

Stochastic Background of Gravitational Waves Generated by Compact Binary Systems

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Binary Systems are the most studied sources of gravitational radiation, both theoretically and observationally. Theoretically, the mechanisms of emission and the behaviour of the orbital parameters are well known and can be written in analytic form in several cases, which represents a great advantage in terms of computational work; observationally, even though gravitational waves have not been detected yet, the strongest indication of its existence have arisen from the observation of binary systems, more specifically the Hulse-Taylor binary pulsar. When one begins to detect gravitational radiation, one of the observed pattern of the signals will probably be of stochastic background nature because there will be in some cases a superposition of the signals emitted by a plenty of sources around the Universe. So, our task here is to develop a method of summing the gravitational waves emitted by cosmological binary compact systems in order to find out the shape of the spectra, that is, we will develop a method for obtaining the stochastic background generated by these systems. We use here an analogy with a problem of statistical mechanics in order to perform this sum, besides taking into account the temporal variation of the orbital parameters of the systems.

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1. Introduction

Binary systems are well studied sources of gravitational radiation. Particularly, we are interested in systems that are formed by neutron stars and black holes, which in general emit gravitational waves in higher amplitudes than the other types of binaries and, by this reason, the probability of detection could be increased. Besides, if we consider that the Universe is filled with a large population of compact binary systems emitting gravitational waves, we may suppose that in many cases the superposition of the signals will generate a smooth spectrum of characteristic shape and of stochastic type. So, the aim of this paper is to present a method for the calculation of the stochastic background generated by compact binary systems in circular orbits. First we will present a brief description of the theory concerning the binary systems and their emitted waves; after we will present a method for the calculation of the background and, finally, we will show some results.

2. Power Emitted in Gravitational Waves

The total luminosity of gravitational radiation emitted by a binary system is given by

$$\frac{dE}{dt} = \frac{32G^{7/3}}{5c^5} \frac{(m_1 m_2)^2}{(m_1 + m_2)^{2/3}} \omega^{10/3} F(e) \quad (2.1)$$

where m_1 and m_2 are the masses of the components of the system and ω is the angular orbital frequency. We can separate the total power in a sum of terms, where each one of them correspond to one harmonic:

$$F(e) = \sum_{n=1}^{\infty} g(n, e) \quad (2.2)$$

where $g(n, e)$ are very complicated functions of the harmonic number n and the eccentricity e and may be found in [1]. In the present case, we shall restrict ourselves to circular orbits, where $F(e) = 1$, and deduce an expression for the temporal variation of the orbital frequency of the system. So, we may use the chain rule

$$\frac{d\omega}{dt} = \frac{d\omega}{dr} \frac{dr}{dE} \frac{dE}{dt} \quad (2.3)$$

and get

$$\frac{d\omega}{dt} = \frac{96m_1 m_2}{5c^5} G^{5/3} (m_1 + m_2)^{-1/3} \omega^{11/3} \quad (2.4)$$

Solving this differential equation, we have

$$\omega^{-8/3} = \omega_0^{-8/3} - \frac{8}{3} K (t - t_0) \quad (2.5)$$

where

$$K = \frac{96m_1 m_2}{5c^5} G^{5/3} (m_1 + m_2)^{-1/3} \quad (2.6)$$

Moreover, we may write the angular frequency ω in terms of the orbital frequency Ω :

$$\omega = 2\pi\Omega \quad (2.7)$$

3. Distribution Functions

Binary systems have different values for the orbital parameters such as eccentricity and semi-major axis, then we can use distribution functions to describe them. An example of such a function may be found in [2] and is given by:

$$F(a, e)dade = \frac{3}{2} \frac{a^{1/2}}{a_{\max}^{3/2}} e(1 - e^2)^{-3/2} dade \quad (3.1)$$

here, a is the semi-major axis, e is the eccentricity and a_{\max} is the maximum orbital distance observed in a given population of systems. Besides, we can obtain separate distribution for the a and e , say $F(a)$ and $F(e)$, by performing convenient integration over $F(a, e)$. More specifically, in order to obtain $F(a)$ we integrate over e , and the same is valid to $F(e)$. So we have to $F(a)$:

$$F(a)da = \frac{3}{2} \left[\left(\frac{a}{a_{\max}} \right)^{3/4} - \left(\frac{a}{a_{\max}} \right)^{3/2} \right] \frac{da}{a} \quad (3.2)$$

We can use the Kepler Third Law and write Equation 3.2 in terms of the orbital frequency Ω :

$$F(\Omega) = \left[\left(\frac{\Omega_{\min}}{\Omega} \right)^{1/2} - \frac{\Omega_{\min}}{\Omega} \right] \frac{1}{\Omega} \quad (3.3)$$

where Ω_{\min} is the minimum value of Ω . But the orbital frequency changes, according to Equation 2.4. So, we have to change Equation 3.3 in order to include the temporal dependence. Using Equation 2.4, we have after some algebra:

$$F(\Omega, t, t_0) = (\Omega_{\min}^{1/2} \Omega_0^{13/6} - \Omega_{\min} \Omega_0^{5/3}) \Omega^{-11/3} \quad (3.4)$$

where we identified the frequency in Equation 3.3 with the initial frequency Ω_0 in Equation 2.4. It is worth mentioning that for binary systems in circular orbits the gravitational radiation is of quadrupolar pattern, where the frequency of the emitted waves ν is related to Ω by:

$$\nu = 2\Omega \quad (3.5)$$

4. The Stochastic Background Calculation

In a general way, the stochastic background may be calculate by Equation 4.1, according to [3]:

$$h_{\text{BG}}^2 = \frac{1}{\nu} \int h_{\text{source}}^2 dR \quad (4.1)$$

where h_{BG} is the amplitude of the spectrum, h_{source} is the amplitude of a single source, ν is the observed frequency and dR is the emission rate of gravitational waves. To binary systems h_{source} is given by, according to [4] and [10]:

$$h_{\text{source}} = 8.7 \times 10^{-21} \left(\frac{\mu}{M_{\odot}} \right) \left(\frac{M}{M_{\odot}} \right)^{2/3} \left(\frac{100\text{pc}}{d_L} \right) \left(\frac{\text{mHz}}{\nu} \right)^{-2/3} \quad (4.2)$$

where μ is the reduced mass, M is the total mass and d_L is the luminosity distance. So, the main issue is to obtain an expression for dR . We are developing a method for this, based on an elemental problem of Statistical Mechanics.

4.1 The Statistical Problem

Let a gas contained in a rectangular box and in thermal equilibrium. An elementar problem is to calculate how many particles (or molecules) of the gas reach a given wall of area A of the box in a time interval, i.e., the aim is to calculate the flux of particles. According to [5], let dV be a differential volume element adjacent to the wall, as showed in Figure 1.

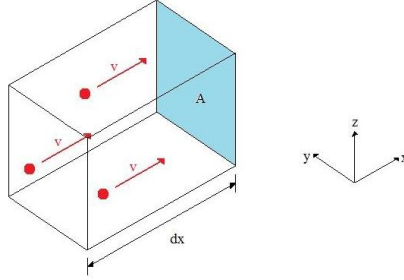


Figure 1: The differential volume element

The volume element is written as

$$dV = Av_x dt \quad (4.3)$$

where all the particles with velocity v_x will reach the area A in the time interval dt . So, given the total volume of the box and considering a spatially homogeneous distribution of particles, the fraction of particles in that volume dV an with velocities in the interval d^3v is given by

$$dm = \frac{dV}{V} \eta(v) d^3v \quad (4.4)$$

An integration over velocities (but considering only positive values to the v_x because we are considering only particles which are going to the area A) yields

$$m = \frac{Adt}{V} \int_0^\infty dv_x \int_{-\infty}^\infty dv_y \int_{-\infty}^\infty dv_z v_x \eta(v) \quad (4.5)$$

In the special case where the box has transversal section A , Equation 4.4 becomes

$$dm = \frac{dx}{L} \eta(v) d^3v \quad (4.6)$$

In the more general case where the spatial distribution of particles is not constant, the equation above becomes

$$dm = f(x) \eta(v) d^3v dx \quad (4.7)$$

where $f(x)$ can be written as

$$f(x) = \frac{\varphi(x)}{\int \varphi(x) dx} \quad (4.8)$$

where $\varphi(x)$ is the spatial distribution of particles of the gas. Now let us define a new element, which describes the temporal variation of the frequency of the system (note that we use here the frequency of the waves ν instead of the orbital frequency Ω):

$$\nu = \frac{d\nu}{dt} \quad (4.9)$$

Now, let us look at the previous problem and substitute the velocity by v defined above and substitute the spatial coordinate by the frequency ν . In this case, the frequency distribution $\varphi(\nu)$ is given by:

$$\varphi(\nu) = \varepsilon \int N(t_o) f(\nu, t, t_o) dt_o \quad (4.10)$$

where $N(t_o)$ is the binary formation rate by comoving volume element, $f(\nu, t, t_o)$ represents the fraction of systems at a given instant t and which originated at the instant t_o and ε represents the efficiency of formation of compact binaries in a given population of progenitor stars. Now, let us rewrite Equation 4.7 as

$$\frac{dR}{dV} \equiv \frac{dn}{dV dt} = \left(\frac{\varphi(\nu) \frac{d\nu}{dt}}{\int \varphi(\nu) d\nu} \right) \xi(\nu) d\nu \quad (4.11)$$

Then it is sufficient to integrate over ν . Besides, it is worth mentioning that there is only one value of ν for each value of ν , as we may conclude by observing Equation 2.4. So, we consider the function $\xi(\nu)$ as a Dirac's delta multiplied by the total number of systems. A further fact we should consider is that the denominator of the term between parenthesis in Equation 4.11 represents the total number of systems, what we write as N . So, performing the integral over ν , we get

$$\frac{dR}{dV} \equiv \frac{dn}{dV dt} = \varphi(\nu) \frac{d\nu}{dt} \quad (4.12)$$

Finally, we write the integral 4.1 as

$$h_{BG}^2 = \frac{1}{\nu} \int h_{source}^2 \frac{dR}{dV} dV \quad (4.13)$$

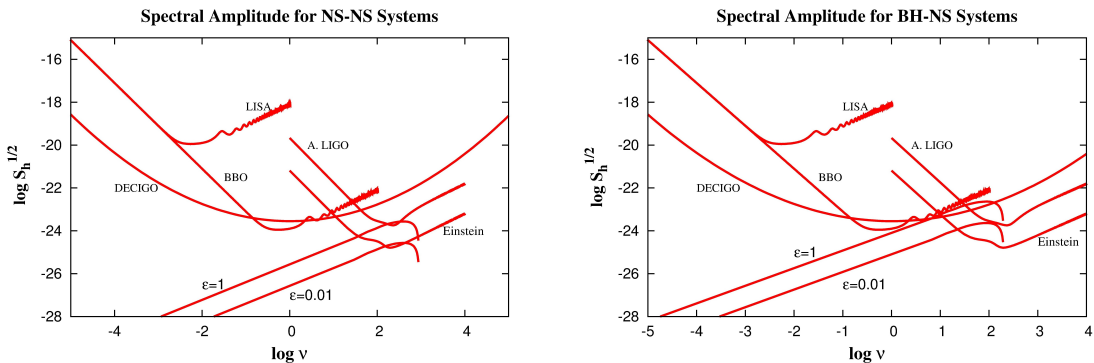
and use Equation 4.12.

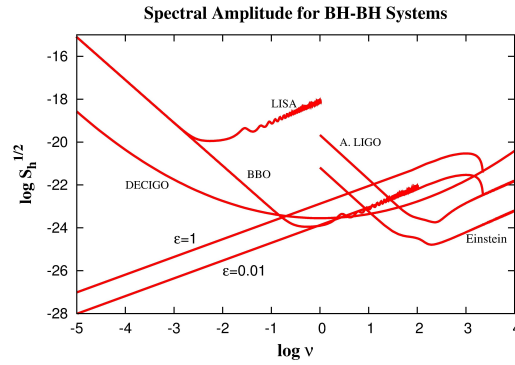
Moreover, it is convenient define the strain amplitude, which may be written as a function of h_{BG} :

$$S_h = \frac{h_{BG}^2}{\nu} \quad (4.14)$$

5. The spectra

We show here the calculated spectra, where S_h is written in function of ε :





As a general result one conclude that LISA would not be affected by any of the compact binary system noises studied here. Observing the curves for the NS-NS systems, one notes that the background could be in principle be detected by Einstein only if $\epsilon \sim 1$. BBO, DECIGO and ALIGO would not be affected by such a putative background. BH-NS binaries would not mar the sensitivity of LISA and would only mar ALIGO and marginally mar the DECIGO in the case of $\epsilon \sim 1$. For more realistic values of ϵ the BH-NS would not mar the BBO, DECIGO and ALIGO sensitivity curves. For Einstein, however, only if $\epsilon \ll 1$ the confusion noise would not be relevant. BH-BH binaries would not be important for BBO and DECIGO only if $\epsilon \ll 0.01$. For Einstein, however, even for a very small value of ϵ , say 10^{-6} , such a binary system would be important.

6. The Binary Coalescence Rates

The estimate of the merging rates of binaries is an interesting way of testing the method developed here because there are several known values for these rates and some articles show, in a independent way, results that can be used for comparison (see, e.g., [6], [7], [8] or [9]). In order to calculate the merging rate, we can use our method in a straightforward way: all we need to do is to calculate the rate dR , using particular values for the frequency and redshift. For example, in order to calculate the merging rate for a given type of binary system in the neighbourhood of the Galaxy, we calculate dR at the maximum frequency of the gravitational waves emitted by these systems and at redshifts $z \ll 1$. The following table show some results:

system	NS-NS	BH-NS	BH-BH
merger rate ($\text{Mpc}^{-3}\text{yr}^{-1}$)	$\epsilon 5 \times 10^{-5}$	$\epsilon 4 \times 10^{-8}$	$\epsilon 6 \times 10^{-4}$

We can note the influence of the efficiency ϵ : for higher values of ϵ we have higher coalescence rates, and this influence is stronger than in the case of the amplitude S_h .

7. Conclusions

A new method for the calculation of the stochastic background has been tested for binary systems in circular orbits and the results are promising. In order to submit the method to an additional test we calculated the binary merger rates and compared these values to some present in the literature. In the future we will include the effects of the eccentricity in the calculation of the stochastic background.

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