PROCEEDINGS OF SCIENCE

Taylor-Lagrange Renormalisation: electron and gauge-boson self-energies

Bruno Mutet and Pierre Grangé*

Laboratoire de Physique Théorique et Astroparticules
Université de Montpellier II, CNRS, IN2P3,
Pl. E. Bataillon, Montpellier, Cedex 05, F-34095,France
E-mail: pierre.grange@lpta.univ-montp2.fr,
E-mail: bruno.mutet@lpta.univ-montp2.fr

Ernst Werner

Institut für Theoretische Physik
D-93040, Universität Regensburg - Germany
E-mail: Ernst.Werner@physik.uni-regensburg.de

It is shown that no IR/UV divergences occur in the calculations of the electron and gauge-boson self-energies when performed in the Taylor-Lagrange renormalisation scheme. The Lorentz structure of the gauge boson self-energy to one loop is shown to emerge naturally in this scheme, directly at the physical dimension $D = 4$. Possible consequences on the fate of quadratic divergences in the Standard Model are pointed out.

Light Cone 2010: Relativistic Hadronic and Particle Physics
June 14-18, 2010
Valencia, Spain

*Speaker.

PoS(LC2010)026

1. Introduction

Following the analysis presented and developed in previous LC meetings [1] we have advocated recently [2] the use of the Taylor-Lagrange renormalization scheme (TLRS) in the treatment of the Yukawa model. In particular Mathiot reported in this meeting [3] that this scheme is very well adapted to any calculation in Light Front Dynamics (LFD). It is systematic, can treat singularities of any type on the same footing, conserves rotationnal symmetries, and moreover does not require to perform any infinite scale limit. Here we want to elaborate on specific aspects of self-energies calculations which, because of time, could not be treated in Mathiot's presentation. We shall be concerned with the evaluation of the electron and gauge-boson self-energies in TLRS and their comparison with usual results from dimensional regularisation (DR). The second section presents a simple understanding of IR and UV treatments from the mathematical structure of partition of unity (PU) test-functions. In the next section we apply the TLRS formalism to the calculation of the electron self-energy and field renormalisation and compare with well known DR results. The fourth section is concerned with the Lorentz structure of the gauge boson self-energy with emphasis on distinctive facts from TLRS. Some concluding remarks form the last section.

2. Understanding simply IR and UV treatments from mathematical structure of partition of unity (PU) test-functions.

2.1 Structure of PU test-functions.

Whe shall first motivate the use of PU test-functions and explain what they are.
It has long been known that fields are operator valued distributions (OPVD) acting on specific test functions (Infinitely differentiable, and decreasing at infinity faster than any inverse power of the variables). In constructing a field theory there are two basic requirements to satisfy:

i) the construction of the physical fields should be independent of the form of the test functions used and,

ii) it should preserve the basic Poincaré and Lorentz invariances.

These requirements are only fulfilled by PU test-functions.
A PU on $[a,b]$ is build from a set of functions $\beta_i(X)$ such that

$$\sum_{j=1}^{N} \beta_j(X) = 1 \quad \text{for any} \quad X \in [a,b].$$

The most simple realization is given by $u(X)$ such that

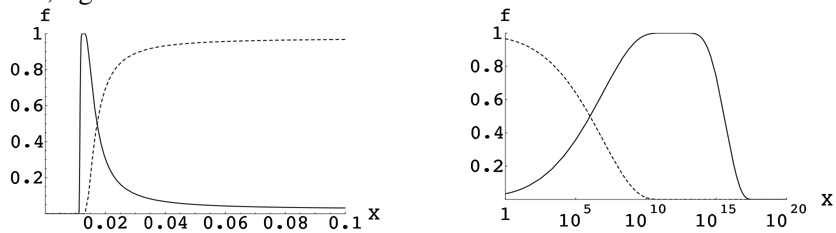
$$u(X) + u(h(X) - X) = 1 \quad \text{for any} \quad X \in [0, h(X)],$$

where $h(X) = \eta^2 X^\alpha + (\alpha - 1)$ with real parameters $\eta > 1$ and $\alpha < 1$.

$$\beta_i(X) \equiv u(|X - ih(X)|) \quad \text{for} \quad i = 0,1 \quad \text{and} \quad |X - ih(X)| < h(X).$$

Introducing $h(X)$ instead of a constant value h permits to go beyond the usual cut-off like treatment: this is known as a running support extension. It provides an infinitesimal drop-off of the test function in the limit $\alpha \to 1^-$. A particular construction of a PU with only two elementary functions

$\beta_0(X) = u(X)$ and $\beta_1(X) = u(h(X) - X)$ is shown in the following two figures (taken from [2]) $i = 0$ (dashed line) and $i = 1$ (solid line); $\alpha = 0.95$ and $\eta^2 = 2$.
Left curve: IR ; right curve: UV .



The end points of the two components are given by the solutions of
$x = 2h(x)$ $\;x0_1 = \frac{2(1-\alpha)}{2\eta^2-1}$; $xm_2 = (2\eta^2)^{\frac{1}{1-\alpha}}$ and of $x = h(x)$ $\;x0_2 = \frac{(1-\alpha)}{\eta^2-1}$; $xm_1 = (\eta^2)^{\frac{1}{1-\alpha}}$.
When $\alpha \to 1_-$, the couples $\{x0_1, x0_2\} \to 0$ and $\{xm_1, xm_2\} \to \infty$: there is a step function behaviour at the origin and an infinetisimal drop-off at infinity and the PU extends then to the whole integration domain.

2.2 Singular distributions revisited

Let $f(X)$ be a super regular PU test function (SRTF) $\in \mathbb{S}(\mathbb{R}^d)$ and $T(X)$ a distribution defined in $\mathbb{S}'(\mathbb{R}^d - \{0\})$ which we want to extend to the whole $\mathbb{S}'(\mathbb{R}^d)$. The singular order k of $T(X)$ at the origin of $(\mathbb{R}^d)$ is such that

$$k = inf\{s : \lim_{\lambda \to 0} \lambda^s T(\lambda X) = 0\} - d$$

For a SRTF f is identical to its Taylor remainder $R^k(f)$ of any order since $f^{(n)}(0) = 0 \quad \forall n > 0$. We use the property that $R^k(f)$ is given by Lagrange's formula. The most simple case is for $d = 1$, $T(X) = \frac{1}{X}$ and we have

$$R^0 f(X) = \int_0^1 dt \partial_{(t)} f(tX) \equiv X \int_0^1 \frac{dt}{t} \partial_{(X)} f(tX)$$

When $\alpha \to 1_-$ $\;f(Xt) \to \theta\left[(1-\alpha)X/t - h((1-\alpha)X/t)\right]$ i.e. $t > X\left(\eta^2 - 1\right) \equiv \tilde{\eta} X$.. After partial integration on X

$$< \frac{1}{X}, R^0 f(X) >= - < \partial_X \int_{\tilde{\eta}X}^1 \frac{dt}{t}, f(Xt) >=< \widetilde{\frac{1}{X}}, f(X) >$$

The extension $\widetilde{\frac{1}{X}}$ is then obtained by identification (up to $\delta(X)$)

$$\widetilde{\frac{1}{X}} = \partial_X \left[\log(\tilde{\eta} X)\right],$$

the derivative here is that of the log as a *distribution* [4]. Then

$$\widetilde{\frac{1}{X}} = Pseudofunction(\frac{1}{X}) \equiv Pf(\frac{1}{X}),$$

with the property $\int_0^a Pf(\frac{1}{X}) dX = \lim_{\varepsilon \to 0} \int_\varepsilon^a \frac{dX}{X} + \log(\varepsilon)$.
The UV treatment is particurlarly simple for $T(X) = \frac{1}{X+a}$. In this case Lagrange's formula writes

$$f(X)) \equiv - \int_1^\infty dt \partial_t f(Xt) = -X \int_1^\infty \frac{dt}{t} \partial_X f(Xt)$$

and since $f(X)$ is of finite support we have $t \leq \lim_{\alpha \to 1_-} [\frac{h(X)}{X}] = \eta^2$. After partial integration on $Y = Xa$ one obtains

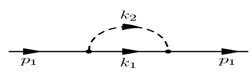
$$
\begin{aligned}
< \frac{1}{X+a}, f(X) > &\equiv < \frac{Y}{Y+1}, f(aY) > \\
&= < \partial_Y [\frac{Y}{Y+1}] \int_a^{\eta^2} \frac{dt}{t}, \lim_{\alpha \to 1_-} f(Yt) > \\
&= \log(\eta^2) - \log(a)
\end{aligned}
$$

3. The electron self-energy within TLRS and DR

3.1 TLRS at $D = 4$

The electron self energy to one loop is,

$$\Sigma^{(f)}(p) =$$



Including the test functions carried by each propagating line, it writes

$$
\begin{aligned}
\Sigma^{(f)}(p) =& -ie^2 \int_0^1 dx \int \frac{d^4\mathbf{k}}{(2\pi)^4} \{ \frac{\gamma_\mu (\not p (1-x) - \not k + m) \gamma^\mu}{\left[k^2 - m^2 x + p^2 x(1-x)\right]^2} f\left[(k+px)^2\right] f\left[(k-p(1-x))^2\right] \} \\
\equiv & A(p^2) + (\not p - m) B(p^2)
\end{aligned}
$$

Note that in the UV $f\left[(k+px)^2\right] \simeq f(k^2)$, but the denominator exhibits an on-shell pole when $k = 0$ and $x \to 0$. However in this case $f\left[(k+px)^2\right] \to f(p^2 x^2)$ and the SRTF nature of f will take care of the singularity at $x = 0$, which is crucial for field renormalization. It is then legitimate to replace the product $f\left[(k+px)^2\right] f\left[(k-p(1-x))^2\right]$ by $f(p^2x^2) f(k^2)$, for this product is what remains anyhow in the UV and IR regimes where the potential divergences occur. We first perform the calculation in the limit $\alpha \to 1_-$ for the factor $f(p^2x^2)$.

$$
\begin{aligned}
\Sigma^{(f)}(p) =& -\frac{e^2}{16\pi^2} \int_0^1 dx \left[2\not p (1-x) - 4m\right] \int_0^\infty dX \frac{X f(X)}{\left[X + \underbrace{x - \frac{p^2}{m^2} x(1-x)}_{g(x,p^2)}\right]^2} \\
=& -\frac{e^2}{16\pi^2} \int_0^1 dx \left[2\not p (1-x) - 4m\right] \int_0^\infty \frac{Y dY}{(Y+1)^2} \underbrace{f\left[Y g(x,p^2)\right]}_{-Y \partial_Y \int_{g(x,p^2)}^{\eta^2} \frac{dt}{t} f(Yt)} \\
=& -\frac{e^2}{16\pi^2} \int_0^1 dx \left[2\not p (1-x) - 4m\right] \int_0^\infty \partial_Y \left[\frac{Y^2}{(Y+1)^2}\right] dY \log\left[\frac{\eta^2}{g(x,p^2)}\right] \\
=& \frac{e^2}{16\pi^2} \left[(-\not p + 4m) \log(\eta^2) + 2 \int_0^1 dx (\not p (1-x) - 2m) \log(g(x,p^2))\right]
\end{aligned}
$$

As expected the TLRS contribution is finite.

3.2 DR with $D = 4 - \varepsilon$

The result is standard (from textbook)

$$\Sigma^{(D)}(p) = \frac{e^2}{8\pi^2\varepsilon}(-\not{p} + 4m) + \frac{e^2}{16\pi^2}\Big[(-\not{p} + 4m)\underbrace{\log\Big(\frac{4\pi\mu^2\exp(-(\gamma+1))}{m^2}\Big)}_{\mu \text{ arbitrary} \simeq \log(\eta^2)}$$
$$+2m + 2\int_0^1 dx(\not{p}\,(1-x) - 2m)\log(g(x,p^2))\Big]$$

We observe that the occurence of the arbitrary mass scale μ, introduced in relation to the dimensionality of the coupling constant at $D = 4-\varepsilon$, is in direct correspondance with the arbitrary scale η present in the test function. This is not really surprising if one recalls that the argument of the scalar test function f ought to be dimensionless, that is of the form $\frac{p^2}{\Lambda^2}$ with Λ, hence η, arbitrary.

3.3 Field renormalisation

It is very instructive to push further the comparison between the TLRS and DR schemes by looking at the field renormalisation constant $Z_2^{(1)}$.
For on-shell DR or TLRS one has $Z_2^{(1)} = 2mA'(m^2) + B(m^2)$. One finds the following DR expression

$$A'(m^2) = \frac{e^2}{8\pi^2 m}\underbrace{\int_0^1 dx \frac{(1-x^2)}{1-\frac{p^2}{m^2}(1-x)}}_{\text{diverges like } \frac{1}{x} \text{ when } p^2 = m^2}$$

The usual pratice is to give a mass λ to the photon such that

$$\begin{aligned} A'(m^2) &= \frac{e^2}{8\pi^2 m}\int_0^1 dx \frac{x(1-x^2)}{x^2 + \frac{\lambda^2}{m^2}(1-x)} \\ &= \lim_{\lambda\to 0}\frac{e^2}{16\pi^2 m}\Big[-1 + \log\Big(\frac{m^2}{\lambda^2}\Big) + \mathcal{O}(\lambda^2)\Big] \end{aligned}$$

For the TLRS expression the test function $f(p^2x^2)$ is still there. One has instead (argument of f is dimensionless)

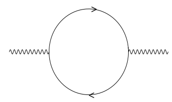
$$\begin{aligned} A'(m^2) &= \frac{e^2}{8\pi^2 m}\int_0^1 dx \frac{(1-x^2)}{x} f(x^2) \\ &= \frac{e^2}{8\pi^2 m}\{\lim_{\varepsilon\to 0}\int_\varepsilon^1 dx \frac{1}{x} + \log(\varepsilon) - \frac{1}{2}\} \\ &= -\frac{e^2}{16\pi^2 m} \end{aligned}$$

finally $\quad Z_2^{(1)} = 2mA'(m^2) + B(m^2) = -\frac{e^2}{16\pi^2}(5 + \log(\eta^2))$
This agrees with the finite part of the DR calculation (with the identification $\eta^2 = \frac{4\pi\mu^2\exp(-(\gamma+1)}{m^2}$)).
One can check that gauge dependent TLRS contributions to $Z_2^{(1)}$ also have no IR or UV divergences.

4. TLRS and DR photon self-energy

4.1 TLRS at $D=4$

$$\Pi^{(f)}_{\mu,\nu}(q) \;=$$



$$=\frac{e^2}{2}\int_0^1 dx\int\frac{d^4\mathbf{p}}{(2\pi)^4}\frac{8x(1-x)(q_\mu q_\nu-\delta_{\mu\nu}q^2)+\delta_{\mu\nu}(\frac{p^2}{2}+m^2+q^2x(1-x))}{\left[p^2+\underbrace{m^2+q^2x(1-x)}_{m^2(x)}\right]^2}f(p^2)$$

We evaluate first the two contributions from the longitudinal term

$$\frac{1}{2}\int\frac{d^4\mathbf{p}}{(2\pi)^4}\frac{p^2f(p^2)}{\left[p^2+m^2(x)\right]^2} = -\frac{m^2(x)}{(4\pi)^2}\log\left[\frac{\eta^2}{m^2(x)}\right]$$

$$m^2(x)\int\frac{d^4\mathbf{p}}{(2\pi)^4}\frac{f(p^2)}{\left[p^2+m^2(x)\right]^2} = \frac{m^2(x)}{(4\pi)^2}\log\left[\frac{\eta^2}{m^2(x)}\right]$$

The sum is zero: $\Pi^{(f)}_{\mu,\nu}(q)$ is transverse at $D=4$!

4.2 DR at $D=4-\varepsilon$

The mechanism is well known. The equivalent longitudinal contribution at arbitrary D is

$$\begin{aligned}
&\frac{1}{2}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{\left[p^2(1-\frac{2}{D})+m^2(x)\right]}{\left[p^2+m^2(x)\right]^2}\\
&=\frac{(1-\frac{2}{D})\frac{D}{2}\Gamma\left[1-\frac{D}{2}\right]}{(4\pi)^{\frac{D}{2}}(m^2(x))^{(1-\frac{D}{2})}}+\frac{m^2(x)\Gamma\left[2-\frac{D}{2}\right]}{(4\pi)^{(\frac{D}{2})}(m^2(x))^{(2-\frac{D}{2})}}\\
&=\frac{1}{(4\pi)^{\frac{D}{2}}(m^2(x))^{(1-\frac{D}{2})}}\{-\Gamma\left[2-\frac{D}{2}\right]+\Gamma\left[2-\frac{D}{2}\right]\}=0
\end{aligned}$$

$\Pi^{(D)}_{\mu,\nu}(q)$ is therefore transverse $\forall D<4$
The question is then: what really makes things work with test functions ?
At variance with a cut-off procedure quadratic divergences violating gauge invariance are absent. Let us look at the fate of quadratic divergences with TLRS

$$\int\frac{d^4\mathbf{p}}{(2\pi)^4}\frac{p^2f(p^2)}{\left[p^2+m^2(x)\right]^2}\text{ gives a term }\int\frac{d^4\mathbf{p}}{(2\pi)^4}\frac{f(p^2)}{\left[p^2+m^2(x)\right]}$$

in turn

$$\int\frac{d^4\mathbf{p}}{(2\pi)^4}\frac{f(p^2)}{\left[p^2+m^2(x)\right]}\text{ reduces to }\int_0^\infty\frac{XdXf(X)}{X+1}\text{ which gives the term }\int_0^\infty dXf(X).$$

In the absence of the test fuction this contribution diverges quadratically with the value of the cut-off as upper limit of the X-integral. However with the test function we have

$$\int_0^\infty dX f(X) \equiv \int_0^\infty \frac{dX}{X^2} f(\frac{1}{X^2}) = \int_0^\infty dX Pf(\frac{1}{X^2})$$
$$= \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \frac{dX}{X^2} - \frac{1}{\varepsilon} = 0!$$

It is instructive to recall that in DR there is an axiomatic consistency requirement which states that [5]:

$$\forall\alpha(>0 \text{ or } <0) \int d^D\mathbf{p}(\mathbf{p}^2)^\alpha = 0$$

5. Concluding remarks

We have shown the flexibility of the Taylor-Lagrange renormalization scheme and its simplicity to work with at the physical dimension $D = 4$. A systematic treatment of the UV-domain is achieved (substractions of PV-type are mathematicaly justified). IR divergences are mathematically treated by Pf-distributions, a result known for sometimes to dedicated mathematicians in their rigorous formulation of QFT (*cf* A. Connes, R. Estrada...). Symmetry violating quadratic divergences appear to be spurious (this is known since the 80's [6]). A particular important aspect of the TLRS ability to work directly at $D = 4$ resides in the ousting of the problematic extension encountered in DR with γ_5 and the Levi-Civita symbol $\varepsilon^{\mu\nu\rho\sigma}$. This certainly bears some important consequences [7] with respect to the fine-tuning problem in SSM and in super-symmetric theories.

References

[1] P. Grange, E. Werner, *Nucl. Phys.(Proc. Suppl.)* **B161** (75) 2006 [math-ph/0510052].

[2] P. Grangé, J.-F. Mathiot, B. Mutet, E. Werner, Phys. Rev. **D80**, 105012 (2009); Phys. Rev. **D82** (2010) 025012.

[3] J.-F. Mathiot, *Field Theory on the Light Front*, PoS(LC2010) .

[4] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1963.

[5] J. Collins, *Renormalization*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1987 p.71.

[6] O.E. Foda, Phys. Lett. **124B** (1983) 192

[7] B. Mutet, ,PoS(LC2010) .