PROCEEDINGS OF SCIENCE

Thermodynamics of $SU(N)$ gauge theories in $2+1$ dimensions in the $T < T_c$ regime

Michele Caselle[a], **Luca Castagnini**[b], **Alessandra Feo**[*a], **Ferdinando Gliozzi**[a] **and Marco Panero**[c]

[a] *Dipartimento di Fisica Teorica, Università di Torino and INFN Torino, Via Giuria 1, 10125 Torino, Italy*

[b] *Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany*

[c] *Department of Physics and Helsinki Institute of Physics, University of Helsinki, FIN-00014 Helsinki, Finland*

E-mail: caselle@to.infn.it, luca.castagnini@physik.uni-regensburg.de, feo@to.infn.it, gliozzi@to.infn.it, marco.panero@helsinki.fi

We present Monte Carlo results for the thermodynamics of pure $SU(N)$ gauge theories with $N = 2,...,6$ in $2+1$ dimensions. We focus on the confined phase region $T < T_c$ and study thermodynamics variables such as the trace of the energy-momentum tensor, pressure, energy and entropy density using the integral method. We also investigate scaling properties with N of the different observables. We compare our results with a gas of free glueballs and the bosonic string predictions for the Hagedorn spectrum.

The XXVIII International Symposium on Lattice Field Theory, Lattice2010
June 14-19, 2010
Villasimius, Italy

*Speaker.

PoS(Lattice 2010)184

1. Introduction

The exploration of the phase diagram of QCD and its thermodynamics are challenging problems and central goals of lattice simulations at finite temperature and density. See [1] for a review. In this work we present Monte Carlo results for the thermodynamics of $SU(N)$ gauge theories with number of colors, $N = 2,...,6$, in $2+1$ dimensions. These theories are closely related with those in $3+1$ dimensions and are more numerically feasible. We focus on the confined phase $T < T_c$ and study thermodynamic variables such as the trace of the energy-momentum tensor, pressure, energy and entropy density using the integral method. We also investigate scaling properties with N of the different observables and compare our results with the predictions obtained assuming that the thermodynamics of the system could be described as a gas of free glueballs. We shall show that a relevant improvement in the comparison near the critical point is obtained including also higher orders in the glueball spectrum and assuming for these terms a bosonic string description.

2. Thermodynamics on the lattice

Before discussing the thermodynamics of $SU(N)$ lattice gauge theories in $2+1$ dimensions, we sketch some basic thermodynamics relations in the continuum. From the partition function $Z(T,V)$ we get the free energy density as,

$$f = -\frac{T}{V}\log Z(T,V)\,, \tag{2.1}$$

where T is the temperature and V is the spatial volume. In the thermodynamic limit the pressure is related to the free energy density as,

$$p = -\lim_{V\to\infty} f\,. \tag{2.2}$$

In the following we will assume to have a large, homogeneous system, so that the pressure can be identified as minus the free energy. Once the pressure is calculated as a function of the temperature $p(T)$, the other thermodynamics variable are derived. For example, the trace of the energy-momentum tensor $\varepsilon - 2p$ is,

$$\frac{\varepsilon - 2p}{T^3} = T\frac{\partial}{\partial T}\left(\frac{p}{T^3}\right)\,. \tag{2.3}$$

The energy density $\varepsilon = T^2 \frac{\partial}{\partial T}(p/T)$ is then obtained by adding $2p/T^3$ to this result while the entropy is given by,

$$s = \frac{\varepsilon + p}{T} = \frac{\partial p}{\partial T}\,. \tag{2.4}$$

On the lattice the temperature and volume of the thermodynamic system are determined by the lattice size $N_\tau \times N_s^2$ and the lattice spacing a,

$$V = (aN_s)^2\,, \qquad T = \frac{1}{aN_\tau}\,. \tag{2.5}$$

In this work, we perform a non-perturbative study of $SU(N)$ Yang-Mills theories with $N = 2,3,4,5,6$ colors regularized on a finite lattice, with lattice spacing a, with N_s points along the two space-like directions and N_τ points along the time-like direction. We use the Wilson action for a generic $SU(N)$ gauge group,

$$S_W(U_\mu(x)) = \sum_P S(U_P)\,, \qquad S(U_P) = \beta\left(1 - \frac{1}{N}\mathrm{ReTr}U_P\right), \tag{2.6}$$

where P denotes one of the $3N_\tau \times N_s^2$ plaquettes on the lattice and U_P is the product of the U_μ-matrices (with $\mu = 0,1,2$) around each 1×1 plaquette. On the lattice the partition function is given by,

$$Z = \int \prod_{x,\mu} dU_\mu(x)\exp(-S_W(U_\mu(x)))\,. \tag{2.7}$$

In the continuum limit eq. (2.6) becomes the standard Yang-Mills action provided that,

$$\beta = \frac{2N}{ag^2}\,. \tag{2.8}$$

In $2+1$ dimensions g^2 has dimensions of mass and sets the scale.

Although in principle all thermodynamics variables can be calculated from the free energy density, in practice, a direct computation of the partition function on the lattice is not possible. Here we use the integral method of Refs. [2, 3], as in Ref. [4]. We first calculate the action, i.e., the derivative of the partition function with respect to the bare coupling β. Up to an integration constant, resulting from the lower integration limit β_0, the pressure is then obtained by integrating,

$$\frac{p(\beta,N_\tau,N_s)}{T^3} = N_\tau^3 \int_{\beta_0}^{\beta} d\beta' \Delta S(\beta',N_\tau,N_s) \tag{2.9}$$

where in 2+1 dimensions

$$\Delta S(\beta',N_\tau,N_s) = 3\langle P_0\rangle_\beta - \langle P_s + 2P_\tau\rangle_\beta\,. \tag{2.10}$$

Here $P_{s,\tau}$ denote the expectation values of space-space, space-time plaquettes, respectively and P_0 is the plaquette value on symmetric lattices N_s^3. Using eqs. (2.9) and (2.10) we can write the trace of the energy-momentum tensor (2.3) as,

$$\frac{\varepsilon - 2p}{T^3} = T\frac{\partial}{\partial T}\left(\frac{p}{T^3}\right) = N_\tau^3 \Delta S\left(\beta\left(\frac{T}{T_c}\right), N_\tau, N_S\right) T\frac{d\beta}{dT}\,. \tag{2.11}$$

In order to obtain eq. (2.11) as a function of T/T_c, where T_c is the critical temperature of the continuum theory, we need to relate T/T_c to β, for any value of N,

$$\beta = \beta(T/T_c)\,. \tag{2.12}$$

In practice, $T\frac{d\beta}{dT}$ is determined through a parametric fit (similar to the one performed in [5] for $SU(3)$) but for generic N. A good choice for β_0 [5] is $\beta_0 = \beta(T/T_c = 0.6)$, after checking from the measurements that N_τ^3 times the integrand in (2.11) is negligible at this temperature.

3. $SU(N)$ gauge theories at large N: scaling properties

Let us investigate the large-N limit of eq. (2.12) in $SU(N)$ gauge theories in $2+1$ dimensions. To do so we need to relate some dimensionless ratios that in this limit become constant [6, 7, 8], say,

$$\frac{m_0}{\sqrt{\sigma}} = 4.108(20) + \frac{c}{N^2} + \dots, \qquad \frac{T_c}{\sqrt{\sigma}} = 0.903(3) + \frac{0.88}{N^2} + \dots, \tag{3.1}$$

where c is a constant. Here $\sqrt{\sigma}$ is the square root of the string tension at zero temperature in the continuum theory.

Considering also that, if we keep g^2N fixed, β scales as N^2 (2.8) and from [6, 7]

$$\frac{\sqrt{\sigma}}{g^2N} = 0.1975 - \frac{0.12}{N^2} + \dots, \tag{3.2}$$

we get

$$\sqrt{\sigma} = \frac{0.395N^2}{a\beta} - \frac{0.24}{a\beta} + \dots. \tag{3.3}$$

Combining these expressions we obtain the dependence of β in terms of the temperature T. To get $\beta(T)$, it is particularly convenient to set the temperature scale using the $\sqrt{\sigma}/T_c$ ratio. To the first order in β we have

$$\frac{T}{T_c} = \frac{T}{\sqrt{\sigma}}\frac{\sqrt{\sigma}}{T_c} = T\frac{a\beta}{(0.395N^2 - 0.24)\left(0.903 + \frac{0.88}{N^2}\right)} \tag{3.4}$$

and using eq. (2.5) gives,

$$\beta = N_\tau \frac{T}{T_c}\left(0.357N^2 + 0.13 - 0.211/N^2\right), \tag{3.5}$$

which for $N = 3$ gives $\beta = 0.34$ (to be compared with the expression given in Bialas et al. [5], $\beta = 3.3N_\tau\frac{T}{T_c} + 1.5 + O(1/N_\tau)$, which gives $\beta = 0.33$). Combining Eq. (3.5) with the data from [8] to get the correction to the scaling in the large-N limit we obtain,

$$\beta = N_\tau \frac{T}{T_c}(0.357N^2 + 0.13 - 0.211/N^2) + (0.22N^2 - 0.5), \tag{3.6}$$

which gives the dependence of β on the temperature T up to a first order correction to be used in eq. (2.11).

4. Numerical results and discussions

We are now ready to evaluate the trace energy-momentum tensor in eq. (2.11) and check the validity of the scaling dependence in eq. (3.6) by plotting the right hand side of eq. (2.11) vs. $t \equiv \frac{T}{T_c}$. This plot is expected not to be dependent on N (and also on N_τ).

The numerical simulations were performed using the Chroma library [10] plus our own programs (for $SU(2)$ and $SU(4)$). We evaluated ΔS for $N_\tau = 6$ (and for SU(2) and SU(3) also for $N_\tau = 8$) and spatial volumes N_S^2 such that the aspect ratio was always $N_s/N_\tau \geq 8$. In agreement with Ref. [5], we may safely assume that, in this temperature range, this condition is enough to

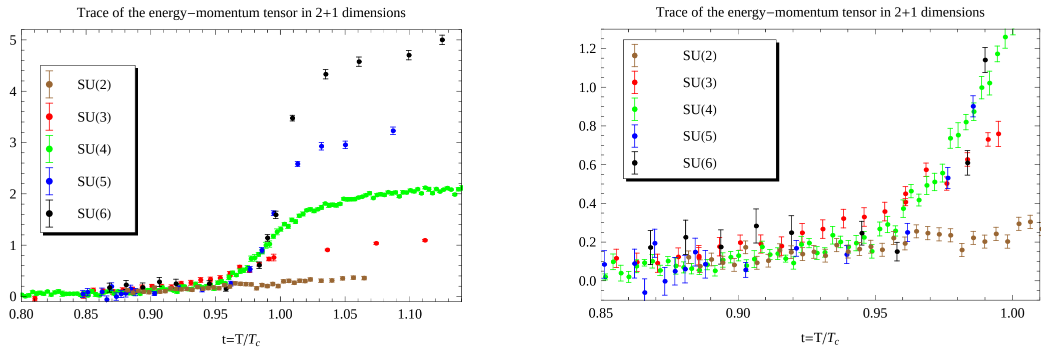

Figure 1: Left:The trace of the energy-momentum tensor vs. $t = T/T_c$ for $SU(N = 2,3,4,5,6)$. Right: Magnified view, of the same, in the low temperature region.

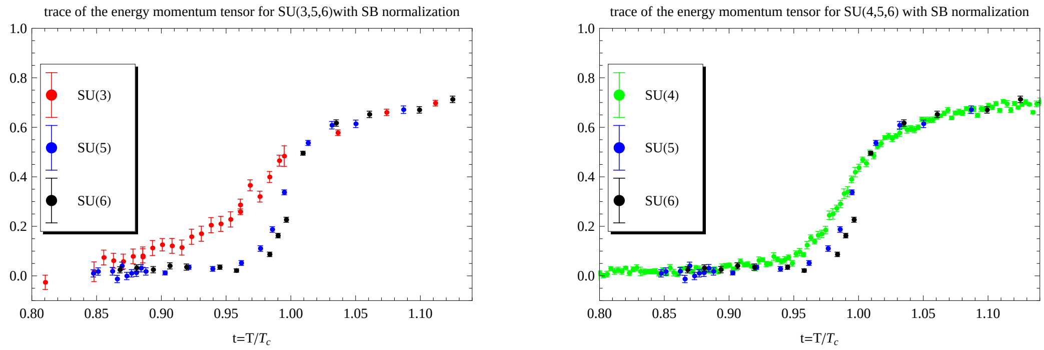

Figure 2: The trace of the energy-momentum tensor normalized to the lattice Stefan-Boltzmann (SB) limit vs. $t = T/T_c$.

eliminate finite size effects in the spatial directions. A detailed description of our results and of the algorithms we used will be reported elsewhere [13].

We report in fig. 1 our estimates for the trace of the energy-momentum tensor. The $SU(3)$ data are in perfect agreement with the one in Ref. [5]. Below the critical temperature T_c there is a good scaling with N. Above T_c, the different curves split up and they appear to be ordered according to the high temperature scaling law for the value of N. See fig. 2 for the gauge groups $SU(3,4,5,6)$.

As mentioned in the introduction our main goal was to compare the $T < T_c$ data with a glueball gas model. We performed this comparison in three steps. First we assumed the gas to be dominated by the lowest glueball state, then we included all the glueballs below the two-particle threshold (for which very precise numerical estimates exist) and finally, following the suggestion of [11], we try to compare our data with the whole glueball spectrum, assuming for the glueballs an ansatz inspired by the effective bosonic string model. Details on the calculations can be found in [13].

The pressure associated with a single non-interacting, relativistic particle species of mass m reads,

$$p = \frac{mT^2}{2\pi} \sum \frac{1}{k^2} \exp\left(-k\frac{m}{T}\right)\left(1+\frac{T}{km}\right) \tag{4.1}$$

From Eq. (4.1) we can reconstruct all thermodynamics quantities as explained earlier. In particular, the trace of the energy-momentum tensor (2.3) can be written as

$$\frac{\varepsilon - 2p}{T^3} = \frac{m^2}{2\pi T^2} \sum \frac{1}{k} \exp\left(-k\frac{m}{T}\right) . \tag{4.2}$$

This observable is particularly suited for this comparison since it can be calculated by numerical simulations without being integrated over β (see Eq. (2.11)). From the above equation we obtain:

$$\Delta S = \frac{m^2 a^2}{2\pi\beta N_\tau} \sum \frac{1}{k} \exp\left(-k\frac{m}{T}\right) . \tag{4.3}$$

For the first two stages of the comparison we used the numerical values of the glueball masses reported in [9]. Given the precision of the data it is mandatory to keep into account scaling corrections in this comparison. The most effective way to do this is to rewrite m/T as

$$\frac{m}{T} = \frac{m}{\sqrt{\sigma}} \frac{\sqrt{\sigma}}{T} = \frac{m}{\sqrt{\sigma}} (aN_\tau \sqrt{\sigma}) , \tag{4.4}$$

and then use the scaling functions reported in [9]. We write here these corrections explicitly in the $SU(3)$ case, for the lowest mass m in the case of a lattice size $N_\tau = 8$ (the generalization to any value of N is straightforward). Using,

$$a\sqrt{\sigma} = \frac{3.367(50)}{\beta} + \frac{4.1(1.7)}{\beta^2} + \frac{46.5(11.0)}{\beta^3} , \tag{4.5}$$

and $\frac{m}{\sqrt{\sigma}} = 4.329(41)$ we obtain,

$$\frac{m}{T} = 8 \times 4.329 \times \left(\frac{3.367}{\beta} + \frac{4.1}{\beta^2} + \frac{46.5}{\beta^3}\right) . \tag{4.6}$$

Higher masses can be treated in the same way, using the data for the ratios $m_i/\sqrt{\sigma}$ reported in [9]. We compare the results of this analysis with the data for the trace of the energy-momentum tensor in fig. 3 for $N = 2$ and $N \geq 3$. The blue and red lines correspond to the inclusion of the lightest mass and the first eight masses, respectively. It is easy to see that these fits fail to reproduce the data and suggest the necessity of taking into account the full spectrum of glueballs using for instance a string inspired ansatz.

To compare our results with the bosonic string predictions for the Hagedorn spectrum (in the same spirit as in Ref. [11] in $4d$) we extended to arbitrary dimensions the computation of the density of states of the closed bosonic string (following [12]).

We found the following expression,

$$\tilde{\rho}_{d-2}(M) = \left(\frac{\pi}{3}\right)^{d-1} \frac{1}{T_H} (d-2)^{\frac{d}{2}-1} \left(\frac{T_H}{M}\right)^{d} e^{M/T_H} . \tag{4.7}$$

Inserting this expression in eq. (4.2) we found a remarkable agreement with the data, even in the region near the critical transition [13]. This comparison is reported in fig.3-left for $SU(2)$ and in fig. 3-right for $N \geq 3$.

We think that this type of analysis (which we plan to further improve in the future) will give us the opportunity to test the string inspired glueball models (like for instance the Isgur-Paton one

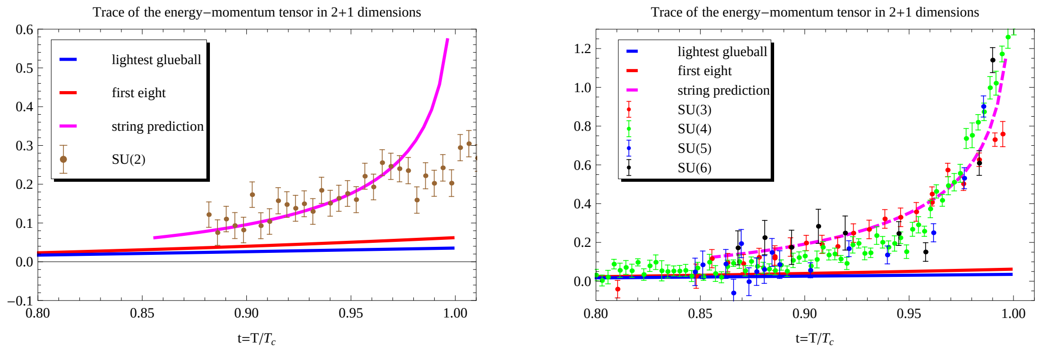

Figure 3: Comparison between contribution of the glueballs spectrum, the string predictions with respect to the trace energy-momentum tensor of $SU(2)$ (Left) and $SU(N \geq 3)$ (Right).

[14]) and also to better understand the many non trivial features of effective string models which have been up to now addressed only looking at observables related to the interquark potential or to the width of the flux tube.

Acknowledgements

Numerical simulations were partially performed on the INFN Milano-Bicocca TURING cluster. L.C. acknowledges partial support from Deutsche Forschungsgemeinschaft (Sonderforschungsbereich/Transregio 55) and the European Union grant 238353 (ITN STRONGnet). M.P. acknowledges financial support from the Academy of Finland project 1134018.

References

[1] Kazuyuki Kanaya, "Finite Temperature QCD", plenary talk, this conference.

[2] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier and B. Petersson, Nucl. Phys. B **469**, 419 (1996) [arXiv:hep-lat/9602007].

[3] J. Engels, J. Fingberg, F. Karsch, D. Miller and M. Weber, Phys. Lett. B **252**, 625 (1990).

[4] M. Panero, Phys. Rev. Lett. **103**, 232001 (2009) [arXiv:0907.3719 [hep-lat]].

[5] P. Bialas, L. Daniel, A. Morel and B. Petersson, Nucl. Phys. B **807**, 547 (2009) [arXiv:0807.0855 [hep-lat]].

[6] M. Teper, Acta Phys. Polon. B **40**, 3249 (2009) [arXiv:0912.3339 [hep-lat]].

[7] B. Lucini and M. Teper, Phys. Rev. D **66**, 097502 (2002) [arXiv:hep-lat/0206027].

[8] J. Liddle and M. Teper, arXiv:0803.2128 [hep-lat].

[9] M. J. Teper, Phys. Rev. D **59**, 014512 (1999) [arXiv:hep-lat/9804008].

[10] R. G. Edwards and B. Joó [SciDAC Collaboration and LHPC Collaboration and UKQCD Collaboration], Nucl. Phys. Proc. Suppl. **140**, 832 (2005) [arXiv:hep-lat/0409003].

[11] H. B. Meyer, Phys. Rev. D **80**, 051502 (2009) [arXiv:0905.4229 [hep-lat]].

[12] B. Zwiebach (2004), A First Course in String Theory Cambridge, UK: Univ. Pr. 558 p.

[13] M. Caselle, L. Castagnini, A. Feo, F. Gliozzi and M. Panero, in preparation.

[14] N. Isgur and J. E. Paton, Phys. Rev. D **31**, 2910 (1985).