PROCEEDINGS OF SCIENCE

On the Stability of Rotating Nuclear Matter Cores of Stellar Dimensions

Kuantay Boshkayev, Michael Rotondo and Remo Ruffini
Physics Department, University of Rome «Sapienza», Square Aldo Moro 5, I-00185 Rome, Italy
ICRANet, Square of the Republic 10, I-65122 Pescara, Italy
E-mail: kuantay@mail.ru, michael.rotondo@icra.it and ruffini@icra.it

A globally neutral system of stellar dimensions consisting of degenerate neutrons, protons and electrons in beta equilibrium is considered using the ultra-relativistic solution of the Thomas-Fermi equation. Such a system at nuclear density having mass numbers $A \approx 10^{57}$ can exhibit a charge distribution different from zero. The analysis to investigate the magnetic field induced by the rotation of the system as a whole rigid body and its stability is presented in the framework of classical electrodynamics.

25th Texas Symposium on Relativistic Astrophysics - TEXAS 2010
December 06-10, 2010
Heidelberg, Germany

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1. Introduction

In [1], [2] a degenerate globally neutral system at nuclear density of N_n neutrons, N_p protons and N_e electrons in beta equilibrium constrained to a constant density distribution for the protons is described using the relativistic Thomas-Fermi equation. Then the results have been extended from heavy nuclei to the case of nuclear matter cores of stellar dimensions i.e. globally neutral systems composed of degenerate neutrons, protons and electrons in beta equilibrium with a mass number $A \approx 10^{57}$ at nuclear density kept together by gravity. Despite the global neutrality the charge distribution turned out to be different from zero inside and outside these cores and consequently they exhibit an overcritical electric field near their surface. In [2] the stability against the Coulomb repulsion of such configurations is shown within the Newtonian theory of gravity and a new island of stability is found. In [3] the analysis to investigate the magnetic field induced by the charge distribution of a nuclear matter core of stellar dimensions when the system is allowed to rotate as a whole rigid body with constant angular velocity around the axis of symmetry has been presented in the framework of classical electrodynamics. In particular it is shown for a rotating massive core with a period of 10 milliseconds the existence of an overcritical magnetic field near its surface. In the present work the special attention is given to the stability of such rotating nuclear matter cores of stellar dimensions extending the results for stability given in [2].

2. The Relativistic Thomas-Fermi equation

It is well known that at mass densities larger than the "melting" density of $\rho_m = 1.5 \cdot 10^{14} g/\text{cm}^3$, all nuclei disappear (see [4]). This allows us to adopt in the description of nuclear matter in bulk the three Fermi degenerate gases of neutrons, protons and electrons. Further

- we take the radius of the core proportional to the total number N_p of protons $R_c = \Delta\,[\hbar/m_\pi c]\,N_p^{1/3}$ and $n_p = \frac{1}{3\pi^2\hbar^3}(P_p^F)^3 = \frac{3N_p}{4\pi R_c^3}\theta(R_c - r)$ (for details see [2]);
- we solve the Thomas-Fermi (T-F) equilibrium configuration assuming $\mathscr{E}_e^F = [(P_e^F c)^2 + m^2c^4]^{1/2} - mc^2 - eV = 0$;
- being the electron number density and the proton number density known, we determine the number of neutrons by the beta equilibrium equation and we compute, on the basis of this general principle, the relation between the proton number N_p and the mass number A.

Using the Poisson equation and introducing the variable $x = r/(\hbar/m_\pi c)$, ($x_c = x(r = R_c)$), we obtain the relativistic T-F equation for extended nuclear matter (for details see [1]):

$$\frac{1}{3x}\frac{d^2\chi(x)}{dx^2} = -\alpha\left\{\frac{1}{\Delta^3}\theta(x_c - x) - \frac{4}{9\pi}\left[\frac{\chi^2(x)}{x^2} + 2\frac{m}{m_\pi}\frac{\chi}{x}\right]^{3/2}\right\}, \tag{2.1}$$

where χ is a dimensionless function defined by $\chi/r = eV/c\hbar$ and α is the fine structure constant. The boundary conditions of the function $\chi(x)$ are $\chi(0) = 0$, $\chi(\infty) = 0$, $\quad N_e = \int_0^\infty 4\pi r^2 n_e(r)dr$. These equations together with the beta equilibrium, form a close set of non-linear boundary value

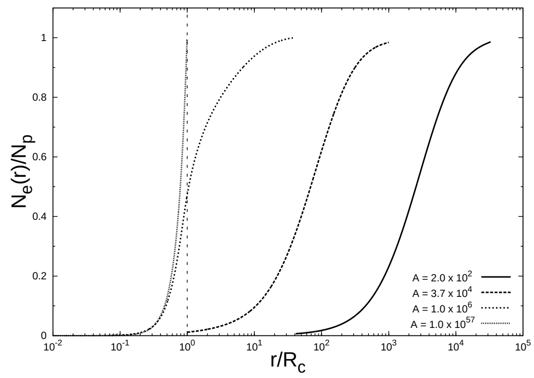

Figure 1: The electron number in the unit of the total proton number N_p, for selected values of A, as function of radial distance is shown in logarithmic scale. It is clear how by increasing the value of A the penetration of electrons inside the core increases (this figure is reproduced from [1]).

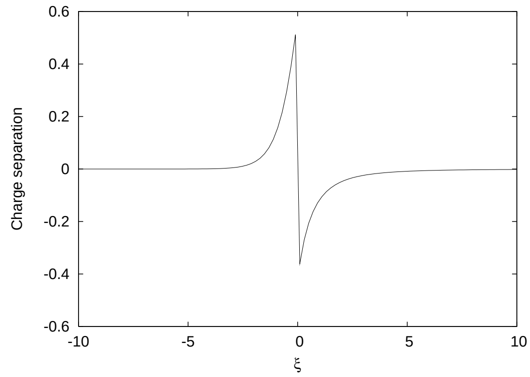

Figure 2: The normalized charge separation $(n_p - n_e)/n_p$ is plotted as function of the dimensionless radial coordinate ξ. The maximum charge separation happens near the surface of the core where a transition layer with an uncompensated charge is located.

problem for a unique solution for the Coulomb potential V and electron distribution n_e, as functions of the parameter Δ, i.e., the proton number-density n_p. A relevant quantity for exploring the physical significance of the solution is given by the number of electrons within a given radius r, $N_e(r) = \int_0^r 4\pi(r')^2 n_e(r')dr'$. This allows to determine, for selected values of the mass number A, the distribution of the electrons within and outside the core and to follow the progressive penetration of the electrons in the core at increasing values of A (see Fig.1). We can then evaluate the net charge inside the core $N_{\rm net} = N_p - N_e(R_c) < N_p$, and consequently determine the electric field at the core surface, as well as within and outside the core.

3. The ultra-relativistic analytic solutions

In the ultra-relativistic limit with the planar approximation the relativistic T-F equation admits an analytic solution. Introducing the new function ϕ defined by $\phi = 4^{1/3}(9\pi)^{-1/3}\Delta\chi/x$ and the new variables $\hat{x} = (12/\pi)^{1/6}\sqrt{\alpha}\Delta^{-1}x$, $\xi = \hat{x} - \hat{x}_c$, where $\hat{x}_c = (12/\pi)^{1/6}\sqrt{\alpha}\Delta^{-1}x_c$, then Eq. (2.1) becomes

$$\frac{d^2\hat{\phi}(\xi)}{d\xi^2} = -H(-\xi) + \hat{\phi}(\xi)^3, \tag{3.1}$$

where $\hat{\phi}(\xi) = \phi(\xi + \hat{x}_c)$. The boundary conditions on $\hat{\phi}$ are: $\hat{\phi}(\xi) \to 1$ as $\xi \to -\hat{x}_c \ll 0$ (at the nuclear matter core center) and $\hat{\phi}(\xi) \to 0$ as $\xi \to \infty$. The function $\hat{\phi}$ and its first derivative $\hat{\phi}'$ must be continuous at the surface $\xi = 0$ of the nuclear matter core of stellar dimensions. Hence equation (3.1) admits an exact solution

$$\hat{\phi}(\xi) = \begin{cases} 1 - 3\left[1 + 2^{-1/2}\sinh(a - \sqrt{3}\xi)\right]^{-1} & \xi < 0, \\ \dfrac{\sqrt{2}}{(\xi + b)}, & \xi > 0, \end{cases} \tag{3.2}$$

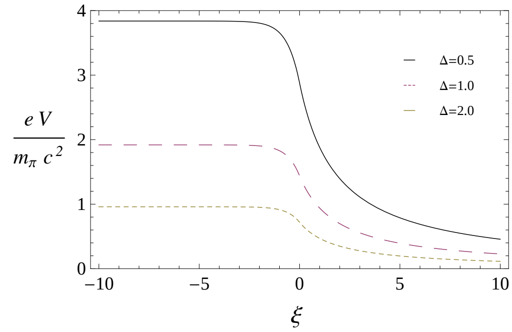

Figure 3: The proton Coulomb potential energy eV, in units of pion mass m_π is plotted as a function of the radial coordinate $\xi = \hat{x} - \hat{x}_c$, for selected values of the density parameter Δ.

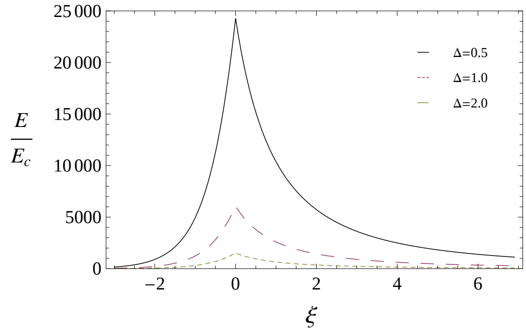

Figure 4: The electric field is plotted in units of the critical field E_c as a function of the radial coordinate ξ, showing a sharp peak at the core radius, for selected values of Δ.

where the integration constants a and b have the values $a = \operatorname{arccosh}(9\sqrt{3}) \approx 3.439$, $b = (4/3)\sqrt{2} \approx 1.886$. The charge distribution inside and outside the core is defined by

$$\rho(\xi) = \begin{cases} \frac{3e}{4\pi}\left(\frac{\Delta\hbar}{m_\pi c}\right)^{-1}\left[1 - \hat{\phi}(\xi)^3\right], & \xi < 0, \\ \dfrac{3e}{4\pi}\left(\dfrac{\Delta\hbar}{m_\pi c}\right)^{-1}\left[-\hat{\phi}(\xi)^3\right], & \xi > 0, \end{cases} \tag{3.3}$$

details are given in Fig. 2. The Coulomb potential and electric field functions in terms of $\hat{\phi}(\xi)$ given by

$$V(\xi) = \left(\frac{9\pi}{4}\right)^{1/3}\frac{m_\pi c^2}{\Delta e}\hat{\phi}(\xi), \qquad E(\xi) = -\left(\frac{3^5\pi}{4}\right)^{1/6}\frac{\sqrt{\alpha}}{\Delta^2}\frac{m_\pi^2 c^3}{e\hbar}\hat{\phi}'(\xi). \tag{3.4}$$

Details are given in Figs. 3 and 4.

4. Rotating Nuclear Matter Cores of Stellar Dimensions in Classical Electrodynamics

The magnetic field of a rotating spherically symmetric configuration along the z axis with constant angular velocity ω is defined by $\mathbf{B}(\mathbf{r}) = B_r \mathbf{e}_r + B_\theta \mathbf{e}_\theta$, where

$$B_r = \frac{2\omega}{c^2}\frac{F}{r}\cos\theta, \quad B_\theta = -\frac{2\omega}{c^2}\left[\frac{F}{r} + \frac{r}{2}\frac{d}{dr}\left(\frac{F}{r}\right)\right]\sin\theta, \tag{4.1}$$

$$F(r) = \frac{1}{r^2}\int_0^r r'^2 \frac{d}{dr'}[r'V(r')]dr'. \tag{4.2}$$

B_r is the radial component and B_θ is the angular component of the magnetic field, $F(r)$ is the superpotential, θ is the angle between r and z axis, and $\mathbf{e}_\theta$ is the unit vector along θ (for details see [3] and [5]). Consequently, the expression for the magnitude (the absolute value) of the magnetic

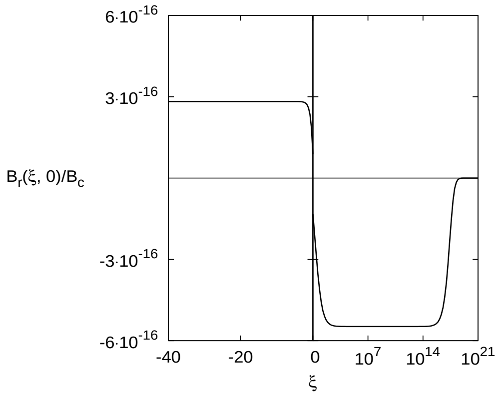

Figure 5: The radial component of the magnetic field B_r is plotted as a function of the radial coordinate ξ in units of the critical field $B_c = m_e^2 c^3/e\hbar \approx 4.5\times 10^{13}$ G. Here the period is taken to be $P = 10$ ms, $\theta = 0$, $\Delta = 1$ and the radius of the core $R_c = 10$ km. Note that B_r is considered at the poles of star, where it has maximum value. Outside the star B_r has very small negative value and it tends to zero.

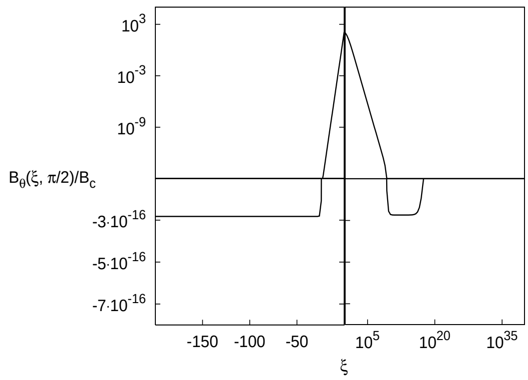

Figure 6: The angular component of the magnetic field B_θ is plotted in units of the B_c. Here $P = 10$ ms, $\theta = \pi/2$, $\Delta = 1$ and $R_c = 10$ km. Note that B_θ is considered at the equator, where it has maximum value. Inside the star it has very small constant negative value. Outside the star first it becomes negative (the value is very small) then eventually it tends to zero.

field can be written as

$$B(r,\theta) = \frac{\omega r}{c^2}\sqrt{\left(\frac{2F}{r^2}\right)^2 + \left\{\frac{4F}{r^2}\frac{d}{dr}\left(\frac{F}{r}\right) + \left[\frac{d}{dr}\left(\frac{F}{r}\right)\right]^2\right\}\sin^2\theta}. \tag{4.3}$$

Using the relation between r and ξ, $r = R_c + \left(\frac{\pi}{12}\right)^{1/6}\frac{\Delta}{\sqrt{\alpha}}\frac{\hbar}{m_\pi c}\xi$, one may estimate the value of the magnetic field. Details are given in Figs. 5, 6,7,8 and 9. Examining the Fig. 5 one can see very small value of B_r which almost does not make a significant contribution to the magnitude of the field, except for the poles of the star. On the contrary, B_θ has values exceeding the critical magnetic field near the surface of the core although localized in a narrow region between positively and negatively charged shells as expected (see Fig. 6). Outside the core the magnetic field becomes negative with very small magnitude. Fig. 7 shows spacial distribution of the magnetic field on the surface of the nuclear matter core. According to the figure the magnetic field has its maximum value at the equatorial plane and minimum at the poles. Fig. 8 represents magnetic lines of force inside, outside and on the surface of the star. It turned out that the lines of force of the overcritical magnetic field are appressed between two shells along the surface of the core. In Fig. 9 the magnitude of the magnetic field is presented as a function of the rotational period P on the surface of the core at the equator. Practically it demonstrates the upper limit of possible values of the magnetic field in the range between 10ms and 100s.

5. The stability of rotating nuclear matter cores of stellar dimensions

In the work [2] the gravitational stability against the Coulomb repulsion of a nuclear matter

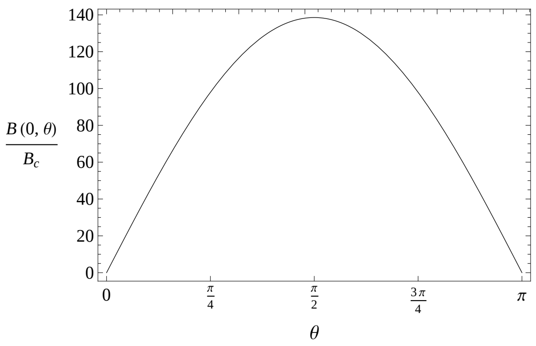

Figure 7: The magnitude of the magnetic field $B(\xi = 0, \theta)$ as given by (4.3) in the units of the critical magnetic field B_c is shown as a function of the angular variable θ. It is seen from the picture that the magnetic field has its maximum at the equatorial plane and minimum at the poles.

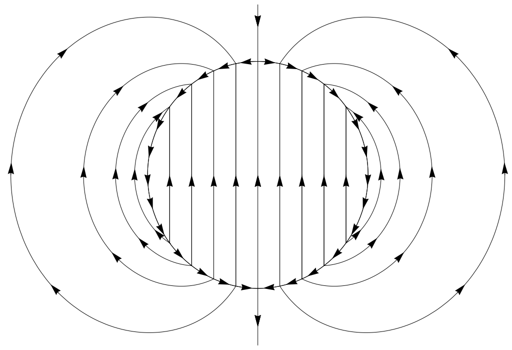

Figure 8: A schematic illustration of the magnetic lines of forces. Outside the star the magnetic field looks like a dipole field. Extra lines (arrows) along the surface of the star indicate an overcritical value of the field between positively and negatively charged shells.

core of stellar dimensions has been analyzed. In particular since in this system the gravitational energy increases proportionally to $A^{4/3}$ and the Coulomb energy increases proportionally to $A^{2/3}$ the two cross at

$$A_R^{\omega=0} = 0.039 \left(\frac{m_{Planck}}{m_n}\right)^3 \left(\frac{N_p}{A}\right)^{1/2}, \tag{5.1}$$

where m_{Planck} is the Planck mass and m_n is the neutron mass. This establishes a lower limit for the mass number A_R necessary for the existence of stable nuclear matter cores of stellar dimensions.

We consider now the analysis of the gravitational stability against the Coulomb repulsion of a nuclear matter core of stellar dimensions when the system is allowed to rotate as a whole rigid spherical object.

We know that the Coulomb energy, mainly distributed within a thin shell of width $\delta R_c \approx \hbar\Delta/(\sqrt{\alpha} m_\pi c)$ with a proton number $\delta N_p \approx 4\pi n_p R_c^2 \delta R_c$ at the surface, is given by

$$\mathscr{E}_{\rm el} \approx 0.15 \frac{3(3\pi)^{1/2}\hbar c}{4\sqrt{\alpha}} \left(\frac{\Delta\hbar}{m_\pi c}\right)^{-1} \left(\frac{N_p}{A}\right)^{2/3} A^{2/3}, \tag{5.2}$$

while the magnetic energy evolving due to rotation is given by

$$\mathscr{E}_{\rm mag} \approx 0.223 \frac{m_\pi c^2}{\Delta\sqrt{\alpha}} \left(\frac{N_p}{A}\right)^{4/3} A^{4/3} \left(\frac{\Delta\hbar}{m_\pi c}\right)^2 \frac{\omega^2}{c^2}. \tag{5.3}$$

and the rotational kinetic energy of that thin proton shell is given by

$$\mathscr{E}_{\rm rot} \approx \frac{m_n c^2}{\sqrt{\alpha}} \left(\frac{N_p}{A}\right)^{4/3} A^{4/3} \left(\frac{\Delta\hbar}{m_\pi c}\right)^2 \frac{\omega^2}{c^2}. \tag{5.4}$$

To ensure the stability of the system, the magnitude of the attractive gravitational energy of the thin proton shell

$$\mathscr{E}_g \approx \frac{3Gm_n^2}{\sqrt{\alpha}} \left(\frac{\Delta\hbar}{m_\pi c}\right)^{-1} \left(\frac{N_p}{A}\right)^{1/3} A^{4/3}, \tag{5.5}$$

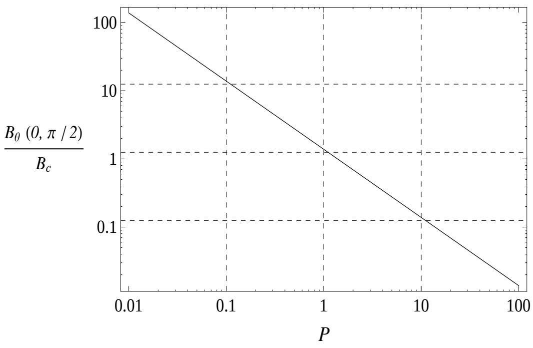

Figure 9: The magnitude of the magnetic field is plotted as a function of the period of the star P in the units of the critical field B_c at the surface of the core on the equator in the logarithmic scale. Here $R_c = 10\ km$ and $\Delta = 1$.

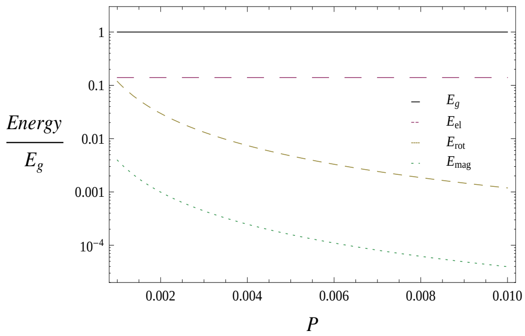

Figure 10: Energies of the system in the units of the gravitational energy of the thin shell plotted as a function of the period of the star P for $\Delta = 1$ in the range between 1ms and 10ms in logarithmic scale.

must be larger than the repulsive Coulomb energy (5.2), the magnetic energy (5.3) and the repulsive rotational energy (5.4). Indeed, it is shown in the Fig. 10 that for the periods more than 1ms, the condition $\mathscr{E}_g > \mathscr{E}_{el} + \mathscr{E}_{mag} + \mathscr{E}_{rot}$ is valid. This leads to

$$A_R^{\omega\neq 0} \approx A_R^{\omega=0}\left[1 + \left(0.112 + 0.5\Delta\frac{m_n}{m_\pi}\right)\left(\frac{m_{Planck}}{m_n}\right)^2 \frac{N_p}{A}\left(\frac{\Delta\hbar}{m_\pi c}\right)^2 \frac{\omega^2}{c^2}\right], \tag{5.6}$$

which generalizes the relation given by Eq.(5.1). We can see that the correction term $\left(0.112 + 0.5\Delta\frac{m_n}{m_\pi}\right)\left(\frac{m_{Planck}}{m_n}\right)^2 \frac{N_p}{A}\left(\frac{\Delta\hbar}{m_\pi c}\right)^2 \frac{\omega^2}{c^2}$ for the pulsars with the period more than 1ms is of the order of 10^{-3}, so in this case the system is stable.

6. Conclusions

In this paper we have investigated the stability of rotating nuclear matter cores of stellar dimensions against the rotational kinetic energy and induced magnetic energy. In fact the whole system is gravitationally bound and stable even for the 1ms period. But for the periods less than 1ms the centrifugal repulsive forces will prevail over the gravitational force. In that case the system can no longer be stable.

Since the electric field and the magnetic field are mainly concentrated in the very thin shell on the surface, compared to the radius of the object, we have considered the energies of the system only in that region. The magnetic energy has the order of one tenth of the rotational energy irrespective of the value of the period.

The results obtained in the work can have important consequences on the understanding of physical processes in neutron stars as well as on the initial conditions leading to the formation of a black hole.

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