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A study of $\mathcal{N}=2$ Landau-Ginzburg model by lattice simulation based on a Nicolai map

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The two-dimensional $\mathcal{N}=2$ Wess-Zumino model with a quasi-homogeneous superpotential is believed to provide a Landau-Ginzburg description of the two-dimensional $\mathcal{N}=2$ superconformal minimal model. For the cubic superpotential $W=\lambda\Phi^3/3$, it is expected that the Wess-Zumino model describes A_2 model and the chiral superfield Φ shows the conformal weight $(h,\bar{h})=(1/6,1/6)$ at the IR fixed point. We study this conjecture by a lattice simulation, extracting the weight from the finite volume scaling of the susceptibility of the scalar component in Φ. We adopt a lattice model with the overlap fermion, which possesses a Nicolai map and a discrete R-symmetry. We set $a\lambda=0.3$ and sample the scalar configurations by solving the Nicolai map on each $L\times L$ lattices, with $L=18,20,22,24,26,28,30,32$. To solve the map, we use the Newton-Raphson algorithm with various initial configurations. About 640 configurations are analyzed on each L, and the fermion determinants are explicitly evaluated. The result is $1-h-\bar{h}=0.660\pm0.011$, which is consistent with the conjecture.

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1. Introduction

Two-dimensional $\mathcal{N}=2$ Wess-Zumino (WZ) model with a quasi-homogeneous superpotential is expected to describe, at the infrared (IR) fixed point, the $\mathcal{N}=2$ superconformal minimal models [1]. This conjecture has been tested in various aspects. If one could calculate correlation functions of the WZ model directly in the IR region, it would give us a further test of the conjecture. Since the coupling of the WZ model becomes strong in the IR region, this certainly requires non-perturbative techniques.

The purpose of this study is to provide a non-perturbative numerical evidence for the above conjecture in the case of the simplest cubic superpotential $W(\Phi)=\lambda\Phi^3/3$ by simulating the lattice WZ model. The WZ model was first formulated on the lattice in [2] with the Wilson fermion. But since in the present work [3] we hope to treat the massless model, we adopt the lattice model constructed in [4] with the overlap fermion, which possesses the Nicolai map [5] and a discrete R-symmetry. For the cubic superpotential, it is expected that the WZ model describes the A_2 model ($c=1$) and the chiral superfield Φ shows the conformal weight $(h,\bar{h})=(1/6,1/6)$ at the IR fixed point. We will extract the conformal weight of the chiral superfield Φ from the finite volume scaling of the susceptibility of the scalar component ϕ in Φ. Interestingly, the A_2 model is also realized by the Gaussian model ($c=1$) with the coupling constant K at the $\mathcal{N}=2$ supersymmetric points, $K=1/12\pi$ or $3/4\pi$ [6]. We will extract the coupling constant K of the Gaussian model by identifying the phase factor of the scalar component ϕ as the 2π-periodic Gaussian field. This will provide a clear numerical evidence of the full recovery of $\mathcal{N}=2$ supersymmetry in the IR limit.

2. Lattice formulation of WZ model

We adopt the lattice WZ model which possesses a Nicolai map and a discrete R-symmetry[4]. The model is formulated with overlap Dirac operator$D=(1/a)\left[1+X/\sqrt{X^\dagger X}\right]$, where $X=1-(1/2)\left[(1+\gamma_\mu)a\nabla_\mu^+ +(1-\gamma_\mu)a\nabla_\mu^-\right]$ with the first forward, backward difference operators $\nabla_\mu^\pm$. D satisfies the Ginsparg-Wilson relation, $D\hat{\gamma}_3+\gamma_3 D=0$ with $\hat{\gamma}_3=\gamma_3(1-aD)$, and it can be expressed as $D=\gamma_1 S_1+\gamma_2 S_2+aT$ in a spinor decomposition, defining three difference operators S_1, S_2 and T which satisfy the relation $2T=-(S_1)^2-(S_2)^2+(aT)^2$. Then, the bosonic and fermionic actions are given by

$$\begin{aligned} S_{\rm B} &= a^2\sum_x\left\{\phi^*(2T)\phi+W^{*\prime}(1-a^2T/2)W'+W'(-S_1+iS_2)\phi+W^{*\prime}(-S_1-iS_2)\phi^*\right\},\\ S_{\rm F} &= a^2\sum_x\bar{\psi}\left(D+F(\phi)\right)\psi, \end{aligned} \tag{2.1}$$

where $F(\phi)=\frac{1+\gamma_3}{2}W''\frac{1+\hat{\gamma}_3}{2}+\frac{1-\gamma_3}{2}W^{*\prime\prime}\frac{1-\hat{\gamma}_3}{2}$ and $W'=\partial W(\phi)/\partial\phi$, $W''=\partial^2W(\phi)/\partial\phi^2$. In the finite lattice of the volume $L\times L$, we adopt the periodic boundary condition for both bosonic and fermionic fields.

Throughout this article we consider the cubic superpotential $W(\Phi)=\lambda\Phi^3/3$. The coupling λ, which has the mass dimension one, is the unique mass parameter of our model, besides the lattice spacing a. Thus λ gives the scale under which the WZ model reduces to a conformal field theory. To see the conformal behavior on the lattice, we should prepare a large lattice size $L/a\gg(a\lambda)^{-1}$, while the continuum limit is $L/a\to\infty$ and $a\lambda\to 0$.

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We summarize the symmetries of our lattice model. First we note that the WZ model possesses the Nicolai map [5], which is explicitly given by

$$\eta = W' + a(\phi - \frac{a}{2}W')T + (\phi^* - \frac{a}{2}W^{*\prime})(S_1 + iS_2). \tag{2.2}$$

The Jacobian of this map from $\{\phi,\phi^*\}$ to $\{\eta,\eta^*\}$ precisely cancels the overlap fermion determinant $|D+F(\phi)|$, while the bosonic action $S_{\rm B}$ is identical to the Gaussian weight $a^2\sum_x|\eta(x)|^2$, which allows us to interpret η, η^* as the random white noises. Thanks to this map, the lattice model has the following on-shell nilpotent supersymmetry Q:

$$Q\phi = -\bar\psi_-, \quad Q\phi^* = -\bar\psi_+, \tag{2.3}$$
$$Q\psi_+ = -\eta^*, \quad Q\psi_- = -\eta, \tag{2.4}$$
$$Q\bar\psi_+ = 0, \qquad Q\bar\psi_- = 0, \tag{2.5}$$

where we write $\psi=(\psi_1,\psi_2)^t$, $\bar\psi=(\bar\psi_1,\bar\psi_2)$, $\psi_\pm=(\psi_1\pm\psi_2)/\sqrt{2}$ and $\bar\psi_\pm=(\bar\psi_1\mp\bar\psi_2)/\sqrt{2}$. This lattice model also has a Z_3 R-symmetry. $S_{\rm F}$ has the $U(1)$ R-symmetry under $\phi\to e^{-2i\alpha}\phi$, $\psi\to e^{i\alpha\hat\gamma_3}\psi$, $\bar\psi\to\bar\psi e^{i\alpha\gamma_3}$. But this symmetry is broken to Z_3 ($\alpha=n\pi/3, n\in\mathbf{Z}$) by the last two terms in $S_{\rm B}$, which would be surface terms in the continuum limit. With these symmetries, we can show in the lattice perturbation theory that the desired continuum limit is achieved without extra fine-tunings. The redefined fields $\varphi\equiv\lambda\phi$, $\chi\equiv\lambda\psi$ are helpful for this discussion. In this notation, φ has mass dimension 1 and χ has 3/2, and λ is factorized as the overall factor $1/\lambda^2$ in the action. This overall factor counts the number of loops as the Planck constant does. Consider the generic radiative correction to an operator $\mathcal{O}$ of mass dimension $p\,(\geq 0)$ in the action,

$$\left(\frac{c_0a^{p-4}}{\lambda^2} + c_1a^{p-2} + c_2\lambda^2a^p + \cdots\right)\int \mathrm{d}^2x\,\mathcal{O} \tag{2.6}$$

where $c_0, c_1, ...$ are constants. The first, second and third terms represent the contributions at tree, one-loop and two-loop levels. In the continuum limit $a\to 0$, the corrections terminate at the two-loop level. At the tree level the lattice action agrees with that of the WZ model in the continuum limit. So we have to consider the operators with $p\leq 2$. Such operators which preserve the Z_3 R-symmetry and the fermion number are a constant and $\varphi^*\varphi$. But the constant has no effect on the path-integral and $\varphi^*\varphi$ is forbidden by the supersymmetry Q. Thus we do not need any extra fine-tunings to achieve the desired continuum limit, at least in the perturbation theory.

We also note that by the Z_3 R-symmetry, the cubic quasi-homogeneous superpotential is uniquely singled out. The Yukawa coupling terms $\bar\psi(1+\gamma_3)\phi^n(1+\hat\gamma_3)\psi$ with $n\neq 1$ and their conjugates, which may appear in the models other than the A_2 model, are not allowed by the symmetry. Therefore, we do not need to worry about operator mixings even at finite lattice spacing.

Unfortunately, however, $|D+F|$ can be negative. It is easily shown that $\gamma_1(D+F)\gamma_1 = (D+F)^*$ in the basis $\gamma_1=\sigma_3$, $\gamma_2=-\sigma_2$, implying that every non-real eigenvalue of $D+F$ is doubly degenerated and $|D+F|$ is real. But the possibility of unpaired real negative eigenvalues can not be excluded. Then, $|D+F|e^{-S_{\rm B}}$ can not be interpreted as the safely positive probability weight. Therefore, our lattice WZ model generally faces the so-called sign problem in the standard simulation methods like the hybrid Monte Carlo (HMC) method.

3. Sampling configurations

Next we explain how we sample configrations. We utilize the Nicolai map [7, 8]. Using the Nicolai map Eq.(2.2), one can rewrite the expectation value of the observable $\mathscr{O}$ in the following form,

$$\langle \mathscr{O} \rangle = \frac{\langle \sum_{i=1}^{N(\eta)} \mathscr{O}(\eta, \phi_i(\eta))\, \mathrm{sgn}|D+F(\phi_i(\eta))| \rangle_\eta}{\langle \sum_{i=1}^{N(\eta)} \mathrm{sgn}|D+F(\phi_i(\eta))| \rangle_\eta}, \tag{3.1}$$

where $\langle \cdots \rangle_\eta$ denotes the average over the Gaussian white noise η. $\{\phi_i(\eta)\}_{i=1,\cdots,N(\eta)}$ are the solutions of the Nicolai map Eq.(2.2) for a given η, which contribute to the observable (i.e.$|D+F(\phi_i)| \neq 0$), and $N(\eta)$ denotes the number of the solutions. These solutions exist discretely. To see this, suppose a solution $\phi + \delta\phi$ that is infinitesimally different from another solution ϕ, $|D+F(\phi)| \neq 0$. Then Eq.(2.2) means $0 = (\mathrm{Re}\delta\phi, i\mathrm{Im}\delta\phi)(D+F(\phi)) \Rightarrow \delta\phi = 0$ and therefore the sigma symbol comes in Eq.(3.1).

The expression Eq.(3.1) suggests us a possible simulation method. If one succeeds in solving the Nicolai map Eq.(2.2) for a given η to obtain the sets $\{\phi_i\}$ and $\{\mathrm{sgn}|D+F(\phi_i)|\}$, one can simulate the model by observing the numerator and the denominator separately, provided that signals for the denominator remain the order $\mathscr{O}(1)$. An advantage of this method is that the autocorrelation between samples completely disappears. Of course, one needs to evaluate $\mathrm{sgn}|D+F(\phi_i)|$, but it would not be demading in two dimensions for moderate lattice sizes L/a. To solve the Nicolai map numerically, one may use the Newton-Raphson algorithm. A difficulty is that one do not know $N(\eta)$ a priori. What one may try then is to solve the Nicolai map with various initial configurations.

We sample configurations with the above strategy in this work. For each noise η, we try the Newton-Raphson iterations from 100 initial ϕ configurations. Since almost all noises would be of order $\mathscr{O}(1)$, the solutions should be of order $\mathscr{O}(1)$. So we generate these initial configurations by the standard normal distribution. And we suppose that the configurations obtained in this method exhaust solutions of the Nicolai map.

We check the quality of these samples by two tests. One is the Witten index. Note that the denominator in Eq.(3.1), $\Delta \equiv \langle \sum_{i=1}^{N(\eta)} \mathrm{sgn}|D+F(\phi_i)| \rangle_\eta$, is the partition function normalized into the Witten index. For instance, in the massive free case $W(\Phi) = m\Phi^2/2$, the Nicolai map is just a linear equation with a coefficient matrix $D + m(1 - aD/2)$. Since the determinant of this matrix is positive , the solution is unique and Δ indeed reproduces the known result in the continuum theory: $\langle +1 \rangle_\eta = 1$. In the present cubic case, it must be close to 2 at least in the continuum limit. The other test is the Ward identity of the supersymmetry Q of the lattice model. For the right hand side of Eq.(2.2), the on-shell supercharge acts as $Q\eta = \delta S/\delta\psi_+$, $Q\eta^* = \delta S/\delta\psi_-$. Then $\langle Q(\cdots) \rangle = 0$ and the Schwinger-Dyson equations imply the following Ward identities on the lattice:

$$\langle \eta(x_1)\cdots\eta(x_m)\eta^*(y_1)\cdots\eta^*(y_n) \rangle = \begin{cases} 0 & m \neq n \\ \sum_\sigma \prod_{k=1}^m \delta_{x_k, y_{\sigma(k)}} & m = n. \end{cases} \tag{3.2}$$

For instance, the case m, $n = 1$ reads $\langle S_\mathrm{B} \rangle = L^2$. An ideal case to pass these tests is the case where it happens that $\sum_{i=1}^{N(\eta)} \mathrm{sgn}|D+F| = 2$ over the noises. In this case, the Witten index 2 is exactly reproduced. In addition, from Eq.(3.1), $\langle \eta\cdots\eta^*\cdots \rangle$ reduces to $\langle \eta\cdots\eta^*\cdots \rangle_\eta$, and the latter reproduces the right hand side of Eq.(3.2) because the noise is the standard normal distribution.

Table 1: Samples

L	18	20	22	24	26	28	30	32
A	316	319	319	316	316	314	307	316
B	3	0	1	3	4	6	10	4
C	1	1	0	0	0	0	1	0
D	0	0	0	1	0	0	2	0
Δ	1.997	1.997	2	2.003	2	2	1.994	2
δ [%]	0.3	0.0	0.1	0.4	0.4	0.4	0.4	0.2

The sampling was carried out at $a\lambda = 0.3$, on different $L\times L$ lattices with $L = 18,20,22,24,26,28,30,32$. We generated 320 noises on each L. The results, shown in TABLE 1, are classified into following 4 patterns. **A**: 2 solutions with positive fermion determinants. **B**: 4 solutions, one with negative and the rest with positive determinants. **C**: 1 solution with positive determinant. **D**: 3 solutions with positive determinants. Since almost all the noises belong to **A** or **B** where $\sum_{i=1}^{N(\eta)} \mathrm{sgn}|D+F| = 2$, the summation is almost 2 over the noises. The Witten index Δ and the ratio $\delta = (\langle S_B\rangle - L^2)/L^2$ over these samples are also shown in TABLE 1. From these results, one can estimate the systematic error in our simulation method as less than 0.5%.

4. Numerical results

With these configrations of the scalar component ϕ and the random source η, one can calculate the various correlation functions in the WZ model. To extract the conformal weight of the chiral superfield Φ, we observe the finite volume scaling of the susceptibility of the scalar component ϕ,

$$\chi_\phi \equiv a^2 \sum_{|x|\geq 3a} \langle \phi(x)\phi^*(0)\rangle. \tag{4.1}$$

Here we omit the contribution from the correlations at the distances shorter than $\lambda^{-1} = a/0.3 \simeq 3a$, as an improvement of the observable. If the chiral superfield Φ really shows the scaling behavior with the conformal weight $(h,\bar{h})$, and the correlation function $\langle\phi(x)\phi^*(0)\rangle$ scales in the IR region, the finite volume scaling of χ_ϕ in the continuum limit should read

$$\chi_\phi \propto \int_V d^2x \frac{1}{|x|^{2h+2\bar{h}}} \propto V^{1-h-\bar{h}}, \tag{4.2}$$

where $V = L^2$ is the system volume. Using this relation, we read $1-h-\bar{h}$ from the slope of the $\ln\chi_\phi$-$\ln L^2$ plots.

The numerical results are shown in FIG. 1. The error bars show the statistical errors. The solid line in FIG. 1 is a fit to the plots by least-square-method and the slope gives $1-h-\bar{h} = 0.660\pm 0.011$. Our result is consistent with the value expected by the conjecture, $1-h-\bar{h} = 2/3 = 0.666\ldots$. This is our main result.

For a further support of the conjecture, we will also extract the coupling constant K of the Gaussian model as follows. In the continuum limit, the WZ model has $U(1)$ R-symmetry. Although

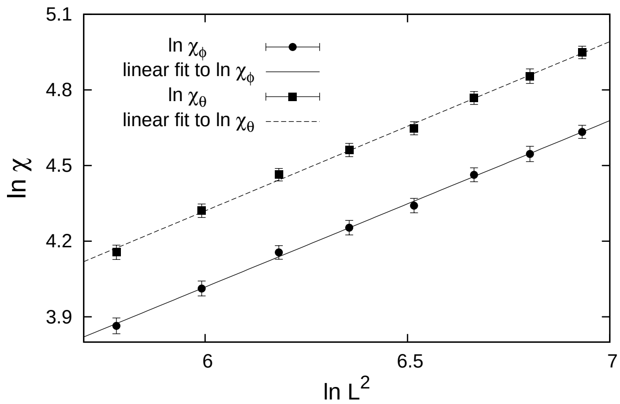

Figure 1: $\ln\chi$ - $\ln L^2$ plots

this chiral symmetry is not broken spontaneously, according to Coleman's theorem, there may appear massive fermions and bosons in the spectrum and may decouple in the IR limit, leaving only massless degrees of freedom[9]. If one writes $\phi = |\phi|e^{i\theta}$, the R-symmetry is given by $\theta \to \theta - 2\alpha$. Then, the modulus $|\phi|$ and the new fermion $\chi = e^{i\gamma_3\theta/2}\psi$, $\bar{\chi} = \bar{\psi}\, e^{i\gamma_3\theta/2}$ are all singlets and may aquire masses. These degrees of freedom may decouple in the IR limit, leaving θ as low energy degrees of freedom. The IR effective action of θ may be given by [10]

$$S_{\rm eff} = \frac{K}{2}\int d^2x(\nabla\theta)^2 \qquad (\theta \sim \theta + 2\pi), \tag{4.3}$$

where K is an effective coupling constant. Other chiral symmetric terms are irrelevant.

The value of K will provide us a criterion for the full recovery of $\mathcal{N}=2$ superconformal symmetry in the IR limit. The Gaussian model can reproduce $\mathcal{N}$=2 superconformal algebra, because both bosonic and fermionic components of the chiral superfield can be constructed from the single bosonic field θ, and the model contains indeed the superconformal stress-energy tensor at the two values $K = 1/12\pi$ and $3/4\pi$ [6]. U(1) R-charges of ϕ and ψ in the WZ model suggest that $K = 3/4\pi$ is realized in our case.

To extract the value of the coupling constant K, we again plot the finite volume scaling of the susceptibility

$$\chi_\theta \equiv a^2 \sum_{|x|\geq 3a} \langle e^{i\theta(x)} e^{-i\theta(0)}\rangle. \tag{4.4}$$

If the above scenario works, in the IR region, the operator $e^{i\theta}$ becomes the vertex operator with the weight $(h,\bar{h}) = (1/8\pi K, 1/8\pi K)$. Then, one expects $\chi_\theta \propto V^{0.666\ldots}$. The scaling dimension is identical with χ_ϕ.

The numerical results are shown also in FIG. 1. The solid line is again a fit to the plots. The slope is 0.671 ± 0.014 and therefore $K = 0.242 \pm 0.010$, which is consistent with $K = 3/4\pi = 0.238\ldots$. Thus, our numerical results support the conjecture, including the fact $c = 1$. In addition, since $K = 3/4\pi$ is the $\mathcal{N}=2$ supersymmetric point of the Gaussian model, this result implies the restoration of all supersymmetries in the IR limit (at long distances) without fine tunning.

5. Summary and Outlook

We have sampled configurations in the lattice WZ model by solving the Nicolai map through the Newton-Raphson algorithm. We have tested these configurations by evaluating the Witten index and the supersymmetric Ward-Takahashi identity. Then we have extracted the conformal weight of the chiral superfield. Our results are consistent with the conjecture that the WZ model with the cubic superpotential describes the $\mathcal{N}=2$ superconformal A_2 minimal model. This provides a new non-perturbative evidence for the conjecture.

It would be interesting to check the conjecture for the cases with other quasi-homogeneous superpotentials. For instance, the quartic potential case is expected to correspond to the A_3 minimal model. If one can check the conjecture in this case, it automatically gives a check of the E_6 minimal model, because the tensor product of A_2 and A_3 models gives the E_6 model. In addition, the WZ model with two supermultiples are expected to describe the D-series.

If one can define the c-function on the lattice, it would be possible to observe directly the central charges through it. It will give us further evidences for the conjecture.

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