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Spherical Vortices, Fractional Topological Charge and Lattice Index Theorem in SU(2) LGT

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We report on our studies of topological properties of classical spherical center vortices with the low-lying eigenmodes of the Dirac operator in the fundamental and adjoint representations using both the overlap and asqtad fermion formulations. We confirm the discrepancy between the topological charge and the index of the Dirac operator, which was already found in a previous work of our group [Phys. Rev. D 77, 14515 (2008)] for overlap fermions, also for staggered fermions and adjoint representations. Furthermore, the index theorem of the adjoint fermions gives some evidence for fractional topological charge. During cooling the spherical center vortex on a $40^3 \times 2$ lattice we find an object with topological charge $Q = 1/2$ which we identify as a Dirac monopole without an antimonopole. For more details see [arXiv: hep-lat:1005.1015].

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1. Introduction

The center vortex model [1, 2, 3, 4, 5, 6] seems to be the most promising candidate to explain quark confinement in non-abelian gauge theories. Center vortices, quantized magnetic flux lines, compress the gluonic flux into tubes and cause a linearly rising potential at large separations. Numerical simulations have indicated that vortices could also account for phenomena related to chiral symmetry, such as causing topological charge fluctuations and spontaneous chiral symmetry breaking (SCSB) [7, 8, 9, 10]. These features of the QCD vacuum are intimately linked to the properties of the low-lying spectrum of the Dirac operator. In Ref. [11] we found strong correlations between the center vortices and the density distribution of low-lying Dirac eigenmodes.

The Atiyah-Singer index theorem [12, 13, 14, 15] states that the topological charge of a gauge field configuration equals the index of the Dirac operator in this gauge field background. In Ref. [16] we investigated the lattice index theorem and the localization of the zero modes for thick classical center vortices. For non-orientable spherical vortices, the index of the overlap Dirac operator turned out to differ from the topological charge.

Here we present our analysis of the topological charge and the lattice index theorem. We work with thick spherical vortices in SU(2) lattice gauge theory and extend our overlap results to (improved) staggered fermions in fundamental and adjoint representations. We also discuss the role of adjoint fermions with respect to fractional topological charge.

2. Fundamental Dirac zero modes and Spherical Center Vortices

We construct the non-orientable spherical vortex as described in section IIIB in [16]. Only the timelike links in one timeslice form a hedgehog-like configuration, resulting in one spherical vortex sheet without any intersections and hence no topological charge. Since only links in the time direction are different from $\mathbb{1}$, the topological charge determined from any lattice version of $F\tilde{F}$ vanishes for this spherical vortex configuration.

According to the Atiyah-Singer index theorem the topological charge is related to the index of the Dirac operator by

$$\text{ind}\, D[A] = n_- - n_+ = Q \tag{2.1}$$

where n_- and n_+ are the number of left- and right-handed zero modes of the Dirac operator [12, 13, 14]. For the spherical vortex configuration, the index of the overlap Dirac operator [15, 17, 18] is nonzero, even for lattice sizes fullfilling the Lüscher condition [19, 20, 21].

Staggered fermions [22, 23] don't have exact zero modes, but a separation between "would-be" zero modes and nonchiral modes is observed for improved staggered quark actions [24]. For our non-orientable spherical vortex configurations this separation is large enough to clearly identify the asqtad staggered zero modes. According to the index theorem, $\text{ind}\, D[A] = n_- - n_+ = 2Q$, the (doubly degenerate) zero modes give exactly the same topological charge as the overlap Dirac operator. The number of zero modes for overlap and asqtad staggered fermions is presented in Table 1. The scalar densities of the would-be zero modes for both types of spherical vortices show a distribution (Fig. 1) that looks very similar to their overlap counterparts (see [16]).

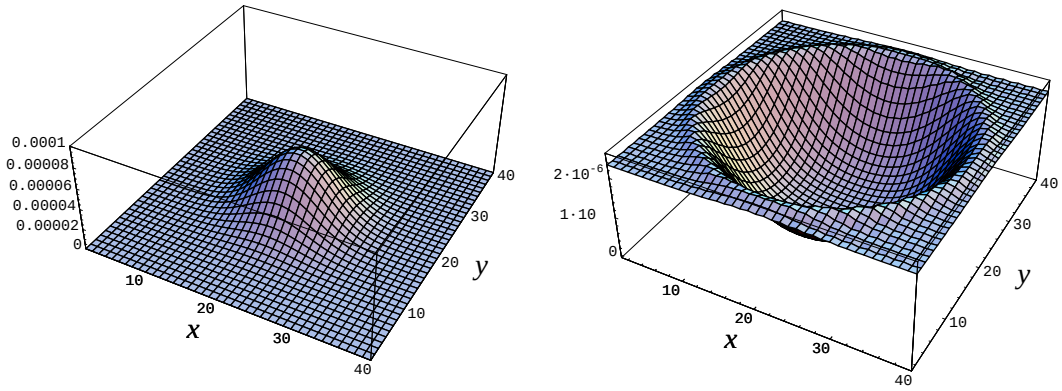

Figure 1: Scalar densities of the single (unpaired) asqtad staggered zero mode of positive chirality for the positive (α_+, left) and negative chirality for the negative (α_-, right) spherical vortex in a time slice of a $40^3 \times 8$ lattice, with antiperiodic boundary conditions (see also [16], compare to Figs.6 and 4).

3. Adjoint fermions, index theorem and fractional topological charge

The advantages of probing the topological background of a gauge configuration with fermions in the adjoint representation were discussed at recent lattice conferences (see e.g. [25]) and also this year in the talk by Alfonso Sastre [26]. The index of the massless Dirac operator in the adjoint representation of the $SU(N)$ gauge group in a background field of topological charge Q is equal to $2NQ$ [27]. Since the fermion is in the real representation, the spectrum of the adjoint Dirac operator is doubly degenerate. Therefore, for $SU(2)$ the index can only be a multiple of four if the gauge field background is made up of classical instantons.

Classical instantons carry an integer topological charge. Thus, in case of a fermion in the fundamental representation of $SU(2)$ there is exactly one zero mode for a one-instanton configuration. Now, if the actual constituents of the QCD vacuum had topological charge $Q = 1/2$, no zero mode would be produced. However, for adjoint fermions configurations with topological charge $Q = 1/2$ are able to create a zero mode. Edwards et al. presented in [27] some evidence for fractional topological charge on the lattice. García-Pérez et al. [25], however, associated this to lattice artefacts, *i.e.,* topological objects of size of the order of the lattice spacing.

We test the lattice index theorem for overlap and asqtad staggered fermions in the adjoint color representation on the spherical vortex configurations. In Table 1 the number of positive and negative overlap and staggered zero modes in the fundamental and adjoint representation for periodic and antiperiodic boundary conditions for spherical vortex configurations on different lattice sizes are summarized. All investigated fermion representations equally violate the lattice index theorem for these special configurations: the index is nonzero while the topological charge vanishes.

The measurements with adjoint fermions also show fractional topological charge, see Table 1, the cases where the adjoint index is not a multiple of four are printed in bold. Very interesting is the case of the cooled $40^3 \times 2$ lattice configuration, which will be discussed next.

	fundamental:				adjoint:			
	overlap:		asqtad:		overlap:		asqtad:	
lattice:	pbc:	apbc:	pbc:	apbc:	pbc:	apbc:	pbc:	apbc:
8^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ **6-**	0+ **2-**	8+ **12-**	0+ **4-**
12^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ **6-**	0+ **2-**	8+ **12-**	0+ **4-**
16^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-
20^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-
40^3x2:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-
cooled:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ **6-**	0+ **2-**	8+ **12-**	0+ **4-**
40^3x4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-

Table 1: Number of positive and negative overlap and staggered zero modes in the fundamental and adjoint representation for periodic and antiperiodic boundary conditions of spherical vortices on different lattice sizes. Numbers which indicate half-integer topological charge are printed in bold.

4. Topological charge after cooling

During cooling the spherical vortex configuration the index of the overlap Dirac operator does not change, but the topological charge quickly changes to a nonzero value according to the index, while the action S reaches a (nonzero) plateau, as shown in Fig. 2a) for a $40^3 \times 4$ lattice. So, the index of the overlap Dirac operator agrees with the topological charge after some cooling, but not on the original vortex configuration. During cooling of a spherical vortex on a $40^3 \times 2$ lattice we find evidence for fractional topological charge as shown in Fig. 2b). It exhibits a second "plateau" for the topological charge with $Q \approx 1/2$ between cooling steps 100 and 130. We analyzed this configuration with adjoint eigenmodes, which indeed measure fractional topological charge, as seen in Table 1. The spherical vortex contracts during cooling, after 78 cooling steps the vortex structure vanishes. In the maximal abelian gauge one can identify a monopole-antimonopole ring after

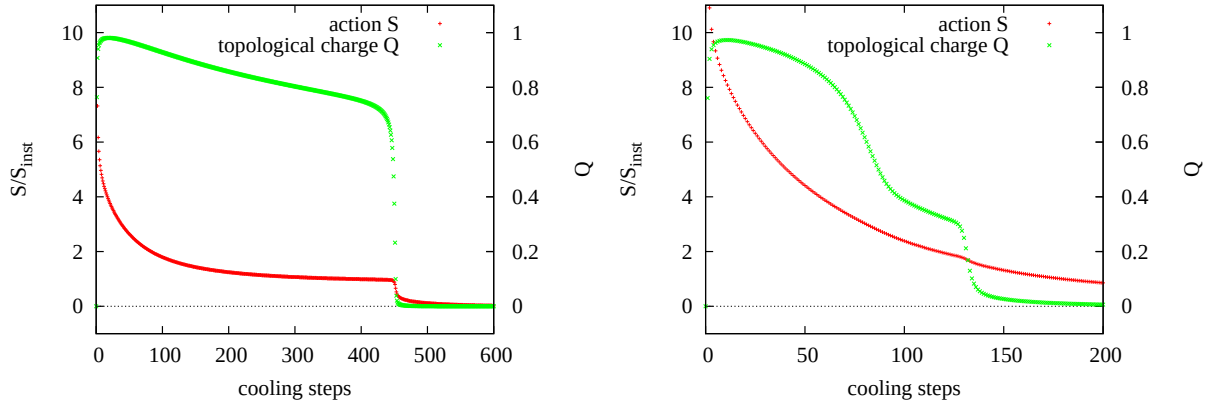

Figure 2: Cooling of a spherical vortex: a) On a $40^3 \times 4$ lattice, the topological charge rises from zero to close to one (right scale) while the action S (in units of the one-instanton action S_{inst}) reaches a plateau at S_{inst} (left scale). b) $40^3 \times 2$ lattice: While the action S decreases slowly, the topological charge first rises from zero to close to one, then decreases to an intermediate plateau of $Q \approx 0.4$ before it vanishes.

projection which again contracts during cooling and disappears after 91 cooling steps. We would like to mention, that the position of the monopole ring on the vortex sphere depends on the $U(1)$ subgroup chosen for the projection, see Fig.3 in [16]. In the interesting region with topological charge $Q = 1/2$ (at cooling step 120), the action and topological charge densities concentrate in the center of the spacial volumes and the gauge field at the lattice boundary is trivial. Landau gauge yields a very symmetric configuration, a static, singular non-abelian monopole without an antimonopole. In Landau gauge both time slices are exactly the same and all time-like plaquettes are trivial. The central cube in every time-slice is a non-abelian representation [28, 29] of a Dirac monopole. Its six plaquettes correspond to rotations by $2\pi/3$ in the fundamental representation of $SU(2)$. In the color frame of the corner with the smallest coordinate values the plaquette color vectors, parallel-transported according to Fig. 2 in [30], point in the $(-1,-1,-1)$-direction. Hence, the six plaquettes sum up to a total rotation of 4π in agreement with the non-abelian lattice Bianchi identity, which states that the product of plaquette rotations in appropriate order results in the unit matrix. Nevertheless the magnetic flux out of the cube is nonvanishing [30]. In the $U(1)$ representation in color direction $(1,1,1)$ the central cube represents a Dirac monopole with plaquette values summing up to 2π. The link variables of the central cube correspond to an $SU(2)$ rotation of a unit vector from one corner to the other, *i.e.*, $\cos(\omega) = 1/3 = (-1,-1,-1)(1,-1,-1)/3$. Due to the non-abelian nature of the $SU(2)$ gauge field for increasing distance from the center the field strength approaches zero. No antimonopole is needed to compensate the monopole. The Polyakov loops around the central cube form a hedgehog. They reflect the color directions of the link variables. A parallel-transport to the above color frame leads to parallel Polyakov lines in color direction $(-1,-1,-1)$. The $SU(2)$ color directions ($\vec{n}$-vectors of $\cos(\alpha) + i\sin(\alpha)\vec{n}\vec{\sigma}$) of link-, plaquette- and Polyakov matrices are depicted in Fig. 3.

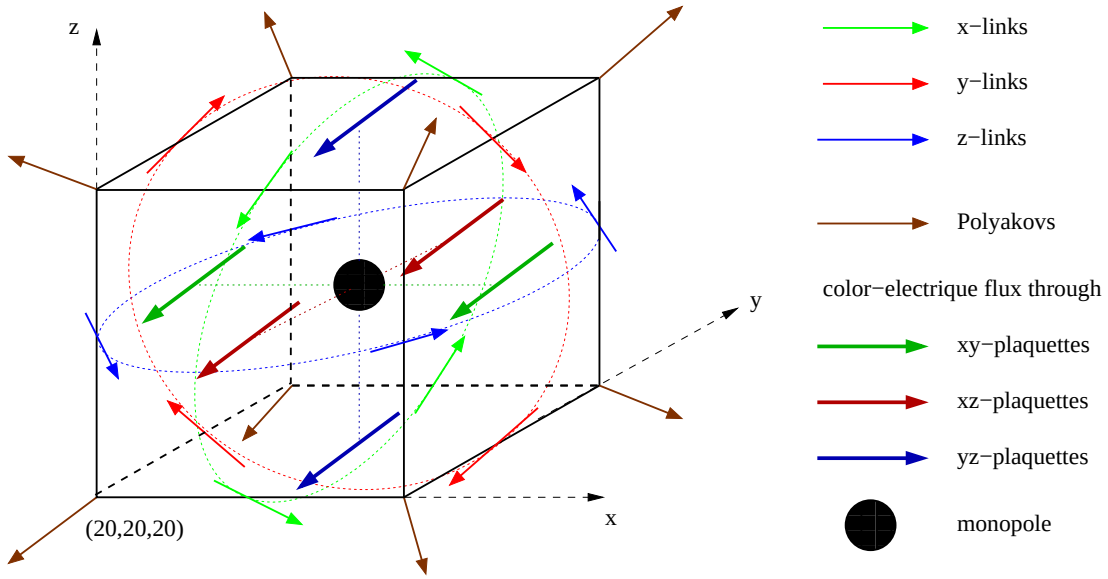

Figure 3: The arrows indicate the rotational vectors of link-, plaquette- and Polyakov-matrices around the central cube of the cooled negative spherical vortex. Link-vectors rotate around the center, color fluxes through plaquettes are aligned parallel in the $(-1,-1,-1)$-direction and Polyakov loops form a hedgehog.

5. Conclusions & Outlook

We reported on violations of the lattice index theorem for smooth, "admissible" gauge configurations of classical, spherical center vortices, for both, overlap and asqtad staggered fermions in the fundamental and adjoint representations. Numerically, the discrepancy equals the winding number of the spheres when they are regarded as maps $S^3 \to SU(2)$. Obviously such windings, given by the holonomy of the time-like loops of the spherical vortex, influence the index theorem [31, 32]. Cooling a spherical vortex on a $40^3 \times 2$ lattice, the index in the adjoint representation indicates evidence for an object with fractional topological charge. This object with $Q = 1/2$ is identified as a Dirac monopole with the wellknown singularity in its center and a gauge field fading away at large distances. Therefore even for periodic boundary conditions it does not need an antimonopole. For more details see [arXiv: hep-lat:1005.1015]. In a forthcoming paper we will analyze other configurations with fractional topological charge.

Acknowledgments

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