

Jost-Lehmann-Dyson Representation and Froissart Bound in Noncommutative Quantum Field Theory

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In the framework of quantum field theory (QFT) on noncommutative (NC) space-time with the symmetry group $O(1,1) \times SO(2)$, we prove that the Jost-Lehmann-Dyson representation, based on the causality condition taken in connection with this symmetry, leads to the mere impossibility of drawing any conclusion on the analyticity of the $2 \rightarrow 2$ -scattering amplitude in $\cos \Theta$, Θ being the scattering angle. Discussions on the possible ways of obtaining high-energy bounds analogous to the Froissart bound on the total cross-section are also presented.

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*This work is a modest contribution dedicated to the memory of Héctor Rubinstein, whom I had the privilege to know and to meet often throughout the last decade, as we were living in neighbouring countries. My additional respect and attachment to Héctor Rubinstein arose from his being also the teacher of my own teacher.

1. Introduction

The development of QFT on NC space-time, especially after the seminal work of Seiberg and Witten [1], which showed that the NC QFT arises as low-energy effective theory from string theory in Kalb-Ramond background field, has triggered lately the interest also towards the formulation of an axiomatic approach to the subject. The power of the axiomatic approach consists in that the results are rigorously derived, with no reference to the specific form of interaction or to perturbation theory. Consequently, in the framework of noncommutative spaces, the analytical properties of scattering amplitude in energy E and forward dispersion relations have been considered [2, 3], Wightman functions have been introduced and the CPT theorem has been proven [4, 5], and as well attempts towards a proof of the spin-statistics theorem have been made [5]¹.

In the axiomatic approach to commutative QFT, one of the fundamental results consisted of the rigorous proof of the Froissart bound on the high-energy behaviour of the scattering amplitude, based on its analyticity properties [9, 10]. In this paper we aim at obtaining the analog of this bound when the space-time is noncommutative. Such an undertaking, besides being topical in itself, would also prove fruitful in the conceptual understanding of subtle issues, such as causality, in nonlocal theories to which the NC QFT's belong.

In the following we shall consider NC QFT on a space-time with the commutation relation

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \quad (1.1)$$

where $\theta_{\mu\nu}$ is an antisymmetric constant matrix (for a review, see, e.g., [11, 12]). Such NC theories violate Lorentz invariance, while translational invariance still holds. We can always choose the system of coordinates, such that $\theta_{13} = \theta_{23} = 0$ and $\theta_{12} = -\theta_{21} \equiv \theta$. Then, for the particular case of space-space noncommutativity, i.e. $\theta_{0i} = 0$, the theory is invariant under the subgroup $O(1, 1) \times SO(2)$ of the Lorentz group. The requirement that time be commutative ($\theta_{0i} = 0$) discards the well-known problems with the unitarity [13] of the NC theories and with causality [14, 15] (see also [6]). As well, the $\theta_{0i} = 0$ case allows a proper definition of the S -matrix [3].

In the conventional (commutative) QFT, the Froissart bound was first obtained [9] using the conjectured Mandelstam representation (double dispersion relation) [16], which assumes analyticity in the entire E and $\cos \Theta$ complex planes. The Froissart bound,

$$\sigma_{tot}(E) \leq c \ln^2 \frac{E}{E_0}, \quad (1.2)$$

expresses the upper limit of the total cross-section σ_{tot} as a function of the CMS energy E , when $E \rightarrow \infty$. However, such an analyticity or equivalently the double dispersion relation has not been proven, while smaller domains of analyticity in $\cos \Theta$ were already known [17].

One of the main ingredients in rigorously obtaining the Froissart bound is the Jost-Lehmann-Dyson representation [18, 19] of the Fourier transform of the matrix element of the commutator of currents, which is based on the causality as well as the spectral conditions (for an overall review, see [20]). Based on this integral representation, one obtains the domain of analyticity of the scattering amplitude in $\cos \Theta$. This domain proves to be an ellipse – the so-called Lehmann's ellipse [17].

¹In the context of the Lagrangean approach to NC QFT, the CPT and spin-statistics theorems have been proven in general in [6]; for CPT invariance in NC QED, see [7, 8].

However, this domain of analyticity in $\cos\Theta$ can be enlarged to the so-called Martin's ellipse by using the dispersion relations satisfied by the scattering amplitude and the unitarity constraint on the partial-wave amplitudes. Using this larger domain of analyticity, the Froissart bound (1.2) was rigorously proven in QFT [10] (for a review, see [21]).

Further on, the analog of the Froissart bound was rigorously obtained for the $2 \rightarrow 2$ -particle scattering in a space-time of arbitrary dimension D [22, 23].

In NC QFT with $\theta_{0i} = 0$ we shall follow the same path for the derivation of the high-energy bound on the scattering amplitude, starting from the Jost-Lehmann-Dyson representation and adapting the derivation to the new symmetry $O(1, 1) \times SO(2)$ and to the nonlocality of the NC theory [24]². In Section 2 we derive the Jost-Lehmann-Dyson representation satisfying the light-wedge (instead of light-cone) causality condition, inspired by the above symmetry. We show that no analyticity of the scattering amplitude in $\cos\Theta$ can be obtained in such a case. Since the causality condition is the key ingredient for the analyticity of the scattering amplitude, in Section 3 we discuss possible causality postulates in the noncommutative case, in relation both with the maximal symmetry of the theory (twisted Poincaré [26]) and with the scale of nonlocality as obtained so far in perturbative calculations. It turns out that by postulating a *finite range of nonlocality*, compatible with the twisted Poincaré symmetry, and by using the global nature of local commutativity, we can obtain from the Jost-Lehmann-Dyson representation a domain of analyticity in $\cos\Theta$, which coincides with the Lehmann ellipse. Further, the extension of this analyticity domain to Martin's ellipse is possible in the case of the incoming particles' momenta orthogonal to the NC plane (x_1, x_2) , which eventually enables us to derive the analog of the Froissart bound (1.2) for the total cross-section. The general configuration of incoming particles' momenta is also discussed, together with the problems which arise in such a case. However, the perturbative calculations performed so far seem to indicate an *infinite range of nonlocality*, in which case the initial causality condition involving the light-wedge should be postulated, leading to the lack of analyticity of the scattering amplitude. The situation is discussed in connection with the perturbative problem of UV/IR mixing in NC QFT. Section 6 is devoted to conclusion and discussions.

2. Jost-Lehmann-Dyson representation

The Jost-Lehmann-Dyson representation [18, 19] is the integral representation for the Fourier transform of the matrix element of the commutator of currents:

$$f(q) = \int d^4x e^{iqx} f(x), \quad (2.1)$$

where

$$f(x) = \langle p' | [j_1(\frac{x}{2}), j_2(-\frac{x}{2})] | p \rangle, \quad (2.2)$$

satisfying the causality and spectral conditions. The process considered is the $2 \rightarrow 2$ scalar particles scattering, $k + p \rightarrow k' + p'$, and j_1 and j_2 are the scalar currents corresponding to the incoming and outgoing particles with momenta k and k' (see also [20, 27]).

²A preliminary work along this line with stronger claims, based on a conjecture, has been previously reported in [25].

For NC QFT with $O(1, 1) \times SO(2)$ symmetry, in [28] a new causality condition was proposed, involving (instead of the light-cone) the light-wedge corresponding to the coordinates x_0 and x_3 , which form a two-dimensional space with the $O(1, 1)$ symmetry. Accordingly we shall require the vanishing of the commutator of two currents (in general, observables) at space-like separations in the sense of $O(1, 1)$ as:

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0, \text{ for } \tilde{x}^2 \equiv x_0^2 - x_3^2 < 0. \quad (2.3)$$

The spectral condition compatible with (2.3) would require now that the physical momenta be in the forward light-wedge:

$$\tilde{p}^2 \equiv p_0^2 - p_3^2 > 0 \text{ and } p_0 > 0. \quad (2.4)$$

The standard spectral condition

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 \geq 0, \quad p_0 > 0.$$

based on Poincaré symmetry or twisted Poincaré symmetry [26] implies the forward light-wedge condition (2.4) as well.

The spectral condition (2.4) will impose restrictions on $f(q)$. Using the translational invariance in (2.2), one can express the matrix element of the commutator of currents, $f(x)$, in the form:

$$\begin{aligned} f(x) &= \int dq e^{-iqx + i(p+p')\frac{x}{2}} G_1(q) - \int dq e^{iqx - i(p+p')\frac{x}{2}} G_2(q) \\ &= \int dq e^{-iqx} \left[G_1\left(q + \frac{1}{2}(p+p')\right) - G_2\left(-q + \frac{1}{2}(p+p')\right) \right], \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} G_1(q) &= \langle p' | j_1(0) | q \rangle \langle q | j_2(0) | p \rangle, \\ G_2(q) &= \langle p' | j_2(0) | q \rangle \langle q | j_1(0) | p \rangle. \end{aligned} \quad (2.6)$$

Comparing (2.5) with the inverse Fourier transformation³, $f(x) = \int dq e^{-iqx} f(q)$, it follows that

$$f(q) = f_1(q) - f_2(q) = G_1\left(q + \frac{1}{2}(p+p')\right) - G_2\left(-q + \frac{1}{2}(p+p')\right). \quad (2.7)$$

Given the way the functions G_1 and G_2 are defined in (2.6), one finds that $f(q) = 0$ in the region where the momenta $q + \frac{1}{2}(p+p')$ and $-q + \frac{1}{2}(p+p')$ are simultaneously nonphysical, i.e. when they are out of the future light-wedge (2.4).

In order to express the condition for $f(q) = 0$, we shall define the $O(1, 1)$ -invariant $\tilde{m}^2 = k_0^2 - k_3^2 = f(m^2, k_1^2 + k_2^2)$, where k is the momentum of an arbitrary state and m is its mass. However, we have to point out that \tilde{m} is only a kinematical variable, invariant with respect to $O(1, 1)$ (but not the mass).

For the *physical* states with momentum $q + \frac{1}{2}(p+p')$, we take \tilde{m}_1 to be the minimal value of the $O(1, 1)$ -invariant quantity above. Then, in the Breit frame, where $\frac{1}{2}(p+p') = (p_0, 0, 0, 0)$,

³Throughout the paper we omit all the inessential factors of $(2\pi)^n$, which are irrelevant for the analyticity considerations.

one finds that $f_1(q) \neq 0$ for all the q values, satisfying the spectral condition $q_0 + p_0 \geq 0$ and $(q_0 - p_0)^2 - q_3^2 \geq 0$. In other words, $f_1(q) = 0$ for $q_0 < -p_0 + \sqrt{q_3^2 + \tilde{m}_1^2}$. Similarly one finds that $f_2(q) = 0$ for $p_0 - \sqrt{q_3^2 + \tilde{m}_2^2} < q_0$ (where \tilde{m}_2 has a meaning analogous to that of \tilde{m}_1 , but for the states with the momentum $-q + \frac{1}{2}(p + p')$).

As a result, due to the spectral condition (2.4), $f(q) = 0$ in the region outside the hyperbola

$$p_0 - \sqrt{q_3^2 + \tilde{m}_2^2} < q_0 < -p_0 + \sqrt{q_3^2 + \tilde{m}_1^2}. \quad (2.8)$$

To derive the Jost-Lehmann-Dyson representation, further we consider the 6-dimensional space-time with the Minkowskian metric $(+, -, -, -, -, -)$. On this space, we define the vector $z = (x_0, x_1, x_2, x_3, y_1, y_2)$. For practical purposes we introduce also the notations for the 2-dimensional vector $\tilde{x} = (x_0, x_3)$ and the 4-dimensional vector $\tilde{z} = (z_0, z_3, z_4, z_5) \equiv (x_0, x_3, y_1, y_2)$. On the 6-dimensional space we define the function

$$F(z) = f(x)\delta(\tilde{x}^2 - y^2) = f(x)\delta(\tilde{z}^2), \quad (2.9)$$

depending on all six coordinates.

When the causality condition (2.3) is fulfilled, i.e. for the physical region, $f(x)$ and $F(z)$ determine each other, since

$$\int dy_1 dy_2 F(z) = f(x)\theta(\tilde{x}^2) = \begin{cases} f(x) & \text{for } \tilde{x}^2 > 0, \\ 0 & \text{for } \tilde{x}^2 < 0. \end{cases} \quad (2.10)$$

The Fourier transform of $F(z)$,

$$F(r) = \int d^6 z e^{izr} F(z), \quad (2.11)$$

can be expressed, using (2.9) and (2.10), as

$$F(r) = \int d^4 q D_1(r - \hat{q}) f(q). \quad (2.12)$$

Denoting the remaining 4-dimensional vector $\tilde{r} = (r_0, r_3, r_4, r_5)$, we have

$$D_1(r) = \int d^6 z e^{izr} \delta(\tilde{z}^2) = \frac{\delta(r_1)\delta(r_2)}{\tilde{r}^2} = \delta(r_1)\delta(r_2)D_1(\tilde{r}), \quad (2.13)$$

with $D_1(\tilde{r}) = \frac{1}{\tilde{r}^2}$.

We define now the "subvector" of a 6-dimensional vector as $\hat{q} = (q_0, q_1, q_2, q_3, 0, 0)$ and we find the relation between $F(\hat{q})$ and $f(q)$ in view of the causality condition (2.3):

$$F(\hat{q}) = \int d^4 x f(x)\theta(\tilde{x}^2)e^{iqx} = f(q). \quad (2.14)$$

$D_1(\tilde{r})$ satisfies the 4-dimensional wave-equation:

$$\square_4 D_1(\tilde{r}) = 0, \quad (2.15)$$

where the d'Alembertian is defined with respect to the coordinates r_0, r_3, r_4, r_5 . Then, due to (2.12), it follows that $F(r)$ satisfies the same equation,

$$\square_4 F(r) = 0. \quad (2.16)$$

It is crucial to note that $F(r)$ depends on all six variables r_0, \dots, r_5 :

$$F(r) = \int d^4 q f(q) D_1(\tilde{r} - \tilde{q}) \delta(r_1 - q_1) \delta(r_2 - q_2),$$

where $\tilde{q} = (q_0, q_3, 0, 0)$.

The solution of (2.16) can be written in the form [31]:

$$F(r') = \int d^3 \Sigma_\alpha \int \int dr_1 dr_2 \left[F(r) \frac{\partial D(\tilde{r} - \tilde{r}')}{\partial \tilde{r}_\alpha} - D(\tilde{r} - \tilde{r}') \frac{\partial F(r)}{\partial \tilde{r}_\alpha} \right] \delta(r_1) \delta(r_2),$$

where $D(\tilde{r})$ satisfies the homogeneous differential equation $\square_4 D(\tilde{r}) = 0$, with the initial conditions

$$D(\tilde{r})|_{r_0=0} = 0 \quad \text{and} \quad \frac{\partial D}{\partial r_0}(\tilde{r})|_{r_0=0} = \prod_{i=1}^3 \delta(r_i).$$

The first condition implies that $D(\tilde{r})$ is an odd function, with the result that:

$$D(\tilde{r}) = \int d^4 z e^{-i\tilde{z}\tilde{r}} \varepsilon(z_0) \delta(\tilde{z}^2) = \varepsilon(r_0) \delta(\tilde{r}^2). \quad (2.17)$$

We note here that the surface Σ is 3-dimensional and not 5-dimensional as it is in the commutative case with light-cone causality condition. Now we can express $f(q)$ using (2.14) as:

$$\begin{aligned} f(q) &= F(\hat{q}) = \int dr_1 dr_2 \delta(r_1 - q_1) \delta(r_2 - q_2) \\ &\times \int d^3 \Sigma_\alpha \left[F(r) \frac{\partial D(\tilde{r} - \tilde{q})}{\partial \tilde{r}_\alpha} - D(\tilde{r} - \tilde{q}) \frac{\partial F(r)}{\partial \tilde{r}_\alpha} \right]. \end{aligned} \quad (2.18)$$

Due to the arbitrariness of the surface Σ , one can reduce the integration over r_4 and r_5 , using the cylindrical symmetry, to the integral over $\kappa^2 = r_4^2 + r_5^2$. Subsequently we change the notation of variables r_i to u_i and use the explicit form of $D(\tilde{r})$ from (2.17) to obtain:

$$\begin{aligned} f(q) &= \int du_1 du_2 \delta(u_1 - q_1) \delta(u_2 - q_2) \int d^1 \Sigma_j d\kappa^2 \\ &\times \left\{ F(u, \kappa^2) \frac{\partial}{\partial \tilde{u}_j} [\varepsilon(u_0 - q_0) \delta((\tilde{u} - \tilde{q})^2 - \kappa^2)] \right. \\ &\left. - \varepsilon(u_0 - q_0) \delta((\tilde{u} - \tilde{q})^2 - \kappa^2) \frac{\partial F(u, \kappa^2)}{\partial \tilde{u}_j} \right\}. \end{aligned} \quad (2.19)$$

Using the standard mathematical procedure [31] for performing the integration in (2.19), we obtain the Jost-Lehmann-Dyson representation in NC QFT, satisfying the light-wedge causality condition (2.3):

$$\begin{aligned} f(q) &= \int d^4 u d\kappa^2 \varepsilon(q_0 - u_0) \delta[(q_0 - u_0)^2 - (q_3 - u_3)^2 - \kappa^2] \\ &\times \delta(q_1 - u_1) \delta(q_2 - u_2) \phi(u, \kappa^2), \end{aligned} \quad (2.20)$$

where $\phi(u, \kappa^2) = -\frac{\partial F(u, \kappa^2)}{\partial \tilde{u}_0}$.

Equivalently, denoting $\tilde{u} = (u_0, u_3)$, (2.20) can be written as:

$$f(q) = \int d^2\tilde{u} d\kappa^2 \varepsilon(q_0 - u_0) \delta[(\tilde{q} - \tilde{u})^2 - \kappa^2] \phi(\tilde{u}, q_1, q_2, \kappa^2). \quad (2.21)$$

The function $\phi(\tilde{u}, q_1, q_2, \kappa^2)$ is an arbitrary function, except that the requirement of spectral condition determines a domain in which $\phi(\tilde{u}, q_1, q_2, \kappa^2) = 0$. This domain is outside the region where the δ function in (2.21) vanishes, i.e.

$$(\tilde{q} - \tilde{u})^2 - \kappa^2 = 0, \quad (2.22)$$

but with \tilde{q} in the region given by (2.8), where $f(q) = 0$. Putting together (2.22) and (2.8), we obtain the domain out of which $\phi(\tilde{u}, q_1, q_2, \kappa^2) = 0$:

$$\begin{aligned} a) & \frac{1}{2}(\tilde{p} + \tilde{p}') \pm \tilde{u} \text{ are in the forward light-wedge (cf. (2.4));} \\ b) & \kappa \geq \max \left\{ 0, \tilde{m}_1 - \sqrt{\left(\frac{\tilde{p} + \tilde{p}'}{2} + \tilde{u}\right)^2}, \tilde{m}_2 - \sqrt{\left(\frac{\tilde{p} + \tilde{p}'}{2} - \tilde{u}\right)^2} \right\}. \end{aligned} \quad (2.23)$$

For the purpose of expressing the scattering amplitude, we actually need the Fourier transform $f_R(q)$ of the retarded commutator,

$$f_R(x) = \theta(x_0) f(x) = \langle p' | \theta(x_0) [j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right)] | p \rangle. \quad (2.24)$$

Using (2.24) and the Fourier transformation $f(x) = \int dq' e^{-iq'x} f(q')$, we can express $f_R(q)$ as follows:

$$\begin{aligned} f_R(q) &= \int dx e^{iqx} f_R(x) = \int dx e^{iqx} \theta(x_0) f(x) \\ &= \int dq' f(q') \int dx e^{i(q-q')x} \theta(x_0). \end{aligned} \quad (2.25)$$

Taking into account that

$$\int dx_0 e^{i(q-q')x} \theta(x_0) = -i \frac{e^{i(\tilde{q}-\tilde{q}')\tilde{x}}}{q_0 - q'_0},$$

eq. (2.25) becomes:

$$f_R(q) = i \int dq'_0 \frac{f(q'_0, \vec{q})}{q'_0 - q_0}.$$

Now in the above formula we introduce the Jost-Lehmann-Dyson representation (2.21), with the result:

$$f_R(q) = i \int \frac{dq'_0}{q'_0 - q_0} \int d^2\tilde{u} d\kappa^2 \varepsilon(q'_0 - u_0) \delta[(q'_0 - u_0)^2 - (q_3 - u_3) - \kappa^2] \phi(\tilde{u}, q_1, q_2, \kappa^2). \quad (2.26)$$

In (2.26) one can integrate over q'_0 , using the known formula of integration with a δ -function, $\int G(x) \delta(g(x)) dx = \sum_i \frac{G(x_{0i})}{\left. \frac{\partial g}{\partial x} \right|_{x=x_{0i}}}$, where x_{0i} are the simple roots of the function $g(x)$. We identify

in (2.26) $G(q'_0) = \frac{\varepsilon(q'_0 - u_0)}{q'_0 - q_0}$ and $g(q'_0) = (q'_0 - u_0)^2 - (q_3 - u_3) - \kappa^2$ (with the roots $q'_0 = u_0 \pm [(q_3 - u_3)^2 + \kappa^2]^{1/2}$).

With these considerations, from (2.26) we obtain the NC version of the Jost-Lehmann-Dyson representation for the retarded commutator:

$$f_R(q) = \int d^2\tilde{u} d\kappa^2 \frac{\phi(\tilde{u}, q_1, q_2, \kappa^2)}{(q_0 - u_0)^2 - (q_3 - u_3)^2 - \kappa^2}. \quad (2.27)$$

Compared to the usual Jost-Lehmann-Dyson representation,

$$f_R^{comm}(q) = \int d^4u d\kappa^2 \frac{\phi(u, \kappa^2)}{(q_0 - u_0)^2 - (\vec{q} - \vec{u})^2 - \kappa^2}, \quad (2.28)$$

the expression (2.27) is essentially different in the sense that the arbitrary function ϕ now depends on q_1 and q_2 . This feature will have further crucial implications in the discussion of analyticity of the scattering amplitude in $\cos \Theta$.

2.1 (Non-)Analyticity of the scattering amplitude in $\cos \Theta$

In the center-of-mass system (CMS) and in a set in which the incoming particles are along the vector $\vec{\beta} = (0, 0, \theta)^4$, the scattering amplitude in NC QFT depends still on only two variables, the CM energy E and the cosine of the scattering angle, $\cos \Theta$ (for a discussion about the number of variables in the scattering amplitude for a general type of noncommutativity, see [29]).

In terms of the Jost-Lehmann-Dyson representation, the scattering amplitude is written as (cf. [20] for commutative case):

$$M(E, \cos \Theta) = i \int d^2\tilde{u} d\kappa^2 \frac{\phi(\tilde{u}, \kappa^2, k + p, (k' - p')_{1,2})}{[\frac{1}{2}(\vec{k}' - \vec{p}') + \vec{u}]^2 - \kappa^2}, \quad (2.29)$$

where $\phi(\tilde{u}, \kappa^2, \dots)$ is a function of its $O(1, 1)$ - and $SO(2)$ -invariant variables: $u_0^2 - u_3^2$, $(k_0 + p_0)^2 - (k_3 - p_3)^2$, $(k_1 + p_1)^2 + (k_2 + p_2)^2$, $(k'_1 - p'_1)^2 + (k'_2 - p'_2)^2, \dots$ The function ϕ is zero in a certain domain, determined by the causal and spectral conditions, but otherwise arbitrary.

For the discussion of analyticity of $M(E, \cos \Theta)$ in $\cos \Theta$, it is of crucial importance that all dependence on $\cos \Theta$ be contained in the denominator of (2.29). But, since the *arbitrary* function ϕ depends now on $(k' - p')_{1,2}$, it also depends on $\cos \Theta$. This makes impossible the mere consideration of any analyticity property of the scattering amplitude in $\cos \Theta$.

Since the Jost-Lehmann-Dyson representation reflects the effect of the causal and spectral axioms, we notice that the hypotheses (2.3) and (2.4) used for the present derivation allow for a much larger physical region, by not at all taking into account the effect of the NC coordinates x_1 and x_2 . One might wonder now whether in the above derivation there is any condition which could be subject to challenge. In that case there might also appear the possibility that an analyticity domain can be obtained, leading to some high-energy upper bound on the scattering amplitude.

⁴The 'magnetic' vector $\vec{\beta}$ is defined as $\beta_i = \frac{1}{2} \varepsilon_{ijk} \theta_{jk}$. The terminology stems from the antisymmetric background field $B_{\mu\nu}$ (analogous to $F_{\mu\nu}$ in QED), which gives rise to noncommutativity in string theory, with $\theta_{\mu\nu}$ essentially proportional to $B_{\mu\nu}$ (see, e.g., [1]).

3. Causality in NC QFT

3.1 Causality and symmetry in NC QFT

In the following, we shall challenge the causality condition (2.3)

$$f(x) = 0, \quad \text{for } \tilde{x}^2 \equiv x_0^2 - x_3^2 < 0, \quad (3.1)$$

which takes into account *only* the variables connected with the $O(1,1)$ symmetry.

This causality condition is suitable in the case when the nonlocality in the NC variables x_1 and x_2 is *infinite*. The fact that in the causality condition (3.1) the coordinates x_1 and x_2 do not enter means that the propagation of a signal in this plane is instantaneous: *no matter how far apart in the noncommutative coordinates two events are*, the allowed region for correlation is given by only the condition $x_0^2 - x_3^2 > 0$, which involves the propagation of a signal only in the x_3 -direction, while the time for the propagation along x_1 - and x_2 -directions is totally ignored.

Recall that we are using an axiomatic approach, in whose commutative counterpart the assumption of locality was a postulate. In our noncommutative case, the postulate of locality has to be replaced by a postulate prescribing the scale of nonlocality. Postulating that the scale of nonlocality in x_1 and x_2 is $l \sim \sqrt{\theta}$, then the propagation of the interaction in the noncommutative coordinates is instantaneous *only within this distance l* . It follows then that two events are correlated, i.e. $f(x) \neq 0$, when $x_1^2 + x_2^2 \leq l^2$ (where $x_1^2 + x_2^2$ is the distance in the NC plane with $SO(2)$ symmetry), provided also that $x_0^2 - x_3^2 \geq 0$ (the events are time-like separated in the sense of $O(1,1)$). Adding the two conditions, we obtain that

$$f(x) \neq 0, \quad \text{for } x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) \geq 0. \quad (3.2)$$

The negation of condition (3.2) leads to the conclusion that the locality condition should indeed be given by:

$$f(x) = 0, \quad \text{for } \tilde{x}^2 - (x_1^2 + x_2^2 - l^2) \equiv x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) < 0,$$

or, equivalently,

$$f(x) = 0, \quad \text{for } x_0^2 - x_3^2 - (x_1^2 + x_2^2) < -l^2, \quad (3.3)$$

where l^2 is a constant proportional to the NC parameter θ . When $l^2 \rightarrow 0$, (3.3) becomes the usual locality condition.

When $x_1^2 + x_2^2 > l^2$, for the propagation of a signal only the difference $x_1^2 + x_2^2 - l^2$ is time-consuming and thus in the locality condition it is the quantity $x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2)$ which will occur. Therefore, we shall have again the locality condition of the form (3.3).

Since there is no noncommutativity in the momentum space, the spectral condition will read now as

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 \geq 0, \quad p_0 > 0. \quad (3.4)$$

At this point we recall that the maximal symmetry of a NC QFT with $\theta_{\mu\nu}$ a constant matrix is not the classical $O(1,1) \times SO(2)$ symmetry, but a quantum symmetry, namely the twisted Poincaré symmetry [26], whose representation content is identical to the usual Poincaré symmetry. Moreover, the usual space-time interval $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$ is invariant under the twisted Poincaré

algebra, as well as the scale of nonlocality l , since the latter is expressed in terms of the twisted Poincaré-invariant θ . Consequently, (3.3) (3.4) are compatible with the twisted Poincaré algebra.

In fact, the consideration of nonlocal theories of the type (3.3), (3.4) was initiated by Wightman [30]. It was proven later [31–33] (see also [34]) that, indeed, in a quantum field theory which satisfies the translational invariance and the spectral axiom (3.4), the nonlocal commutativity

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0, \quad \text{for } x_0^2 - x_1^2 - x_2^2 - x_3^2 < -l^2$$

implies the local commutativity

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0, \quad \text{for } x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0. \quad (3.5)$$

This powerful theorem (stating the "global nature of local commutativity"), which does not require standard Lorentz invariance, but only translational invariance, can be applied in the noncommutative case with postulated finite nonlocality, with the conclusion that the causality properties of a QFT with space-space noncommutativity are physically identical to those of the corresponding commutative QFT.

It is then obvious that the Jost-Lehmann-Dyson representation (2.28) obtained in the commutative case holds also on the NC space for any orientation of the vector $\vec{\beta}$. Consequently, the NC two-particle→two-particle scattering amplitude will have the same form as in the commutative case:

$$M(E, \cos \Theta) = i \int d^4 u d\kappa^2 \frac{\phi(u, \kappa^2, k+p)}{[\frac{1}{2}(k' - p') + u]^2 - \kappa^2}. \quad (3.6)$$

This leads to the analyticity of the NC scattering amplitude in $\cos \Theta$ in the analog of the Lehmann ellipse, which behaves at high energies E the same way as in the commutative case, i.e. with the semi-major axis as

$$y_L = (\cos \Theta)_{max} = 1 + \frac{const}{E^4}. \quad (3.7)$$

3.1.1 Enlargement of the domain of analyticity in $\cos \Theta$ and use of unitarity. Martin's ellipse

Two more ingredients are needed in order to enlarge the domain of analyticity in $\cos \Theta$ to the Martin's ellipse and to obtain the Froissart bound: the dispersion relations and the unitarity constraint on the partial-wave amplitudes [21].

When using the causality condition (2.3), the forward dispersion relation cannot be obtained in NC theory with general direction of the $\vec{\beta}$ -vector [2]. However, the conclusion to which we arrived by imposing the nonlocal commutativity condition (3.3) and reducing it to the local commutativity (3.5) leads straightforwardly to the usual forward dispersion relation also in the NC case with a general $\vec{\beta}$ direction.

As for the unitarity constraint on the partial wave amplitudes, the problem has been investigated in [29], for a general case of noncommutativity $\theta_{\mu\nu}$, $\theta_{0i} \neq 0$. For space-space noncommutativity ($\theta_{0i} = 0$), the scattering amplitude depends, besides the center-of-mass energy, E , on three angular variables. In a system where we take the incoming momentum \vec{p} in the z -direction, these variables are the polar angles of the outgoing particle momentum, Θ and ϕ , and the angle α between the vector $\vec{\beta}$ and the incoming momentum. The partial-wave expansion in this case reads:

$$A(E, \Theta, \phi, \alpha) = \sum_{l, l', m} (2l' + 1) a_{ll'm}(E) Y_{lm}(\Theta, \phi) P_{l'}(\cos \alpha), \quad (3.8)$$

where Y_{lm} are the spherical harmonics and P_l are the Legendre polynomials.

Imposing the unitarity condition directly on (3.8) or using the general formulas given in [29], it can be shown that a simple unitarity constraint which involves single partial-wave amplitudes one at a time can be obtained only in a setting where the incoming momentum is orthogonal to the NC plane (equivalently it is parallel to the vector $\vec{\beta}$). In this case the amplitude depends only on one angle, Θ , and the unitarity constraint is reduced to the well-known one of the commutative case, i.e.

$$\text{Im } a_l(E) \geq |a_l(E)|^2. \quad (3.9)$$

For this particular setting, $\vec{p} \parallel \vec{\beta}$, it is then straightforward, following the prescription developed for commutative QFT, to enlarge the analyticity domain of scattering amplitude to Martin's ellipse with the semi-major axis at high energies as

$$y_M = 1 + \frac{\text{const}}{E^2} \quad (3.10)$$

and subsequently obtain the NC analog of the Froissart bound on the total cross-section, in the CMS and for $\vec{p} \parallel \vec{\beta}$:

$$\sigma_{tot}(E) \leq c \ln^2 \frac{E}{E_0}. \quad (3.11)$$

Thus, the unitarity constraint on the partial-wave amplitudes distinguishes a particular setting ($\vec{p} \parallel \vec{\beta}$) in which the Lehmann's ellipse can be enlarged to the Martin's ellipse and the Froissart bound can be obtained, with the assumption of finite nonlocality. Nevertheless, this does not necessarily exclude the possibility of obtaining a rigorous high-energy bound on the cross-section for $\vec{p} \not\parallel \vec{\beta}$, and the issue deserves further investigation.

3.2 Causality and nonlocality in NC QFT

It was shown in the previous subsection that the violation of Lorentz invariance in itself does not forbid the existence of an analyticity domain of the scattering amplitude in $\cos \Theta$ and the derivation of a high-energy bound, compatible with the twisted Poincaré symmetry.

However, for the derivation of the analog of the Froissart bound the key ingredient was the assumption of finite nonlocality. This issue deserves a more thorough investigation, in the light of the Lagrangean models studied so far. We have to point out from the very beginning that the Lagrangean models have been studied up to at most two-loops and that no definite statement about the renormalizability of NC quantum field theories in general has been made so far.

It is well known that in NC QFT treated with the Weyl-Moyal correspondence (i.e. with the usual product of fields replaced by Moyal \star -product in the Lagrangean) the short distance (UV) effects are related to the long-distance (topological) features of the space-time. This fact was first noticed in [35], where it was shown that noncommutativity leads to UV-regular theories when at most one dimension of the space-time is noncompact. For the NC flat space-time UV-regularity is not achieved, but instead the exotic phenomenon of UV/IR mixing appears [36]. The physical meaning of this mixing is that at quantum level, even very low-energy processes receive contributions from high energy virtual particles. The nonlocality is energy-dependent, and for virtual particles of arbitrarily high energy, the nonlocality is arbitrarily large.

Another investigation leading to the same conclusion was performed in the first paper dealing with the causality in NC QFT in the Lagrangean approach [14]. There it was shown, through the study of a scattering process, that space-space NC ϕ^4 in 2+1 dimensions is causal at macroscopical level. However the incident particles should be viewed as extended rigid rods, of the size θp , perpendicular to their momentum. In other words, the noncommutativity introduces an energy-dependent scale of spatial nonlocality θp .

Judging by the above-mentioned results obtained in specific NC models up to one-loop level, the previous analysis of analyticity and high-energy bounds in axiomatic NC QFT becomes inconclusive. It appears that the finite nonlocality condition (3.3) is solely a conjecture, but only based on this conjecture one can derive rigorously the analyticity properties and high-energy bounds on the scattering amplitude (see Section 3).

Nevertheless, in the case of a *compact* noncommutative space-time, the NC QFT is finite, i.e. there are no UV divergences [35], consequently no UV/IR mixing, and the range of nonlocality is finite. For such NC QFT the finite nonlocality is no more a conjecture and one may reconsider the rigorous axiomatic derivation of the analyticity and high-energy bounds along the lines of Section 3.1.

4. Conclusion and discussions

In this paper we have tackled the problem of high energy bounds on the two-particle→two-particle scattering amplitude in NC QFT. The key issue in the analysis proved to be the scale of nonlocality of the quantum field theory on NC space-time.

We have found that, assuming infinite nonlocality and using the causal and spectral conditions (2.3) and (2.4) proposed in [28] for NC theories with $O(1, 1) \times SO(2)$ symmetry, a new form of the Jost-Lehmann-Dyson representation (2.27) is obtained, which does not permit to draw any conclusion about the analyticity of the scattering amplitude (2.29) in $\cos\Theta$. Therefore the derivation of high-energy bounds on the scattering amplitude is impossible.

However, by postulating that the nonlocality in the noncommuting coordinates is finite, we were led to imposing a new causality condition (3.3), which accounts for the finiteness of the range of nonlocality and prevents the *instantaneous propagation of signals* in the *entire* noncommutative plane (x_1, x_2) . We proved that the new causality condition, compatible also with the twisted Poincaré symmetry, is formally identical to the one corresponding to the commutative case (3.5), using the Wightman-Vladimirov-Petrina theorem.

Thus, with the assumption of finite nonlocality, the scattering amplitude in NC QFT is proved to be analytical in $\cos\Theta$ in the Lehmann ellipse, just as in the commutative case; moreover, dispersion relations can be written on the same basis as in commutative QFT. Finally, based on the unitarity constraint on the partial-wave amplitudes in NC QFT, we can conclude that, for theories with space-space noncommutativity ($\theta_{0i} = 0$), the total cross-section is subject to an upper bound (3.11) identical to the Froissart bound in its high-energy behaviour, when the incoming particle momentum \vec{p} is orthogonal to the NC plane.

The perturbative studies performed so far indicate an infinite range of nonlocality as more plausible. In perturbative terms, this is equivalent to the standing problem of UV/IR mixing. How-

ever, for compact noncommutative spaces, where the range of nonlocality is finite and the NC QFT models do not exhibit UV divergences, we trust that an analog of the Froissart bound holds.

Recently, the validity of the Froissart bound in NC QFT has been studied based on the AdS/CFT correspondence [37]. The original idea appeared in [38], where the AdS/CFT correspondence was used to infer the Froissart bound in high-energy QCD scattering. According to [37], the Froissart bound holds as well in NC QFT. This might look as contradicting the results of the present paper. However, in [37] the Froissart bound was derived in a specific scalar field model, perturbatively and essentially by using an IR cutoff brane. It turns out that in the considered toy model the Froissart bound is saturated in both the commutative and noncommutative directions, however the size of the cross-section is smaller in the commutative directions than in the noncommutative ones, with a ratio which depends only on the noncommutative parameter and the IR cutoff. This strongly suggests that the IR cutoff actually acts as a restriction on the range of nonlocality to a finite region in the noncommutative plane.

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