SU(3) Deconfinement in (2+1)d from Twisted Boundary Conditions and Self-Duality

Nils Strodthoff[*]
Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany
E-mail: nstrodt@crunch.ikp.physik.tu-darmstadt.de

Sam R. Edwards
Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany
E-mail: edwards@crunch.ikp.physik.tu-darmstadt.de

Lorenz von Smekal
Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany
E-mail: lorenz.smekal@physik.tu-darmstadt.de

We study the pure SU(3) gauge theory in 2+1 dimensions on the lattice using 't Hooft's twisted boundary conditions to force non-vanishing center flux through the finite volume. In this way we measure the free energy of spacelike center vortices as an order parameter for the deconfinement transition. The transition is of 2nd order in the universality class of the 2d 3-state Potts model, which is self-dual. This self-duality can be observed directly in the SU(3) gauge theory, and it can be exploited to extract critical couplings with high precision in rather small volumes. We furthermore obtain estimates for critical exponents and the critical temperature in units of the dimensionful continuum coupling. Finally, we also apply our methods to the (2+1)d SU(4) gauge theory which was previously found to have a weak 1st order transition. We nevertheless observe at least approximate $q = 4$ Potts scaling at length scales corresponding to the lattice sizes used in our simulations.

The XXVIII International Symposium on Lattice Field Theory, Lattice2010
June 14-19, 2010
Villasimius, Italy

[*]Speaker.

1. Introduction

The motivation of our study is to see how much we can learn about the deconfinement transitions in pure SU(N) gauge theories at finite temperature from universality and scaling. As these concepts require second order phase transitions, in 3+1 dimensions we are only left with SU(2), where spatial center vortex sheets share their universal behavior with interfaces in the $3d$ Ising model. This has been studied in [1]. In 2+1 dimensions on the other hand, we can apply analogous methods to both SU(2) and SU(3) which then exhibit a 2$^{\text{nd}}$ order deconfinement transition. The latter is in the universality class of the $2d$ Ising model for which by far the largest pool of exact results is available. This was exploited in [2] for a high precision determination of critical couplings and temperature, and in [3] for accurate finite-size scaling, a reflection of self-duality and a precision determination of the behavior of the continuum sting tension and its dual around the phase transition. Here, we report on first results from applying the same methods to SU(3) and gradually also to SU(4) in 2+1 dimensions.

For SU(3), the dimensionally reduced spin model with the same global symmetry and the universal properties of the Z_3 center-symmetry breaking deconfinement transition is the 3-state Potts model. For SU(4), the Z_4 center symmetry alone does not uniquely specify the effective spin model to describe the dynamics of Polyakov loops. SU(4) is a rank-three group and has three fundamental representations, 4, $\bar{4}$ and 6. So even the simplest effective Polyakov-loop model will consist of two distinct real terms, with nearest neighbor couplings between loops in $4/\bar{4}$ representations and between loops in the 6 representation [4]. Depending on the relative weight between the two, the corresponding spin model could be any of the Z_4-symmetric Ashkin-Teller models with three energy levels per link and continuously varying critical exponents between the $q = 4$ Potts model class and that of the planar or vector Potts model which corresponds to two non-interacting Ising models in this case. Even though the more recent studies of the $(2+1)d$ SU(4) gauge theory [5, 6, 7] indicate that the transition is weakly 1$^{\text{st}}$ order, we do find, at least approximately at the length scales corresponding to our spatial lattice volumes, a universal scaling which seems closest to the standard $q = 4$ Potts case. Because this might not seem very natural, it would be interesting to explain that.

One special feature of the q-state Potts models in 2 dimensions is that they are self-dual for all q, no matter whether they have 1$^{\text{st}}$ ($q > 4$) or 2$^{\text{nd}}$ ($q \leq 4$) order transitions. With a 2$^{\text{nd}}$ order transition and scaling, this self-duality is reflected in the $(2+1)d$ gauge theory: the spatial center-vortex free energies are mirror images around criticality of those of the confining electric fluxes [3]. Here we verify this explicitly for SU(3), and show how it can be used to remove the leading (universal) finite-size corrections in the determination of critical couplings from finite volume extrapolations.

2. Concepts and Methods

't Hooft's twisted boundary conditions, center vortices and electric fluxes: In a theory without matter fields where the gauge fields represent the center of the gauge group trivially, the boundary conditions in a finite Euclidean $1/T \times L^d$ volume are only fixed up to center elements giving rise to N^d gauge-inequivalent boundary conditions for a pure SU(N) gauge theory in $d+1$ dimensions. These twisted boundary conditions [8] can be classified either as magnetic twists defined in purely spatial planes or as temporal twists in the planes oriented along the Euclidean time direction. The latter are labeled by a vector $\vec{k} \in Z_N^d$. In the following we will only consider temporal twists because the magnetic twists are irrelevant for the deconfinement transition.

Temporal twist introduces spatial center vortices whose free energies provide order parameters for the deconfinement transition. These vortex free energies F_k (per temperature T) are defined as ratios $R_k(\vec{k}) \equiv Z_k(\vec{k})/Z_k(\vec{0}) = e^{-F_k(\vec{k})}$ of partition functions $Z_k(\vec{k})$ with temporal twist $\vec{k}$ over the periodic ensemble $Z_k(\vec{0})$. Analogously, one defines the electric flux free energies F_e via $R_e(\vec{e}) \equiv Z_e(\vec{e})/Z_e(\vec{0}) = e^{-F_e(\vec{e})}$. These describe gauge-invariant color-singlet free energies of static fundamental charges at some point $\vec{x}$ with mirror anti-charges in a neighboring volume at $\vec{x} + L\vec{e}$ along the direction of the flux $\vec{e}$ relative to the *no-flux* ensemble $Z_e(\vec{0}) = \sum_{\vec{k}} Z_k(\vec{k})$, which is an enlarged ensemble corresponding to fluctuating temporal twists, see [1, 3]. The electric flux and spatial center vortex partition functions are related by a d-dimensional Z_N Fourier transform,

$$R_e(\vec{e}) = \frac{1}{N} \left\langle \mathrm{tr}\big(P(\vec{x})P^\dagger(\vec{x}+\vec{e}L)\big) \right\rangle_{\text{no-flux}} = \left(\textstyle\sum_{\vec{k}} e^{2\pi \mathrm{i}\vec{e}\cdot\vec{k}/N} R_k(\vec{k}) \right) / \textstyle\sum_{\vec{k}} R_k(\vec{k}) . \qquad (2.1)$$

Universality and self-duality: By the Svetitsky-Yaffe conjecture, a $d+1$ dimensional gauge theory with second order deconfinement transition has the same universal properties as a d dimensional spin model with the same global Z_N symmetry [9]. The Polyakov loop correlators of the gauge theory near criticality behave in the same way as those of spins in the spin model. Spatial center vortices correspond to spin interfaces, which are frustrations where the coupling of adjacent spins favors cyclically shifted spin states rather than parallel ones for the usual ferromagnetic couplings. Consequently, the center-vortex free energies F_k show the universal behavior of interface free energies. For the 2+1 dimensional SU(3) gauge theory the corresponding spin model is the $2d$ 3-state Potts model. Like all q-state Potts models in 2 dimensions, it is self-dual.

Kramers-Wannier duality is of course a very well-known concept from statistical physics [11]. It provides exact maps between the spin systems and their dual theories in terms of disorder variables on the dual lattice. In 2 dimensions, just as for the Ising model ($q = 2$) these dual theories of the q-state Potts models are again q-state Potts models, but at a dual temperature $\widetilde{T}$ which is swapped around criticality at T_c as compared to the original model. Duality transformations in a finite volume do not preserve boundary conditions, however. Periodic boundary conditions on one side generally correspond to fluctuating boundary conditions on the other [12]. This was explicitly demonstrated for the duality between the $3d$ Ising and the Z_2-gauge model in [13]. The exact finite-volume duality transformation for the $2d$ q-state Potts models is given in [3]. Here it suffices to note that its structure is precisely that in (2.1) with $q = N$. It expresses the partition function of the dual q-state Potts model with certain set boundary conditions at a temperature $\widetilde{T}$ as a $2d$ Z_q-Fourier transform over Potts models with all possibilities of cyclically shifted boundary conditions at temperature T. In the $(2+1)d$ gauge theory, the temperature is the same on both sides of the Z_N-Fourier transform (2.1). But within the universal scaling window around a 2nd order phase transition, as a consequence of the self-duality of the spin model, the free energies of spatial center vortices and those of the confining electric fluxes are mirror images of one another around T_c.

Numerical procedure: To implement 't Hooft's twists on the lattice we multiply a stack of plaquettes by the corresponding center element $z \in Z_N$ so as to fix the corresponding amount of center flux through the planes with twisted boundary conditions. For the temporal $\vec{k}$-twists this introduces thin spatial center vortices perpendicular to $\vec{k}$ which separate regions where fundamental Polyakov loops differ by the center phase $z = e^{2\pi i k/N}$. They are thus like spin interfaces in the Potts model.
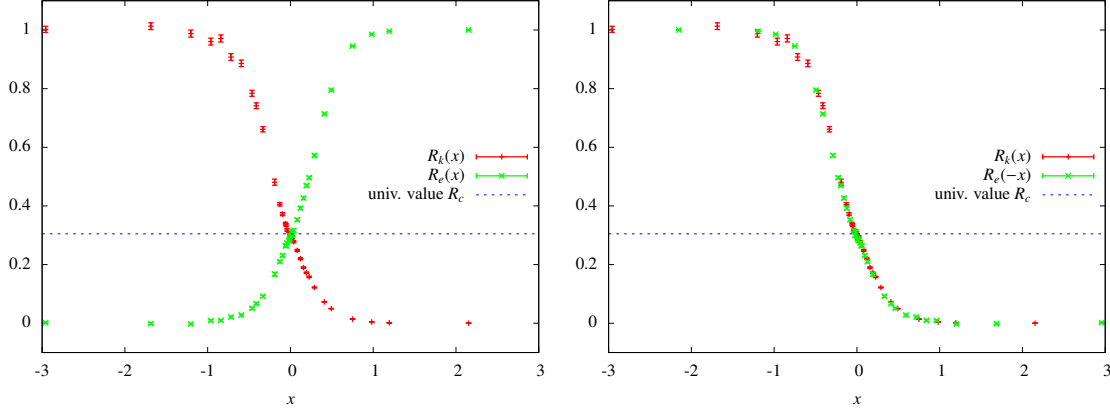
Figure 1: Self-duality in SU(3) ($N_s = 24$, $N_t = 2$)

We use the snake algorithm [14] to compute the ratios $R_k(\vec{k}) = Z_k(\vec{k})/Z_k(\vec{0})$ via products of ratios of partition functions which differ by only one phase rotated plaquette to successively form a vortex that wraps around the spatial volume perpendicular to $\vec{k}$. We use the standard Wilson gauge action and a heatbath algorithm supplemented by a number of overrelaxation steps. SU(N) updates are implemented as updates of SU(2) subgroups in the usual way [15].

3. Results

Self-duality in SU(3): As for $SU(2)$ in 2+1 dimensions [3] the self-duality of the 3-state Potts model is reflected in SU(3). The spatial center vortex and electric flux partition functions are related by a Z_3-Fourier transform as in (2.1) whose structure is the same as that of the finite volume duality transformation of the Potts model. Its self-duality therefore implies that electric fluxes on one side of the phase transition should resemble center vortex ensembles on the other.

Swapping the temperature $\widetilde{T} \leftrightarrow T$ in the spin model near criticality amounts to $x \leftrightarrow -x$ where $x = \pm L/\xi_\pm$ is the finite-size scaling variable given by the ratio of the finite size L over the correlation lengths $\xi_\pm = f_\pm(\pm t)^{-\nu}$, with ξ_+ for $x > 0$ and $t > 0$ above T_c, and ξ_- for $x, t < 0$, below. For the gauge theory we use $x = \pm T_c L(\pm t)^\nu \propto L/\xi^\pm$ and control the reduced temperature $t = T/T_c - 1$ by changing the lattice coupling. Within the universal scaling window, as functions of x, we should then find that $R_k(x) = R_e(-x)$ for matching pairs of twist $\vec{k}$ and flux $\vec{e}$. This is indeed the case also for SU(3), as demonstrated in Fig. 1, where we compare the ratios of partition functions R_k and R_e for one unit of temporal twist and one unit of electric flux, *i.e.*, both $\vec{k}$ and $\vec{e}$ either $(1,0)$ or $(0,1)$, over the phase transition.

Critical couplings: There is a long history of methods to extract critical couplings or temperatures from simulations in finite volumes, going back to using pairwise intersections of Binder cummulants on sucessively larger lattices [16]. Hasenbusch later demonstrated that the ratios of partition functions with different boundary conditions could be used in the same way to obtain a much more rapid convergence with very good estimates already from rather small lattices [17]. At criticality, these ratios tend to universal values $0 < R_c < 1$ in the thermodynamic limit. In [2] it was therefore shown how to obtain critical couplings for gauge theories from intersecting the ratios R_k of finite volume partition functions with these universal fixed points, once their values are known. For (2+1)d SU(2) this led to an even faster convergence than their pairwise intersections. For the
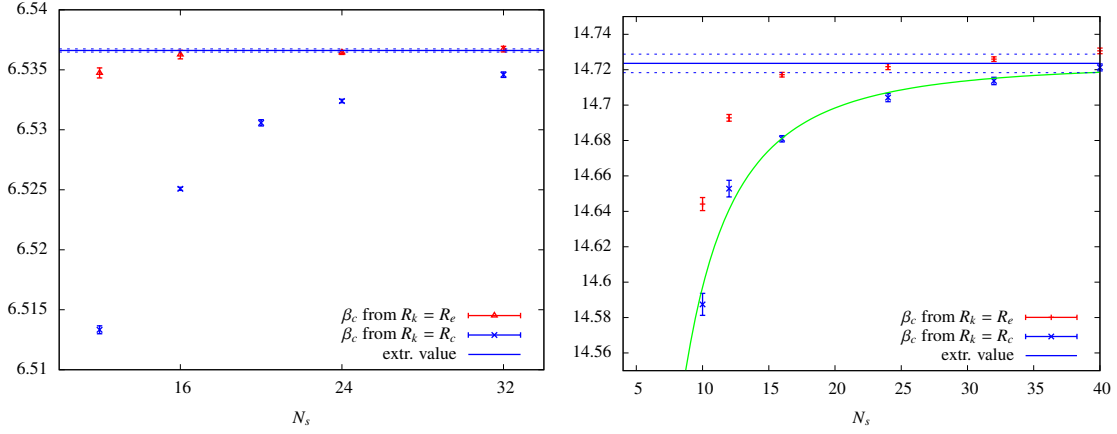
Figure 2: Convergence to β_c ($N_t = 4$) from $R_k(\beta) = R_e(\beta)$ and $R_k(\beta) = R_c$ in SU(2) (left) and SU(3) (right).

$2d$ Potts models with 2nd order transition, *i.e.*, for $q = 2, 3$ and 4, the universal numbers $R_c^{(m,n)}$ have been obtained exactly, in terms of Jacobi theta functions, for all cyclic boundary conditions (with $m, n = 0, 1, \ldots q - 1$) in [10]. For $q = 3$ on a symmetric lattice they are,

$$R_c^{(1,0)} = R_c^{(2,0)} = 0.30499982\ldots, \quad \text{and} \quad R_c^{(1,1)} = R_c^{(2,2)} = R_c^{(1,2)} = 0.19500018\ldots. \tag{3.1}$$

Using a finite-size scaling ansatz for the vortex ensemble ratios R_k around criticality of the form

$$R_k(\beta) = R_c + b(\beta - \beta_c)N_s^{1/\nu} + cN_s^{-\omega} + \cdots, \tag{3.2}$$

we define pseudo-critical couplings $\beta_c(N_s, N_t)$ in a finite volume by requiring that $R_k(\beta) = R_c$,

$$\beta_c(N_s, N_t) = \beta_c(N_t) - (c/b)N_s^{-(\omega + 1/\nu)} + \cdots. \tag{3.3}$$

These extrapolate to $\beta_c(N_t)$ from large spatial lattice sizes N_s at fixed numbers of time slices N_t. As a byproduct this method gives numerical estimates of the correction to scaling exponent ω.

With self-duality, however, there is a yet more efficient method to determine β_c [3]. This is based on the simple observation that one must then have $R_e(\beta) = R_k(\beta)$ for like $\vec{e}$ and $\vec{k}$ at $\beta = \beta_c$. In fact, one can easily convince oneself that with self-duality,

$$R_e(\beta) = R_c - b(\beta - \beta_c)N_s^{1/\nu} + cN_s^{-\omega} + \cdots,$$

with the same coefficients b and c as in (3.2).

N_t	$\beta_c(R_k = R_e)$	$\beta_c(R_k = R_c)$	Lit.
2	8.15309(11)	8.15297(57)	8.1489(31)†
4	14.7262(9)	14.7194(45)	14.717(17)†
6	21.357(25)	-	21.34(4) ‡
8	27.84(12)	-	-

Table 1: SU(3) critical couplings from self-duality (weighted means), intersection with the universal value (extrapolated), and literature values from †[7], ‡[18].

Therefore, the leading finite-size corrections to β_c when defined by $R_e = R_k$ cancel. At criticality, $R_e(\beta_c) = R_k(\beta_c) = R_c + cN_s^{-\omega} + \cdots$, so the leading corrections only move the intersection point vertically without shifting the so defined critical coupling. The gain is illustrated for both SU(2) and SU(3) in Fig. 3. When intersecting $R_e = R_k$, we form weighted means from sufficiently large aspect ratios N_s/N_t where we assume that the estimates have converged within errors. These are compared to the extrapolated values from intersecting $R_k(\beta)$ with the universal value R_c in Tab. 1.
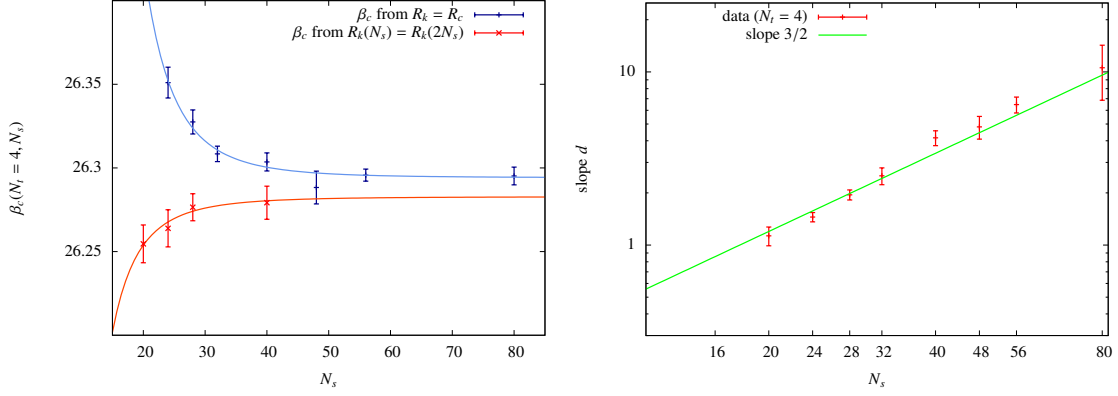
Figure 3: SU(4) critical couplings (left), and slopes $d(N_s)$ compared to a power law with $v = 2/3$ (right).

Critical temperature and correlation length exponent v: In 2+1 dimensions the critical couplings grow linearly with N_t to leading order at large N_t. The slope is given by the critical temperature in units of the dimensionful continuum coupling, $\beta_c(N_t)/(2N_c) = (T_c/g_3^2)N_t + \cdots$.

From our values for $N_t = 4$, 6 and 8 we then obtain $T_c/g_3^2 = 0.5475(3)$ corresponding to $T_c/\sqrt{\sigma} = 0.9938(9)$ with a zero temperature string tension $\sqrt{\sigma}/g_3^2 = 0.5509(4)$ from a weighted average of the four values in [19]. This is consistent with $T_c/\sqrt{\sigma} = 0.9994(40)$ from [7].

Moreover, because the spatial center vortex free energies F_k for sufficiently large L depend only on $L^{1/v}t$, and $t \propto (\beta - \beta_c)$, when expanding $F_k(\beta) = -\ln R_c + d(N_s)(\beta - \beta_c) + \cdots$, we can expect the slope at β_c to behave as $d(N_s) \sim N_s^{1/v}$. The result from fitting our slopes for SU(3) with $N_t = 4$ then gives $v = 0.82(4)$ as compared to $v = 5/6 \approx 0.833$ for the $2d$ 3-state Potts model.

Results for SU(4):   The present conclusion from a sequence of studies of the $(2+1)d$ SU(4) gauge theory [5, 7, 6] is that the deconfinement transition is weakly 1st order. Especially the detailed analysis in [6] was consistent with first order volume scaling laws. Here we assess to what extent Potts model scaling describes the transition, at least approximately, and whether we find indications of where our methods start to fail as we go to larger and larger volumes.

First, we extract critical couplings from the pairwise intersections of the F_k's for pairs of lattices with N_s ratios of 2:1. This method is independent of Potts scaling and yields $\beta_c = 26.283(9)$ for $N_t = 4$. Then we compare the so extrapolated value to that obtained from intersecting the F_k's with the 4-state Potts universal value from [10]. The latter has a smaller error because we have more points to fit; it gives the consistent value $\beta_c = 26.294(2)$. Pseudo-critical couplings and fits for each method are shown in Fig. 3. Both extrapolated values are consistent with

Figure 4: Check of Potts model scaling ($N_t = 4$)

$\beta_c = 26.228(75)$ from [7] but deviate with some significance from $\beta_c = 26.251(16)$ given in [6], where first order scaling was assumed in the infinite volume extrapolation of the critical coupling.
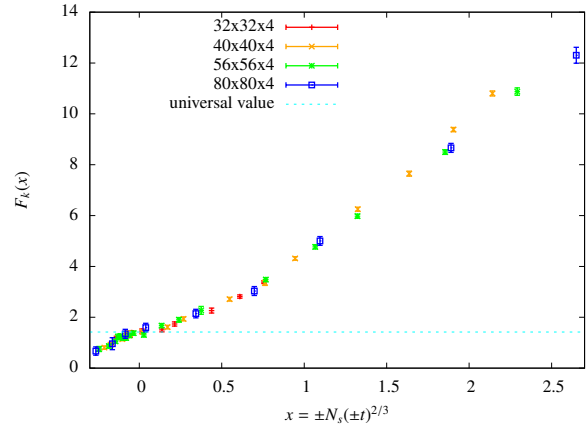
If we furthermore extract a critical exponent ν from the slopes $d(N_s)$ of $F_k(\beta)$ at β_c as before, see Fig. 3, we obtain $\nu = 0.60(2)$ from our data for $N_t = 4$ with $\beta_c = 26.283$. Some systematic uncertainty arises from what precise value is used here, however. Repeating the analysis for different values in the one-σ interval around $\beta_c = 26.283$, we find $\nu = 0.59(5)$. Of all the Z_4-symmetric Ashkin-Teller models, which have continuous $\nu \in [2/3, 1]$, this seems to be at best consistent with the lower bound $\nu = 2/3$ for the $q = 4$ Potts model, in agreement with the earlier conclusion in [5]. It is a general trend of our method, observed also for SU(2) and SU(3), that it underestimates the critical exponent due to subleading finite-size effects, however. Finally, our present $N_t = 4$ data with spatial lattice sizes up to $N_s = 80$ shows reasonably good Potts scaling as seen in Fig. 4 where we plot the center-vortex free energy $F_k(x)$ over the scaling variable $x = \pm T_c L(\pm t)^\nu$ with $\nu = 2/3$.

4. Conclusions

We have studied the deconfinement transition in the pure SU(3) gauge theory in 2+1 dimensions on the lattice. Using 't Hooft's twisted boundary conditions we have measured center-vortex free energies and demonstrated that the self-duality of the associated Potts model is directly reflected in SU(3): the free energies of the confining electric fluxes are mirror images around T_c of those of spatial center vortices. We demonstrated how this can be exploited to remove the leading finite-size corrections in the determination of critical couplings from numerical simulations. We do not yet have the data necessary to compute electric fluxes and to test self-duality analogously in SU(4), but our available data does not show any significant violations of the $q = 4$ Potts scaling.

References

 [1] Ph. de Forcrand and L. von Smekal, Phys. Rev. D **66** (2002) 011504; Nucl. Phys. (PS) **106** (2002) 619.

 [2] S. Edwards and L. von Smekal, Phys. Lett. B **681** (2009) 484.

 [3] L. von Smekal, S. R. Edwards and N. Strodthoff PoS **LAT2010**, 292 (2010);
      N. Strodthoff, S. R. Edwards and L. von Smekal, in preparation.

 [4] C. Wozar *et al.*, Phys. Rev. D **74** (2006) 114501; A. Wipf, private communication.

 [5] Ph. de Forcrand and O. Jahn, Nucl. Phys. Proc. Suppl. **129** (2004) 709.

 [6] K. Holland, M. Pepe and U. J. Wiese, JHEP **0802** (2008) 041.

 [7] J. Liddle and M. Teper, arXiv:0803.2128.

 [8] G. 't Hooft, Nucl. Phys. B **153** (1979) 141.

 [9] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B **210** (1982) 423.

[10] H. Park and M. den Nijs, Phys. Rev. B **38** (1988) 565.

[11] R. Savit, Rev. Mod. Phys. **52** (1980) 453.

[12] C. Gruber, A. Hintermann and D. Merlini, *Group Analysis of Classical Lattice Systems*, Springer, Berlin - Heidelberg 1977; M. Caselle and M. Hasenbusch, private communications.

[13] M. Caselle, M. Hasenbusch, P. Provero and K. Zarembo, Nucl. Phys. B **623** (2002) 474.

[14] Ph. de Forcrand, M. D'Elia and M. Pepe, Phys. Rev. Lett. **86** (2001) 1438.

[15] N. Cabibbo and E. Marinari, Phys. Lett. B **119** (1982) 387.

[16] K. Binder, Z. Phys. B **43** (1981) 119; K. Binder and E. Luijten, Phys. Rept. **344** (2001) 179.

[17] M. Hasenbusch, Physica A **197** (1993) 423.

[18] J. Engels, et al., Nucl. Phys. Proc. Suppl. **53** (1997) 420.

[19] B. Bringoltz and M. Teper, Phys. Lett. B **645** (2007) 383.