

Transport coefficients of causal dissipative relativistic hydrodynamics in quenched lattice simulations

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Transport coefficients of causal dissipative relativistic fluid dynamics (CDR) are studied in quenched lattice simulations. CDR describes the behavior of relativistic non-Newtonian fluids in which the relaxation time appears as a new transport coefficient besides the shear and bulk viscosities. It was recently shown that these coefficients can be given by the temporal-correlation functions of the energy-momentum tensors as in the case of the Green-Kubo-Nakano formula. By using the new formula in CDR, we study the transport coefficients with lattice simulations in pure SU(3) gauge theory. After defining the energy-momentum tensor on the lattice, we extract a ratio of the shear viscosity to the relaxation time which is given only in terms of the static correlation functions. The simulations are performed on $24^3 \times 4-16$ lattices with $\beta_{\text{LAT}} = 6.0$, which corresponds to the temperature range of $0.5 \lesssim T/T_c \lesssim 1.8$, where T_c is the critical temperature.

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1. Introduction

Relativistic fluid dynamics is an important model to understand various collective phenomena in astrophysics and heavy-ion collisions, although its theoretical foundation has not yet been established [1]. The relativistic Navier-Stokes theory is, for example, acausal and unstable and inadequate as the theory of relativistic fluids. The reason is that the irreversible currents (the shear stress tensor $\pi^{\mu\nu}$, the bulk viscous pressure Π etc.) are linearly proportional to the thermodynamic forces (the shear tensor $\sigma^{\mu\nu}$, the expansion scalar θ etc.), with the proportionality constant named the shear viscosity coefficient η , the bulk viscosity coefficient ζ etc. Thus, the forces have an instantaneous influence on the currents, which obviously violates causality and leads to instabilities. These problems are solved by, for example, introducing retardation into the definitions of the irreversible currents, leading to equations of motion for these currents which thus become independent dynamical variables. The retardation effect is characterized by the relaxation time. Theories of this type are called causal dissipative relativistic fluid dynamics (CDR). In CDR, the irreversible currents and the thermodynamic forces are no longer in a simple linear relation, and such fluids are called *non-Newtonian*. As a consequence, the transport coefficients for CDR cannot be computed with methods commonly used for *Newtonian* (Navier-Stokes) fluids, such as the Green-Kubo-Nakano (GKN) formula.

Recently, a new microscopic formula to calculate the transport coefficients of CDR from time-correlation functions was proposed [2, 3]. This formula reproduces the ordinary results when it is applied to the classical Navier-Stokes theory and the diffusion equation. The consistency between this new formula and the results obtained from the Boltzmann equation was confirmed in Ref. [4, 5]. Since this formula is derived from quantum field theory, it will be applicable even to dense fluids, differently from the calculations based on the Boltzmann equation.

The purpose of the present study is to calculate the transport coefficients of CDR with lattice QCD simulations by using the new formula. The calculations of the transport coefficients, in general, contain temporal-correlation functions which are very difficult to estimate in lattice simulations [6, 7]. Thus, as a first attempt, we focus on a ratio between the shear viscosity and the corresponding relaxation time, η/τ_π , which is given only by static correlation functions. After defining the correlation functions between the energy-momentum tensor on the lattice, we calculate the ratio in quenched lattice simulations on $24^3 \times 4-16$ lattices with $\beta_{\text{LAT}} = 6.0$, which corresponds to the temperature range $0.5 \lesssim T/T_c \lesssim 1.8$ where T_c is the critical temperature.

This report is organized as follows: in section 2, we introduce formulations of CDR and show that the ratio of transport coefficients can be expressed in terms of static correlation functions between the energy-momentum tensors. In section 3, the energy-momentum tensor is defined on the lattice by using a clover-shaped combination of gauge links. Results of lattice simulations are shown in section 4, and the summary is given in section 5.

2. Causal dissipative relativistic fluid dynamics

We first choose gross variables which are necessary to extract the macroscopic motion of many-body systems. If the chosen variables are not enough, the derived fluid dynamics will show unphysical behaviors, such as instability and the divergent transport coefficients.

For ideal fluid, the energy-momentum tensor $T^{\mu\nu}$ is a function only of the energy density ε and the fluid velocity u^μ , which is normalized as $u^\mu u_\mu = 1$. Then, by applying a Lorentz transformation and using the definition of the energy density and pressure P , we obtain $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - g^{\mu\nu}P$. Note that P is calculated by the equation of state. Since $T^{\mu\nu}$ is conserved, we have

$$\partial_\mu T^{\mu\nu} = 0. \quad (2.1)$$

This is the relativistic Euler equation.

For dissipative fluid, $T^{\mu\nu}$ cannot be expressed only by ε and u^μ . We represent this additional component by another second rank tensor $\Pi^{\mu\nu}$. The most general $T^{\mu\nu}$ is, then, given by $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - g^{\mu\nu}P + \Pi^{\mu\nu}$. Conventionally, $\Pi^{\mu\nu}$ is expressed using the trace part Π and traceless part $\pi^{\mu\nu}$ as $\Pi^{\mu\nu} = \pi^{\mu\nu} - (g^{\mu\nu} - u^\mu u^\nu)\Pi$. Finally $T^{\mu\nu}$ is expressed as

$$T^{\mu\nu} = (\varepsilon + P + \Pi)u^\mu u^\nu - g^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}, \quad (2.2)$$

and Π and $\pi^{\mu\nu}$ are the bulk viscous pressure and the shear stress tensor, respectively, satisfying the orthogonality condition $u_\mu \pi^{\mu\nu} = 0$. In the traditional Landau-Lifshitz theory [8], the viscous terms are induced instantaneously by the corresponding thermodynamic force:

$$\Pi = -\zeta\theta, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \quad (2.3)$$

where ζ and η are the bulk and shear viscosities, respectively. The thermodynamic forces θ and $\sigma^{\mu\nu}$ are defined by

$$\theta = \partial_\mu u^\mu, \quad \sigma^{\mu\nu} = \frac{1}{2} \left(\partial^\mu u^\nu + \partial^\nu u^\mu - \frac{2}{3}(g^{\mu\nu} - u^\mu u^\nu)\theta \right) \equiv \Delta^{\mu\nu\lambda\delta} \partial_\lambda u_\delta. \quad (2.4)$$

When we use these definitions of the viscous terms, we obtain the relativistic Navier-Stokes equation. Because of the instantaneous production of the viscous terms, this equation contains sound propagations with infinite speed.

In order to solve this problem, the retardation effect is taken into account by introducing relaxation times τ_π for the shear stress tensor and τ_Π for the bulk viscous pressure, respectively. Thus the viscous terms satisfying causality are given by

$$\tau_\Pi u^\mu \partial_\mu \Pi + \tau_\Pi \Pi \theta + \Pi = -\zeta\theta, \quad \tau_\pi \Delta^{\mu\nu\lambda\delta} u^\alpha \partial_\alpha \pi_{\lambda\delta} + \tau_\pi \pi^{\mu\nu} \theta + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \quad (2.5)$$

where τ_Π and τ_π are the relaxation times of Π and $\pi^{\mu\nu}$, respectively. Here the projection operator $\Delta^{\mu\nu\lambda\delta}$ is necessary to satisfy the orthogonality relation. These are the equations of CDR. One can easily check that the Navier-Stokes theory is reproduced in the vanishing relaxation time limit. The second terms on the l.h.s. come from the (de)compression of fluid cells which is important to implement stable numerical calculations with ultra-relativistic initial conditions [9].

In fluid dynamics, transport coefficients are inputs which should be calculated from the underlying microscopic dynamics. As was discussed in the introduction, we cannot apply the GKN formula to CDR. The new formula is derived by using the projection operator method [2, 4]. The results are summarized as

$$\frac{\eta}{\beta(\varepsilon + P)} = \frac{\eta_{GKN}}{\beta^2 \int d^3\mathbf{x} (\hat{T}^{0x}(\mathbf{x}), \hat{T}^{0x}(\mathbf{0}))}, \quad \frac{\tau_\pi}{\beta} = \frac{\eta_{GKN}}{\beta^2 \int d^3\mathbf{x} (\hat{T}^{yx}(\mathbf{x}), \hat{T}^{yx}(\mathbf{0}))}, \quad (2.6)$$

$$\frac{\zeta}{\beta(\varepsilon + P)} = \frac{\zeta_{GKN}}{\beta^2 \int d^3\mathbf{x} (\hat{T}^{0x}(\mathbf{x}), \hat{T}^{0x}(\mathbf{0}))}, \quad \frac{\tau_\Pi}{\beta} = \frac{\zeta_{GKN}}{\beta^2 \int d^3\mathbf{x} (\delta\hat{\Pi}(\mathbf{x}), \delta\hat{\Pi}(\mathbf{0}))}, \quad (2.7)$$

where $\hat{\cdot}$ denotes operator, and we define $\hat{\Pi} \equiv \sum_{i=1}^3 \hat{T}^{ii}/3 - c_s^2 \hat{T}^{00}$ and $\delta\hat{A} \equiv \hat{A} - \text{Tr}[\rho_{eq}\hat{A}]$ with the equilibrium density matrix ρ_{eq} . The inner product is defined by Kubo's canonical correlation,

$$(A, B) = \int_0^\beta \frac{d\lambda}{\beta} \text{Tr}[\rho_{eq} A(-i\lambda) B]. \quad (2.8)$$

Here η_{GKN} and ζ_{GKN} are the shear and bulk viscosities of Newtonian fluids which are calculated using the GKN formula (or more precisely, using the Zubarev method). These quantities are given by the temporal (dynamical) correlation functions.

One can see that the new transport coefficients are still calculated from the GKN formula with the normalization factors, which are, on the other hand, given by the static correlation functions. Thus, for example, the ratio of the shear viscosity and corresponding relaxation time is calculated only from the static correlation functions,

$$\frac{\eta}{\tau_\pi(\varepsilon + P)} = \frac{\int d^3\mathbf{x} (\hat{T}^{yx}(\mathbf{x}), \hat{T}^{yx}(\mathbf{0}))}{\int d^3\mathbf{x} (\hat{T}^{0x}(\mathbf{x}), \hat{T}^{0x}(\mathbf{0}))}. \quad (2.9)$$

In the leading order of the weakly interacting bose gas, the above ratio becomes $\frac{\eta}{\tau_\pi(\varepsilon + P)} = \frac{P}{\varepsilon + P}$ which becomes zero ($\frac{1}{4}$) at $T = 0$ ($T \rightarrow \infty$) for massive bosons. In the following, we focus on this ratio and calculate it in quenched lattice simulations.

3. Energy-momentum tensor on the lattice

Let us consider the gluonic matter at finite T , and define the energy-momentum tensor for the SU(3) gauge theory in Euclidean space-time as,

$$T_{\mu\nu}(x) = 2\text{tr} [F_{\mu\alpha}(x)F_{\nu\alpha}(x)] - \frac{1}{2}\delta_{\mu\nu} \left(1 + \frac{\beta(g)}{2g} \right) \text{tr} [F_{\rho\sigma}(x)F_{\rho\sigma}(x)], \quad (3.1)$$

where the trace is taken over color indices, and $\beta(g)$ is a beta function on the lattice [10]. In the standard approach, the field strength tensor squared on the lattice (without the summation over Lorentz indices) is defined from the Hermitian part of the plaquette as

$$a^4 \text{tr} [F_{\mu\nu}(x)F_{\mu\nu}(x)] + O(a^5) = \beta_{\text{LAT}} \left[1 - \frac{1}{3} \text{Re tr } U_{\mu\nu}(x) \right], \quad (3.2)$$

where $\beta_{\text{LAT}} = 6/g^2$ is a lattice gauge coupling. This is utilized to define e.g. the standard gauge action. However this does not tell us anything about the off-diagonal part of the energy-momentum tensor, $T_{\mu\nu}(\mu \neq \nu)$. Therefore, the following equality (valid only in the continuum theory with full O(3) rotational symmetry) has been employed to calculate the correlations of the energy-momentum tensor:

$$\langle T_{ij}(x)T_{ij}(y) \rangle = \frac{1}{2} [\langle T_{ii}(x)T_{ii}(y) \rangle - \langle T_{ii}(x)T_{jj}(y) \rangle], \quad (i, j = 1, 2, 3). \quad (3.3)$$

It was however realized recently that this relation receives large errors due to lattice discretization [11]. Moreover, it does not give us a clue to calculate the correlation of T_{i4} (the denominator of the ratio in Eq. (2.9)) at finite T .

Alternative way to define the field strength would be to take the anti-Hermitian part of the plaquette,

$$a^4 \text{tr} [F_{\mu\nu}(x)F_{\rho\sigma}(x)] + O(a^5) \equiv -\frac{\beta_{\text{LAT}}}{24} \text{tr} \left([Q_{\mu\nu}(x) - Q_{\mu\nu}^\dagger(x)] [Q_{\rho\sigma}(x) - Q_{\rho\sigma}^\dagger(x)] \right), \quad (3.4)$$

which can be used both for the first and second terms of the right hand side of Eq.(3.1). Here we adopt a clover-shaped combination of the plaquette [12]

$$Q_{\mu\nu}(x) \equiv \frac{1}{4} [U_{\mu\nu}(x) + U_{\nu-\mu}(x) + U_{-\mu-\nu}(x) + U_{-\nu\mu}(x)], \quad (3.5)$$

to respect the space-time symmetry. This definition naturally leads to $\langle T_{\mu\nu} \rangle = 0$ for $\mu \neq \nu$. In our simulation, we use the energy-momentum tensor obtained from Eq.(3.4).

In the Euclidean space-time, the Kubo's canonical correlation for the energy-momentum tensors appearing in Eq. (2.9) becomes a susceptibility

$$G_{\mu\nu}(T) = \frac{T^2}{V} \left\langle \left(\int d^3\mathbf{x} \int_0^{1/T} d\tau T_{\mu\nu}(\mathbf{x}, \tau) \right)^2 \right\rangle_T, \quad (3.6)$$

where we have used the translation invariance both in spatial and temporal directions, and $\langle \dots \rangle_T$ denotes the thermal average at temperature T . With $T = 1/(aN_t)$, $V = (aN_s)^3$ and $\int d^3\mathbf{x} \int d\tau \rightarrow a^4 \sum_x$ on the lattice, we can rewrite the static susceptibility $G_{\mu\nu}$ in the lattice unit with zero temperature subtraction as

$$G_{\mu\nu}(T) = \left[\left\langle \frac{1}{N_s^3 N_t^2} \left(\sum_x T_{\mu\nu}(x) \right)^2 \right\rangle_T - \left\langle \frac{1}{N_s^3 N_{t0}^2} \left(\sum_x T_{\mu\nu}(x) \right)^2 \right\rangle_{T=0} \right], \quad (3.7)$$

where N_{t0} means the temporal lattice size at $T = 0$.

4. Results of lattice simulations

We perform quenched lattice simulations employing a standard plaquette gauge action on a isotropic lattice of $24^3 \times N_t$ with $N_t = 4 - 16$. The lattice coupling is taken to be $\beta_{\text{LAT}} = 6.0$, which corresponds to $a = 0.093$ fm with the Sommer scale $r_0 = 0.5$ fm [13]. The range of N_t corresponds to $T/T_c \sim 0.5 - 1.8$, where the critical temperature is located between $N_t = 7$ and $N_t = 8$. The zero-temperature subtraction is performed with $N_t = 24$. We generate pure gauge configurations by the pseudo-heat-bath algorithm and measure correlations using 1000–5000 configurations at every 1000 trajectories after thermalization. Statistical errors are estimated by the jackknife analysis.

In order to see the behavior of the energy-momentum tensor constructed from Eq. (3.4), let us first show results of the trace anomaly,

$$\varepsilon - 3P = \left[\left\langle \frac{1}{N_s^3 N_t} \sum_x \sum_\mu T_{\mu\mu}(x) \right\rangle_{T=0} - \left\langle \frac{1}{N_s^3 N_{t0}} \sum_x \sum_\mu T_{\mu\mu}(x) \right\rangle_T \right]. \quad (4.1)$$

Figure 1(left) shows temperature dependence of the trace anomaly together with the energy density and pressure calculated by the T -integral method [13]. Typical enhancement of $(\varepsilon - 3P)/T^4$ around

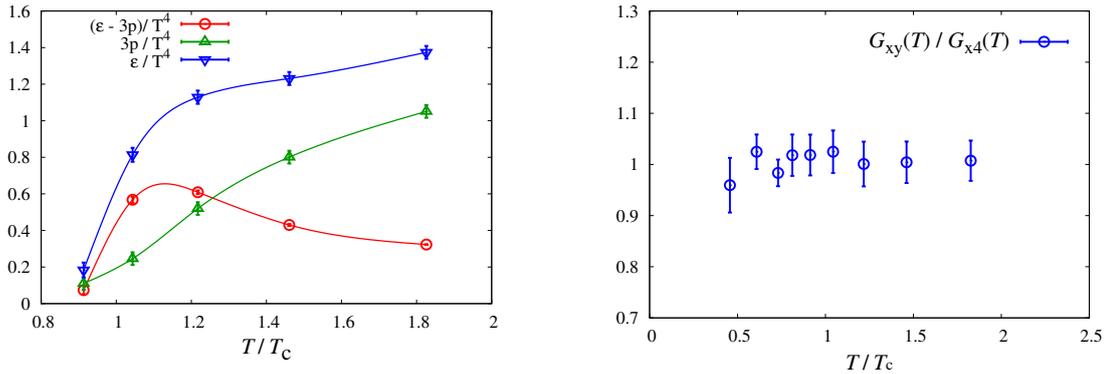


Figure 1: Results of the trace anomaly, energy density and pressure (left) and the ratio of the susceptibilities $G_{xy}(T)/G_{x4}(T)$ (right) as a function of temperature.

T_c , and the rapid (slow) increase of the energy density (pressure) can be seen. The off-diagonal parts of the energy-momentum tensor are found to be zero within the statistical error, $\langle T_{\mu\nu} \rangle_T \simeq 0$, ($\mu \neq \nu$).

We define the averaged static susceptibilities G_{xy} and G_{x4} from Eq.(3.7) as

$$G_{xy}(T) \equiv \frac{1}{3} (G_{12} + G_{13} + G_{23}), \quad G_{x4}(T) \equiv \frac{1}{3} (G_{14} + G_{24} + G_{34}). \quad (4.2)$$

From the simulation, we found that both G_{xy} and G_{x4} increase monotonically with temperature with similar values, so that the ratio G_{xy}/G_{x4} shown in Fig. 1 (right) corresponding to $\eta/\tau_\pi(\epsilon + P)$ is almost unity over the range of temperatures we have explored, $0.5 \lesssim T/T_c \lesssim 1.8$. This behavior is in contrast to that expected from the weakly interacting bose gas mentioned at the end of sec.2, and is worth to be studied further.

5. Summary

We examined transport coefficients of causal dissipative relativistic fluid dynamics (CDR) in quenched lattice simulations. Based on the microscopic formulae proposed in Refs. [2, 3], a ratio between the shear viscosity and the corresponding relaxation time, $\eta/(\tau_\pi(\epsilon + P))$, was computed from the static correlation functions of the energy-momentum tensor. We calculated these static correlation functions in quenched lattice simulations on $24^3 \times 4-16$ lattices with $\beta_{\text{LAT}} = 6.0$, which correspond to the temperature range of $0.5 \lesssim T/T_c \lesssim 1.8$. In this temperature region, the ratio stays constant and close to unity.

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