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Differential Reduction Techniques for the Evaluation of Feynman Diagrams

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Stable reduction methods will be important in the evaluation of high-order perturbative diagrams appearing in QCD and mixed QCD-electroweak radiative corrections at the LHC. Differential reduction techniques are useful for relating hypergeometric functions with shifted values of the parameters. We present a proposition relating the number of master integrals in the expansion of a Feynman diagram to the number of derivatives in a differential reduction.

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PoS(ICHEP 2010)135

Techniques for the evaluation of physical cross sections have traditionally been based on the direct evaluation of Feynman diagrams. For obtaining stable numerical results, a deeper understanding of the analytical structure of Feynman diagrams and Green functions is desirable. In the framework of dimensional regularization,[1] any multi-loop Feynman diagram can be expanded as a Laurent series in powers of the regularization parameter ε. An understanding of the analytic functions in this ε-expansion gives useful information about the Feynman diagram.

The hypergeometric function representation is obtained from a multiple Mellin-Barnes representation[2] of the Feynman integral, which may be written

$$\Phi\left(a_{js}, b_{kr}, c_i, d_j, \gamma\right) = \int_{\gamma+i\mathbb{R}} dz_1 \ldots dz_m \frac{\Pi_{j=1}^{p} \Gamma\left(\sum_{s=1}^{m} a_{js} z_s + c_j\right)}{\Pi_{k=1}^{q} \Gamma\left(\sum_{r=1}^{m} b_{kr} z_r + d_k\right)} z_1^{\alpha_1} \ldots z_m^{\alpha_m} ,$$

where $a_{js}, b_{kr}, c_i, d_j \in \mathbb{R}$, $\alpha_k \in \mathbb{C}$, and z_k are Mandelstam variables. This integral can be written as a sum of multiple residues of the integrated expression, resulting in a linear combination of Horn-type hypergeometric functions of the form

$$H_r(\vec{\gamma};\vec{\sigma};\vec{x}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\Pi_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\Pi_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} ,$$

with $\mu_{ab}, \nu_{ab} \in \mathbb{Q}$, $\gamma_j, \sigma_k \in \mathbb{C}$, and their derivatives with respect to the parameters σ_k, γ_j. (See, for example, Ref. [3].) In general, the hypergeometric representation of a Feynman diagram is not unique, since non-linear transformations of the arguments z_k are possible. This is equivalent to quadratic, cubic, and more complicated transformations of the associated hypergeometric functions. A recent example is considered in Ref. [3, 4].

It is useful to see what can be learned about Feynman diagrams from their representation in terms of Horn-type hypergeometric functions. A series of publications [5] have presented and explored the following proposition:

(i) Each term in the hypergeometric representation of a Feynman diagram should have the same number of derivatives, independent of the type of hypergeometric functions appearing.

(ii) The number of master integrals required to represent a Feynman diagram should coincide with the number of derivatives plus one.

These statements are both based on a *differential reduction algorithm* (Ref. [6] and references therein) which constructs raising and lowering operators which shift the upper and lower parameters of the hypergeometric function by one unit. The reduction procedure can be used to express the hypergeometric functions appearing in the expression for a Feynman diagram in terms of a set of basis functions. Specifically, a differential reduction algorithm can be applied to any Horn-type hypergeometric function, relating functions with shifted arguments according to

$$R(\vec{z}) H(\vec{\beta} + \vec{m}; \vec{\lambda} + \vec{n}; \vec{z}) = \sum_{k=0}^{L} P_k(\vec{z}) \frac{\partial^k}{\partial z_{k_1} \cdots \partial z_{k_r}} H(\vec{\beta}; \vec{\lambda}; \vec{z}) ,$$

where $\vec{m}, \vec{n}$ are set of integer numbers and R, P_k are polynomial functions. (These relations are also useful in constructing the ε-expansion of Hypergeometric Functions [7].)

Consider the standard hypergeometric representation of a Feynman diagram,

$$\Phi(n, \vec{j}; \vec{z}) = \sum_{a=1}^{k} S_a(n, \vec{j}, \vec{z}) H_a(\vec{\beta}_a; \vec{\lambda}_a; \vec{\xi}) \quad (1)$$

PoS(ICHEP 2010)135

where $\vec{j}$ is a list of the powers of the propagators in the Feynman diagram, n is the space-time dimension, $\vec{\xi}$ are the arguments of the hypergeometric functions, which are related the kinematic invariants of the Feynman diagram, $\{\beta_a, \lambda_a\}$ are linear combinations of $\vec{j}$ and n with polynomial coefficients, and S_a are rational functions of the variables $\vec{z}$ with coefficients depending on n and $\vec{j}$.[1] The number of basis elements for the Horn hypergeometric functions H_a can be found by constructing the Pfaff system of differential equations for this function. However, when some of the parameters or differences between parameters of the functions H_a are integers, the number of derivatives is reduced. Our proposition (i) is that, regardless of the type of functions in the r.h.s. of Eq. (1), the number of basis elements is the same (up to a module of rational functions).

Being a sum of holonomic functions, $\Phi(\vec{j};\vec{z})$ is also holonomic. Thus, the number of basis elements on the r.h.s. of Eq. (1) is equal to the number of master-integrals $\Phi_k(\vec{z})$ that may be derived from the l.h.s. by applying the integration-by-parts (IBP) technique [9], which may be written symbolically as $\Phi(n,\vec{j};\vec{z}) = \sum_{k=1}^{h} B_k(n,\vec{j};z)\Phi_k(n;z)$. The number h of nontrivial master integrals following from IBP which are not expressible in terms of gamma functions is then equal to the number of basis elements L for each term of r.h.s. of Eq. (1).

The generalized hypergeometric functions of one variable which have been analyzed in detail in Ref. [5] are all in full agreement with the proposition.

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[1] In general, Eq. (1) may contain derivatives of Horn functions with respect to their parameters. [3] However, there is some evidence [8] that such expressions can again be expressed in terms of linear combination of Horn-type hypergeometric functions.

PoS(ICHEP 2010)135