system is enclosed in a finite volume and at finite temperature [4, 5, 6]. We consider QCD with two light quarks in the isospin limit $m_u = m_d = m$, in a finite volume $V = L_s \times L_s \times L_s \times L_t$ in Euclidean space-time. The Compton wavelength of the pion and the temporal extent of the box are both much larger than the spatial extent of the box, i.e. $M^{-1} \gg L_s$, respectively $L_t \gg L_s$. Here, M is the leading order pion mass in infinite volume

$$M^2 = 2mB\,, \tag{1.1}$$

where B is the low-energy constant related to the quark condensate.

In the chiral limit ($m = 0$) the low-energy excitations are described by an $O(4)$ rotator [7, 8, 9]. At leading order the rotator spectrum is given by

$$E_j = \frac{j(j+2)}{2F^2V_s}\,, \qquad j = 0, 1, \ldots\,, \tag{1.2}$$

where j can be considered as the "angular momentum" in the internal 4-dimensional space and F is the pion decay constant in the chiral limit. The combination F^2V_s in Eq. (1.2) is the leading order term of the moment of inertia Θ. The moment of inertia gets corrections due to ChPT in the delta regime $\Theta = F^2V_s\big(1 + \sim 1/(FL_s)^2 + \ldots\big)$, where the dimensionless expansion parameter

$$\delta^2 = \frac{1}{F^2L_s^2} \ll 1 \tag{1.3}$$

is assumed to be small.

For sufficiently small quark masses the low-energy properties of QCD in the delta regime are still dominated by the rotator, and the symmetry breaking terms will give corrections to the rotator spectrum (1.2). The corresponding small, dimensionless expansion parameter is denoted by

$$r^4 = F^8L_s^{12}M^4 \ll 1\,. \tag{1.4}$$

In our calculations, we assume the two dimensionless expansion parameters to be roughly of the same order, i.e. $\delta^2 \sim r^4$. Considering contributions up to $\mathscr{O}(\delta^4)$ (NNL order), the formula for the energy gap involves the low-energy constants F, B, Λ_1 and Λ_2.

2. Chiral perturbation theory in the delta regime

We use an $O(4)$ non-linear sigma model to describe the partition function 2-flavour QCD

$$Z = \int [\mathrm{D}\vec{S}]\, \delta\big(\vec{S}^2 - 1\big) \exp\Big(-\int \mathrm{d}x\, \mathscr{L}_{\mathrm{eff}}\Big) \tag{2.1}$$

in four-dimensional Euclidean space-time. The effective Lagrangian is expressed in the four-component vector fields $\vec{S}$, where $\vec{S}^2(x) = 1$. We will start from the effective Lagrangian in the p-expansion ($ML_s \gg 1$)

$$\mathscr{L}_{\text{eff}} = \mathscr{L}^{(2)} + \mathscr{L}^{(4)} + \dots\,, \tag{2.2}$$

where

$$\begin{aligned}
\mathscr{L}^{(2)} &= \frac{F^2}{2}\partial_\mu \vec{S}(x)\partial_\mu \vec{S}(x) - F^2 M^2 S_0(x)\,,\\
\mathscr{L}^{(4)} &= -\ell_1\Big(\partial_\mu \vec{S}(x)\partial_\mu \vec{S}(x)\Big)^2 - \ell_2\Big(\partial_\mu \vec{S}(x)\partial_\nu \vec{S}(x)\Big)^2 + \textit{symmetry breaking terms}\,.
\end{aligned} \tag{2.3}$$

The symmetry breaking terms in $\mathscr{L}^{(4)}$ will enter only beyond NNL order in the δ-expansion and can therefore be neglected.

Chiral perturbation series are obtained by expanding the effective action around the classical limit $\vec{S} = 1$. However, in the delta regime, this expansion becomes meaningless due to the presence of very low energy modes.

Since the Compton wavelength is much larger than the spatial extent of the box, collective behaviour sets in. Thus, we can introduce a "global" mode for each time slice, since on a given time slice the field variables are strongly correlated and point almost in the same direction (in the internal space). Due to the fact that the time extent is much larger than the spatial extent, the global mode performs a slow rotation in the internal four-dimensional space. The fluctuations (fast modes) around the direction of the global mode (slow modes) can then be treated by perturbation theory.

We have to incorporate this non-perturbative behaviour of the slow modes in the partition function by introducing a collective variable [10, 11]. The effective action is then expressed in terms of the fast modes and the slow modes. We expand the effective action in the fast modes and integrate them out in the partition function. We are applying dimensional regularisation. By considering only contributions up to NNL order, the partition function reduces to

$$Z \propto \int [\mathrm{D}\vec{e}]\, \exp\Big(-\int \mathrm{d}t\, \frac{\Theta}{2}\dot{\vec{e}}(t)\dot{\vec{e}}(t) - \eta\, e_0(t)\Big)\,, \qquad \vec{e}^{\,2}(t) = 1\,. \tag{2.4}$$

After renormalisation the moment of inertia gets corrections at NL [10] and NNL order [11] and reads[1]

$$\begin{aligned}
\Theta = F^2 L_s^3\Big[1 - \frac{2\bar{G}^*}{F^2 L_s^2} + \frac{1}{F^4 L_s^4}\Big[&0.088431628\\
&+ \partial_0\partial_0\bar{G}^*\frac{1}{3\pi^2}\Big(\frac{1}{4}\log(\Lambda_1 L_s)^2 + \log(\Lambda_2 L_s)^2\Big)\Big]\Big]\,.
\end{aligned} \tag{2.5}$$

The constants $\bar{G}^*$ and $\partial_0\partial_0\bar{G}^*$ are related to the finite volume Green's function $D^*(0)$, respectively $\partial_0\partial_0 D^*(0)$ which enter in perturbation theory for the fast modes

$$\bar{G}^* = 0.2257849591\,, \qquad \partial_0\partial_0\bar{G}^* = 0.8375369106\,. \tag{2.6}$$

[1] The NNL corrections to the moment of inertia have been calculated recently by Niedermayer and Weiermann [12] in lattice regularised ChPT. In order to compare the two results, the matching between the two different regularisation schemes is needed.

Λ_1 and Λ_2 are the intrinsic scales related to the low-energy constants ℓ_1 and ℓ_2 [3]. η which controls the strength of the symmetry breaking:

$$\eta = F^2 L_s^3 M^2 \left[1 - \frac{3\overline{G}^*}{F^2 L_s^2}\right]. \tag{2.7}$$

3. The energy gap

The partition function (2.4) can be interpreted as an $O(4)$ quantum mechanical rotator in an external symmetry breaking potential. The corresponding Hamilton operator reads

$$\mathbf{H} = \frac{1}{\Theta}\left(\frac{\mathbf{L}^2}{2} - (\Theta\eta)\mathbf{e}_0\right), \tag{3.1}$$

where $\mathbf{L}$ is the angular momentum operator in the internal four-dimensional space and Θ and η are given by Eqs. (2.5), (2.7) respectively.

In the chiral limit $(\eta = 0)$ the energy spectrum of the rotator is given by Eq. (1.2), where $F^2 V_s$ has to be replaced simply by Θ. Since $\Theta\eta = r^2(1+\ldots)$ is small, we can calculate the symmetry breaking corrections to the energy spectrum by applying perturbation theory. Considering corrections up to $\mathscr{O}(\delta^4)$ we have to calculate the corrections to the rotator energy (1.2) up to fourth order in perturbation theory.

The energy gap of the system is defined by the energy difference of the first excited state $j = 1$ and the ground state $j = 0$. Due to the presence of the symmetry breaking potential the first excited state splits up into a singlet and a triplet energy state, whereas the triplet provides the lower energy difference. Thus, the energy gap which includes symmetry breaking corrections up to $\mathscr{O}\big((\Theta\eta)^4\big)$ reads

$$E_{L_s} = \frac{3}{2\Theta}\left[1 + \frac{(\Theta\eta)^2}{15} - \frac{193}{120}\frac{(\Theta\eta)^4}{15^2}\right]. \tag{3.2}$$

We recognise the leading symmetry breaking correction $F^8 L_s^{12} M^4/15$ to the energy gap which has already been given in [9].

4. What are the constraints on L_s and M?

The moment of inertia (2.5) is an expansion in the dimensionless parameter $1/(FL_s)^2$. In order to have a reliable expansion, $1/(FL_s)^2$ should be small. Hence, we can estimate that the spatial extent of the box should be about 2.5 fm or larger

$$L_s \gtrsim 2.5\,\mathrm{fm}. \tag{4.1}$$

Since in the delta regime $ML_s \ll 1$, we obtain an estimate for the upper bound on M for a given box size L_s. From Eq. (4.1) we deduce that M should be at least smaller than roughly 80 MeV.

We have assumed that the two dimensionless expansion parameters are of the same order $(r^4 \sim \delta^2)$. This assumption leads to an even smaller upper bound for M for a given box size L_s, in the domain where $1/(FL_s)^2$ is small. For $L_s = 2.5$ fm the corresponding upper bound on M is about 63 MeV, see also Tab. 1.

L_s [fm]	M [MeV]	ML_s
2.0	134	1.4
2.5	62	0.8
3.0	33	0.5
3.5	19	0.3
4.0	12	0.2

Table 1: Here, the upper bounds of M are stated for some selected values of L_s, under the assumption that $F^8 L_s^{12} M^4 \sim 1/(FL_s)^2$.

We want to estimate the size of the corrections to the energy gap (3.2). Therefore, we write

$$E_{L_s} = \frac{3}{2F^2 L_s^3} \left[1 + \Delta_{\mathrm{NL}} + \Delta_{\mathrm{NNL}}\right] , \tag{4.2}$$

where Δ_{NL} is the correction to the leading order energy gap, if we consider contributions up to $\mathscr{O}(\delta^2)$. This means that for the moment of inertia (2.5) we consider only the corrections up to $1/(FL_s)^2$, and for η (2.7) we consider only the leading term[2]. Δ_{NNL} are the additional corrections, if contributions up to $\mathscr{O}(\delta^4)$ are taken into account[3]. For M, we always take the upper bound at a given box size L_s. In Tab. 2 we state some values for Δ_{NL}, respectively Δ_{NNL} for some selected values of L_s.

L_s [fm]	Δ_{NL}	Δ_{NNL}
2.5	-0.20	0.04
3.0	-0.16	0.03
3.5	-0.13	0.02
4.0	-0.10	0.01

Table 2: We give some estimates for the size of the corrections to the energy gap for some specific values of L_s. For M we always assume the upper bound at the given box size L_s. Δ_{NL} are the corrections if we consider only contributions up to $\mathscr{O}(\delta^2)$. Δ_{NNL} are the additional corrections if we take into account contributions up to $\mathscr{O}(\delta^4)$.

5. Summary and Conclusions

For sufficiently small quark masses the low-energy excitations of QCD in the delta regime are still dominated by the rotator spectrum. Up to NNL order the energy gap involves the low-energy constants F, B, Λ_1 and Λ_2.

The delta regime can be probed by lattice QCD simulations. By measuring the energy gap in such simulations and by comparing the corresponding results with the analytic formula (3.2), we can determine the low-energy constants very precisely.

[2]The leading term of $\eta = F^2 L_s^3 M^2$ will give a correction $\sim r^4$ to the energy gap.

[3]We insert $\Lambda_1 = 120\,\mathrm{MeV}$ and $\Lambda_2 = 1200\,\mathrm{MeV}$ in the NNL order corrections of Θ.

The formula for the energy gap contains two expansions, where the corresponding expansion parameters are δ^2, respectively r^4. In order to obtain reliable expansions in these two parameters, it is essential to choose appropriate values for L_s and M. We have seen that the box size should be about 2.5 fm or larger. On the other hand, M has to be chosen between $60-80$ MeV or smaller, depending on the size of L_s. Therefore, QCD simulations with quark masses below their physical value have to be taken into account.

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