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Solution of the Bethe-Salpeter equation in Minkowski space for a two fermion system

J. Carbonell[a] **and V.A. Karmanov***[b]
[a] *Laboratoire de Physique Subatomique et Cosmologie, CNRS/IN2P3,*
53 avenue des Martyrs, 38026 Grenoble, France
E-mail: carbonel@lpsc.in2p3.fr
[b] *Lebedev Physical Institute,*
Leninsky Prospekt 53, 119991 Moscow, Russia
E-mail: karmanov@sci.lebedev.ru

The method of solving the Bethe-Salpeter equation in Minkowski space, developed previously for spinless particles [1, 2], is extended to a system of two fermions. The method is based on the Nakanishi integral representation of the amplitude and on projecting the equation on the light-front plane. The singularities in the projected two-fermion kernel are regularized without modifying the original BS amplitudes. The numerical solutions for the $J = 0$ bound state with the scalar, pseudoscalar and massless vector exchange kernels are found. Binding energies are in close agreement with the Euclidean results. Corresponding amplitudes in Minkowski space are obtained.

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*Speaker.

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1. Introduction

Bethe-Salpeter (BS) equation for a relativistic bound system was initially formulated in the Minkowski space [3]. It determines the binding energy and the BS amplitude. However, in practice, finding the solution in Minkowski space is made difficult due its singular behaviour. The singularities are integrable, but the standard approaches for solving integral equation fail. To circumvent this problem, the BS equation is usually transformed, by means of the Wick rotation, into Euclidean momentum space.

Some attempts have been recently made to obtain the Minkowski BS amplitudes. The approach proposed in [4] is based on the integral representation of the amplitudes and solutions have been obtained for the ladder scalar case [4, 5] as well as, under some simplifying ansatz, for the fermionic one [6]. Another approach [7] relies on a separable approximation of the kernel which leads to analytic solutions.

In previous works [1, 2] we have proposed a new method to find the BS amplitude in Minkowski space and applied it to the system of two spinless particles. Like in the papers [4, 5, 6], it is based on Nakanishi integral representation [8] of the BS amplitude. The main difference between our approach and those followed in [4, 5, 6] is the fact that we use the light-front projection. This eliminates the singularities related to the BS Minkowski amplitudes. The method is valid for any kernel given by the irreducible Feynman graphs.

We present in this contribution a generalisation to the fermion systems of our preceding works [1, 2]. A more detailed version can be found in [9]. We will see that the direct application to the fermionic kernels of the method used in the spinless case, is however married with some numerical difficulties. Although they could be overcome by a proper treatment of the singularities, in this work we propose an alternative method allowing to solve the BS equation for two fermions in Minkowski space with the same degree of accuracy than for the scalar case. The numerical applications are here limited to the $J^\pi = 0^+$ state.

The main steps in deriving system of equations for the Nakanishi weight functions are explained in sect. 2, starting from the original BS equation. Numerical results for the scalar, pseudoscalar and massless vector exchange couplings are presented in sect. 3. Section 4 contains concluding remarks.

2. System of equations

We have considered the following fermion (Ψ, m) - meson (ϕ, μ) interaction Lagrangians:
(*i*) Scalar coupling $\mathscr{L}_{int}^{(s)} = g\bar{\Psi}\Psi\phi$.
(*ii*) Pseudoscalar coupling $\mathscr{L}_{int}^{(ps)} = ig\bar{\Psi}\gamma_5\Psi\phi$.
(*iii*) Massless vector exchange $\mathscr{L}_{int}^{(v)} = \bar{\Psi}\gamma^\mu\Psi\phi_\mu$. with $\Pi_{\mu\nu} = -ig_{\mu\nu}/q^2$ as vector propagator.

Each interaction vertex has been regularized with a vertex form factor $F(k-k')$ by the replacement $g \to gF(k-k')$ and we have chosen F in the form:

$$F(q) = \frac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\varepsilon}. \tag{2.1}$$

Let us first consider the case of the scalar coupling and the corresponding ladder kernel. The BS equation for the amplitude Φ reads:

$$\Phi(k,p) = \frac{i(m+\frac{1}{2}\hat{p}+\hat{k})}{(\frac{1}{2}p+k)^2-m^2+i\varepsilon}\left[\int \frac{\mathrm{d}^4k'}{(2\pi)^4}\,\Phi(k',p)\frac{(-ig^2)F^2(k-k')}{(k-k')^2-\mu^2+i\varepsilon}\right]\frac{i(m-\frac{1}{2}\hat{p}+\hat{k})}{(\frac{1}{2}p-k)^2-m^2+i\varepsilon}, \tag{2.2}$$

where $p=k_1+k_2$, $k=(k_1-k_2)/2$, $k'=(k'_1-k'_2)/2$.

In the case of $J^\pi=0^+$ state, the BS amplitude has the following general form:

$$\Phi(k,p) = S_1\phi_1 + S_2\phi_2 + S_3\phi_3 + S_4\phi_4 \tag{2.3}$$

where S_i are independent spin structures (4×4 matrices) and ϕ_i are scalar functions of k^2 and $p\cdot k$.

The choice of S_i is to some extent arbitrary. To benefit from useful orthogonality properties we have taken

$$S_1=\gamma_5, \quad S_2=\frac{1}{M}\hat{p}\gamma_5, \quad S_3=\frac{k\cdot p}{M^3}\hat{p}\gamma_5-\frac{1}{M}\hat{k}\gamma_5, \quad S_4=\frac{i}{M^2}\sigma_{\mu\nu}p_\mu k_\nu\gamma_5,$$

where $\sigma_{\mu\nu}=\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$. The antisymmetry of the amplitude (2.3) with respect to the permutation $1\leftrightarrow 2$ implies for the scalar functions: $\phi_{1,2,4}(k,p)=\phi_{1,2,4}(-k,p)$, $\phi_3(k,p)=-\phi_3(-k,p)$.

A decomposition similar to (2.3) was used in [6] to solve the BS equation for a quark-antiquark system but the solution was approximated keeping only the first term $S_1\phi_1$.

We substitute (2.3) in eq. (2.2), multiply it by S_i and take traces. As we will see (left panel in fig. 1 below), the kernel in the resulted equation, in contrast to the spinless case, is still singular. These singularities are integrable. They do not prevent from finding numerical solution, but they reduce its precision. This can be avoided by a proper regularization of equation, multiplying both sides of it by the factor

$$\eta(k,p) = \frac{(m^2-L^2)}{\left[(\frac{p}{2}+k)^2-L^2+i\varepsilon\right]}\frac{(m^2-L^2)}{\left[(\frac{p}{2}-k)^2-L^2+i\varepsilon\right]} \tag{2.4}$$

This factor has the form of a product of two scalar propagators with mass L. It plays the role of form factor suppressing the high off-mass shell values of the constituent four-momenta $k^2_{1,2}=(\frac{p}{2}\pm k)^2$ and tends to 1 when $L\to\infty$. In this way, we get the following system of equations for the invariant functions ϕ_i:

$$\begin{aligned}\eta(k,p)\;\phi_i(k,p) \;=\; & \frac{\eta(k,p)}{[(\frac{p}{2}+k)^2-m^2+i\varepsilon][(\frac{p}{2}-k)^2-m^2+i\varepsilon]} \\ & \times\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{ig^2F^2(k-k')}{(k-k')^2-\mu^2+i\varepsilon}\sum_{j=1}^{4}c_{ij}(k,k',p)\phi_j(k',p),\end{aligned} \tag{2.5}$$

Since $\eta(k,p)\neq 0$, the equation thus obtained is strictly equivalent to the original one. We will see however that, due to the presence of the η factor, the light front projection modifies the resulting kernels which become less singular functions. The coefficients c_{ij} are determined by traces and are given in [9].

Then we represent each of the BS components $\phi_i(k,p)$ by means of the Nakanishi integral

$$\phi_i(k,p) = \int_{-1}^{1}\mathrm{d}z'\int_0^\infty \mathrm{d}\gamma'\frac{g_i(\gamma',z')}{\left[k^2+p\cdot k\,z'+\frac{1}{4}M^2-m^2-\gamma'+i\varepsilon\right]^3}. \tag{2.6}$$

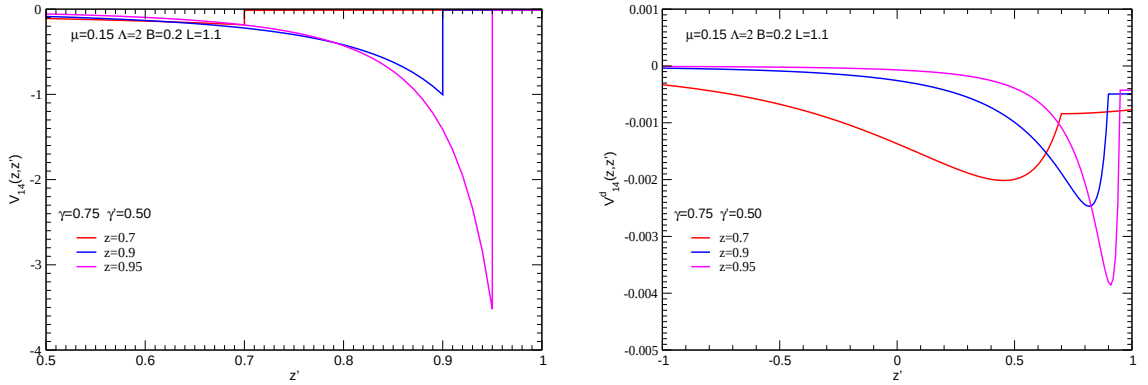

Figure 1: Left: The kernel matrix elements $V_{14}(\gamma,z;\gamma',z')$ for $z=0.7,\ 0.9,\ 0.95$ as a function of z' and fixed values of γ,γ'. The discontinuity is finite at a fixed value of z but diverges when $z\to 1$.
Right: Regularized kernel $V^d_{14}(\gamma,z;\gamma',z')$ v.s. z' for fixed values of $z=0.7,\ 0.9,\ 0.95$ and $L=1.1\,m$.

and apply the light-front projection to the set of coupled equations for the corresponding weight functions $g_i(\gamma,z)$. As mentioned in the Introduction, this projection, which is an essential ingredient of our previous works [1, 2, 10], consists in replacing $k\to k+\frac{\omega}{\omega\cdot p}\beta$ in eq. (2.5) and integrating over β in all the real domain.

The technical details of the light-front projection are similar to those explained in ref. [2] for the case of the spinless particles. We obtain in this way a set of coupled two-dimensional integral equations:

$$\int_0^\infty d\gamma' \int_{-1}^1 dz'\, V^g(\gamma,z;\gamma',z')\, g_i(\gamma',z') = \sum_j \int_0^\infty d\gamma' \int_{-1}^1 dz'\, V^d_{ij}(\gamma,z;\gamma',z') g_j(\gamma',z') \qquad (2.7)$$

The kernel V^g and also V^d_{ij} for all types of couplings and states are given in [9]. These kernels depend on the parameter L. Closer is L to m, smoother is the kernel and more stable are the numerical solutions. However the weight functions $g_i(\gamma,z)$ as well as the binding energies provided by (2.7) are independent of L.

The kernel V_g is finite and it vanishes for $z=\pm 1$. For a fixed values of γ,z and γ', V_g is a continuous function of z' with a discontinuous derivative at $z'=z$.

As already mentioned, without η-factor, most of the kernel matrix elements V^d_{ij} are singular. Namely, they are discontinuous at $z'=z$. In some cases – like *e.g.* V_{14} displayed in fig. 1 (left) – the value of the discontinuity, although being finite at fixed value of z, diverges when $z\to\pm 1$.

For $\eta\neq 1$, *i.e.*, for a finite value of L in (2.4), the z'-dependence of the regularized kernels is much more smooth and therefore better adapted for obtaining accurate numerical solutions. In fig. 1 (right) we plotted the regularized kernel V^d_{14} as a function of z' for the same arguments γ,z,γ' and parameters than in fig. 1 (left), where it was calculated without the $\eta(k,p)$ factor. As one can see, the kernel is now a continuous function of z'.

We would like to emphasize again that despite the fact that the non-regularized and regularized kernels are very different from each other (compare *e.g.* figs. 1, left and right) and that the regularized ones strongly depend on the value of L, they provide – up to numerical inaccuracies

– the same binding energies and weight functions $g_i(\gamma,z)$. We construct in this way a family of equivalent kernels.

Table 1: Left: Coupling constant g^2 as a function of binding energy B for the $J=0$ state with scalar (S), pseudoscalar (PS) and massless vector (positronium) exchange kernels. The vertex form factor parameter is $\Lambda=2$ and the parameter of the η factor $L=1.1$
Right: Coupling constant g^2_{BS} as a function of binding energy B for the positronium $J=0$ state in BS equation in the region of stability without vertex form factor ($\Lambda\to\infty$), *i.e.* $g<\pi$. They are compared to the non relativistic results g^2_{NR}.

	S		PS		positronium
μ	0.15	0.50	0.15	0.50	0.0
B	g^2	g^2	g^2	g^2	g^2
0.01	7.813	25.23	224.8	422.3	3.265
0.02	10.05	29.49	232.9	430.1	4.910
0.03	11.95	33.01	238.5	435.8	6.263
0.04	13.69	36.19	243.1	440.4	7.457
0.05	15.35	39.19	247.0	444.3	8.548
0.10	23.12	52.82	262.1	459.9	13.15
0.20	38.32	78.25	282.9	480.7	20.43
0.30	54.20	103.8	298.6	497.4	26.50
0.40	71.07	130.7	311.8	515.2	31.84
0.50	86.95	157.4	323.1	525.9	36.62

B	g^2_{NR}	g^2_{BS}
0.01	2.51	3.18
0.02	3.55	4.65
0.03	4.35	5.75
0.04	5.03	6.64
0.05	5.62	7.38
0.06	7.95	8.02
0.07	11.24	8.57
0.08	13.77	9.06
0.09	15.90	9.49

3. Numerical results

The solutions of eq. (2.7) have been obtained using the same techniques than in ref [2]. We have computed the binding energies, defined as $B=2m-M$, and BS amplitudes, for the $J=0^+$ two fermion system interacting with massive scalar (S) and pseudoscalar (PS) exchange kernels and for the fermion-antifermion system interacting with massless vector exchange in Feynman gauge. In the limit of an infinite vertex form factor parameter $\Lambda\to\infty$, the later case would correspond to positronium with an arbitrary value of the coupling constant. All the results presented in this section are given in the constituent mass units ($m=1$) and with $L=1.1$.

For the scalar and pseudoscalar cases, the binding energies obtained with the form factor parameter $\Lambda=2$ are given in the left table 1. We present the results for $\mu=0.15$ and $\mu=0.50$ boson masses. They have been compared to those obtained in a previous calculation in Euclidean space [11] using a slightly different form factor. Once taken into account this correction, our scalar results are in full agreement (four digits) with [11]. The pseudoscalar ones show small discrepancies ($\approx 0.5\%$). We have also computed the binding energies by directly solving the fermion BS equation the Euclidean space using a method independent of the one used in [11]. Our Euclidean results are in full agreement with those given in the table 1.

The $B(g^2)$ dependence for the scalar and pseudoscalar couplings is plotted in figs. 2. Notice the different g^2 scales of both dependences. The pseudoscalar binding energies are fast increasing

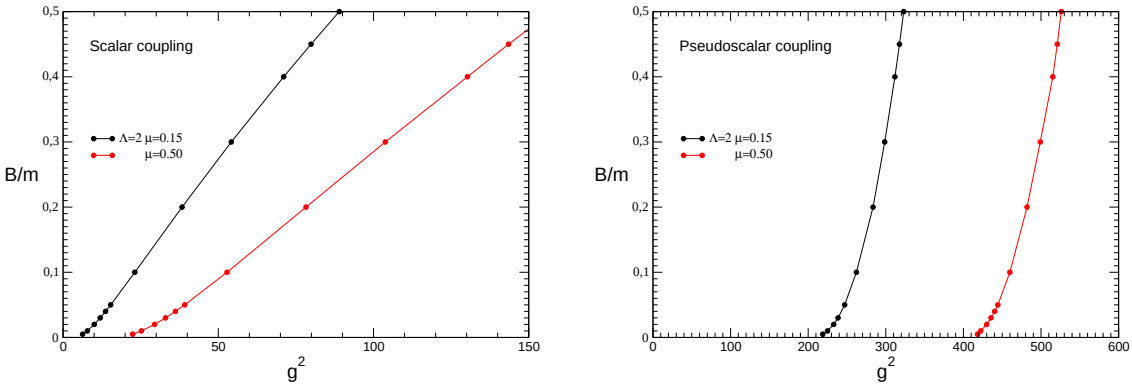

Figure 2: Left: Binding energy for scalar exchange v.s. g^2 for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$. Right: Binding energy for pseudoscalar exchange v.s. g^2 for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$.

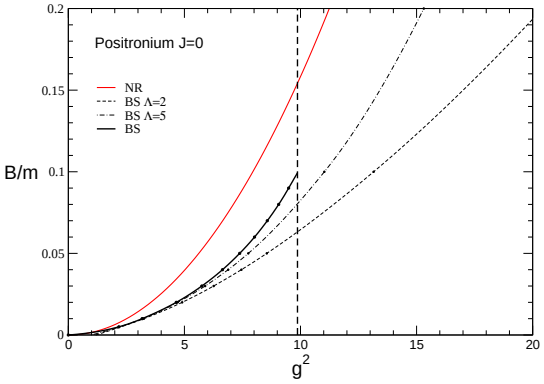

Figure 3: Binding energy for J=0 positronium state versus g^2 (black solid line) in the stability region $g < g_c = \pi$. Dashed and dotted-dashed curves correspond to the results for increasing values of the vertex form factor parameter Λ. They are compared to the non relativistic results (red solid line).

functions of g^2 and thus more sensitive to the accuracy of numerical methods. This sharp behaviour was also exhibit when solving the corresponding light-front equation [12].

In the positronium case, we found the existence a critical value of the coupling constant $g_c = \pi$. For $g < g_c$, solutions with finite binding energy exist without form factor (*i.e.*, at $\Lambda \to \infty$).

The ground state positronium binding energies without vertex form factor are given in table 1 (right) for values of the coupling below g_c, nonrelativistic results $g^2_{NR} = 8\pi\sqrt{B/m}$ are included for comparison. One can see that the relativistic effects are repulsive.

These results are displayed in fig. 3 (black solid line), and compared to the binding energies obtained with two values of the form factor parameter $\Lambda = 2$ (dashed) and $\Lambda = 5$ (dot-dashed). The stability region is limited by a vertical dotted line at $g = g_c = \pi$. Beyond this value the binding energy without form factor becomes infinite and we have found $B(g \to g_c) \approx 0.10$. The inclusion of the form form factor has a repulsive effect, *i.e.* for a fixed value of the coupling constant it provides a binding energy of the system which is smaller than in the $\Lambda \to \infty$ limit (no cut-off).

Finally, in figs. 4 we present some examples of the Nakanishi weigh functions $g_i(\gamma,z)$. They correspond to a $B=0.1$ state with the scalar coupling and the same parameters $\Lambda=2$, $\mu=0.50$ than in table 1. In the left figure is shown the γ-dependence for a fixed value of z and in the right figure – the z-dependence for a fixed γ. Notice the regular behaviour of these functions as well as their well defined parity with respect to z – $g_{1,2,4}$ are even and g_3 is odd.

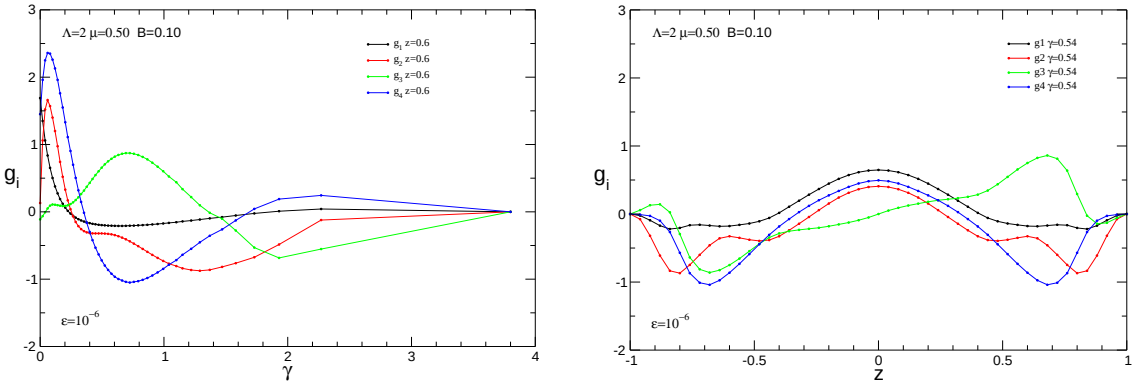

Figure 4: Left: Nakanishi weight functions v.s. γ for $z=0.6$, for scalar exchange for $\Lambda=2$, $L=1.1$, $\mu=0.15$ and $\mu=0.5$. Right: Nakanishi weight functions v.s. z for $\gamma=0.54$.

Corresponding BS amplitudes ϕ_i are displayed in figs. 5. The left panel represents the k_0 dependence of ϕ_i for a fixed value of $|\vec{k}|=0.2$. They exhibit a singular behaviour which corresponds to the pole of free propagators $k_0=\varepsilon_k-\frac{M}{2}$ in r.-h.-side of eq. 2.2. The right panel represents the $|\vec{k}|$ dependence of the amplitudes ϕ_i for a fixed value $k_0=0.04$. For this choice of arguments, the amplitudes are smooth functions of $|\vec{k}|$, though they will be also singular for $k_0>\frac{B}{2}=0.05$.

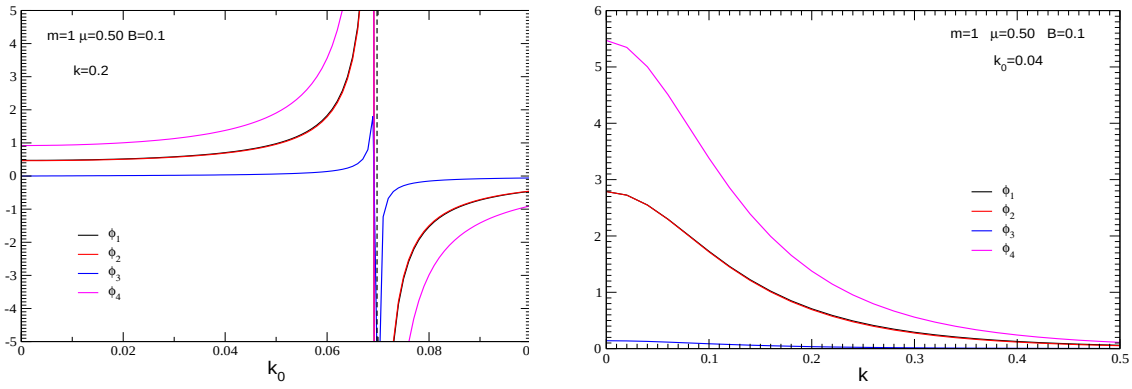

Figure 5: Left: Bethe-Salpeter Minkowski amplitudes, corresponding to fig. 4, v.s. k_0 for $k=|\vec{k}|=0.2$. The amplitudes ϕ_1 and ϕ_2 are indistinguishable. Right: The same as at left v.s. $k=|\vec{k}|$ for k_0=0.04.

4. Conclusions

We have presented a new method to obtain the solutions of the Bethe-Salpeter equation in

Minkowski space for the two-fermion system. It is based on a Nakanishi integral representation of the amplitude and light-front projection and constitutes a natural extension our previous works for the scalar case [1, 2].

A straightforward generalization of this approach, however results into a singular fermionic kernel. In order to smooth this singularities, a proper regularization of the kernels has been proposed. This generates a family of strictly equivalent equations depending on one parameter L. Their solution gives the same binding energies and Nakanishi weight functions $g_i(\gamma, z)$.

The binding energies for the scalar and pseudoscalar exchange kernels and for massless vector exchange (positronium) have been found. They coincide with the ones obtained in Euclidean space, thus providing a validity test of our method. The solutions for the scalar and positronium states without vertex form factor ($\Lambda \to \infty$) are found to be stable below the following critical values g_c of the coupling constant: $g_c = 2\pi$ (scalar) and $g_c = \pi$ (positronium).

The BS amplitudes in Minkowski are obtained in terms of the computed Nakanishi weight functions. They exhibit a singular behaviour due to the poles of the free propagators.

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