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Loop Quantum Gravity – An Introduction

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I present here a short introduction, for non-specialists, to some basic concepts of Loop Quantum Gravity – a non-perturbative canonical quantization of General Relativity.

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1. Introduction

Gravitation theory, expressed classically by General Relativity (GR), is known to be particularly resistent to all attempts of quantization – except in lower dimensional space-times [1, 3]. Perturbative quantum gravity is known to be non-renormalizable – unless it is incorporated in a broader theory like Superstring Theory [2, 4] which is thought to be perturbatively finite [5, 6]. On the other hand, there is much hope in a nonperturbative approach, like the one of Regge Calculus [7], or, more recently, that of Loop Quantum Gravity" (LQG) [8, 9, 10, 11, 12, 13], where some interesting results have being obtained, concerning the general framework, or applications to black hole physics [13] or cosmology [14, 15]. As opposed to String Theory, which proposes a unification of all interactions, LQG stays with the a-priori more modest aim of giving a consistent description of quantum gravity, without attempting unification with the other interactions. Most of the work done in this latter area up to the present day concerns pure gravity, but matter may be added anyhow [9], with couplings independent of the gravitational coupling constant.

The purpose of this talk is to expose the most basic concepts of LQG such as configuration space, holonomies, spin networks, Hilbert space and constraints, and to give some examples of observables. Due to limitation of time, only references to the literature are given for most of the further developments such as spin foams, loop cosmology, etc.

2. Classical General Relativity

General Relativity is a geometrical description of relativistic gravitation theory [16, 17]. Classical four-dimensional space-time is given as a differentiable manifold M_4 and a tangent vector space T_P at each of its points P, equipped with a pseudo-Riemannian metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \qquad x = (t,\mathbf{x})$$

of Lorentzian signature $(-1,1,1,1)$. Most formulations of GR are based on the so-called second order formalism where the basic variables are the components of the metric tensor, and where the field equation – the Einstein equation – is second order in the time derivatives of the metric. The first attempts towards a non-perturbative quantization were performed in this formalism and lead to the well-known Wheeler-DeWitt equation [18, 16, 19], a timeless Schrödinger type equation, of which very few solutions are known, not speaking of the difficulty to construct an appropriate Hilbert space with well-defined operators acting in it.

On the other hand, decisive progress was made using the first order formalism, on which LQG is based, exploiting its strong analogies with Yang-Mills gauge theories. In this formalism indeed the basic variables are, beyond the tetrad field describing the metric, a connection associated to a gauge invariance, namely the invariance under the local Lorentz transformations.

In the first order formalism, one defines in each tangent plane a pseudo orthogonal moving frame ("tetrad" or "vierbein"), as illustrated in Fig. 1. It consists of four vector fields[1] $e_I(x) = e^\mu_I(x)\partial_\mu$, $I = 0,\cdots,3$, obeying the pseudo-orthogonality relations

$$e_I \cdot e_J = g_{\mu\nu}e^\mu{}_I e^\nu{}_J = \eta_{IJ}\,,\ (\eta_{IJ}) = \text{diagonal}\,(-1,1,1,1)\,.$$

[1] Most of our expressions are written in a system of coordinates x^μ, $\mu = 0,\cdots,3$. Covariance under general changes of coordinates is of course assumed.

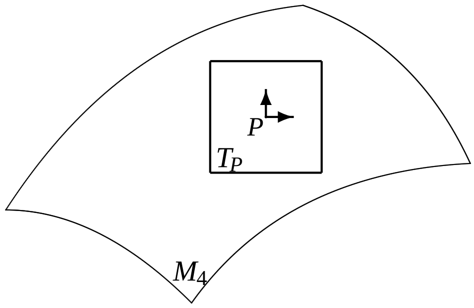

Figure 1: Manifold M_4 and tangent plane T_P with frame at point P.

The dual tangent space T_P^* admits the basis $e^I(x) = e^I{}_\mu(x)dx^\mu$, $I = 0,1,2,3$, with $(e^\mu{}_I) = \left(e^I{}_\mu\right)^{-1}$. The metric then writes $g_{\mu\nu}(x) = \eta_{IJ}\, e^I{}_\mu(x)e^J{}_\nu(x)$. The vierbein basis of T_P and T_P^* are defined up to a (local) Lorentz transformation, e.g.: $e'^I(x) = \Lambda^I{}_J(x)e^J(x)$, with $\eta^{IJ}\Lambda^I{}_K\Lambda^J{}_L = \eta^{KL}$. We have thus two local invariances: Lorentz and diffeomorphisms. The infinitesimal transformations read, in the case of the vierbein:

$$\delta_{\text{Lorentz}} e^I = \varepsilon^I{}_J e^J\,, \quad \text{with } \varepsilon_{IJ} = -\varepsilon_{JI}\,,\ \varepsilon_{IJ} = \eta_{IK}\varepsilon^K{}_J\,,$$
$$\delta_{\text{Diff}} e^I = \mathscr{L}_\xi e^I\,,$$

where $\mathscr{L}_\xi$ is the Lie derivative along the vector field ξ whose components correspond to the infinitesimal coordinate transformations: $\delta x^\mu = \xi^\mu(x)$.

The Lorentz covariant derivative D is defined through the introduction of the connection form ("spin connection") $\omega^I{}_J = \omega^I{}_{J\mu}(x)dx^\mu$, with $\omega_{IJ} - \omega_{JI}$, transforming as

$$\delta_{\text{Lorentz}}\omega^I{}_J = d\varepsilon^I{}_J + [\omega,\varepsilon]^I{}_J$$

under infinitesimal Lorentz transformations. For example, the torsion 2-form is the covariant derivative of the vierbein form:

$$T^I = \frac{1}{2}T^I{}_{\mu\nu}dx^\mu dx^\nu = De^I = de^I + \omega^I{}_J e^J\,.$$

The curvature is another 2-form transforming covariantly under the Lorentz transformations:

$$R^I{}_J = \frac{1}{2}R^I{}_{J\mu\nu}dx^\mu dx^\nu = d\omega^I{}_J + \omega^I{}_K\omega^K{}_J\,.$$

2.1 Palatini-Holst action

The complete action for pure gravity in the first order formalism is the Palatini-Holst action [20]

$$\begin{aligned} S_{\text{PH}} &= -\frac{1}{2k}\int_{M_4} e^I \wedge e^J \wedge \left(\frac{1}{2}\varepsilon_{IJKL}R^{KL} - \frac{1}{\gamma}R_{IJ}\right) \\ &= -\frac{1}{2k}\int_{M_4} d^4x\, \varepsilon^{\mu\nu\rho\sigma} e^I{}_\mu e^J{}_\nu \left(\frac{1}{2}\varepsilon_{IJKL}R^{KL}{}_{\rho\sigma} - \frac{1}{\gamma}R_{IJ\rho\sigma}\right)\,, \end{aligned}$$

where $k = 8\pi G$ with G the Newton constant, and γ is the Barbero-Immirzi [21] parameter. $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor, with $\varepsilon^{0123} = 1$ (and same definition for ε^{IJKL}). The classical theory does not depend on the parameter γ, as shown by the field equations derived from this action: they are the usual null torsion and null Ricci curvature equations

$$T^I = 0\,, \quad R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} = 0\,.$$

The first one allows to express the connection in terms of the vierbein components and their derivatives, the second one is the Einstein equation in the absence of matter.

2.2 Canonical formalism and Ashtekar variables

In order to implement the canonical formalism on which quantization will be based, one has to define some time variable. In GR, this is usually done by introducing a foliation of space-time M_4, defined by a the introduction of a "temporal function" $T(P)$ and the topological assumption that it can be factorized as $M_4 = \mathbb{R}_T \times M_3$, where M_3 represents the space slice at constant "time" $T = T(P)$, as illustrated in Fig. 2.

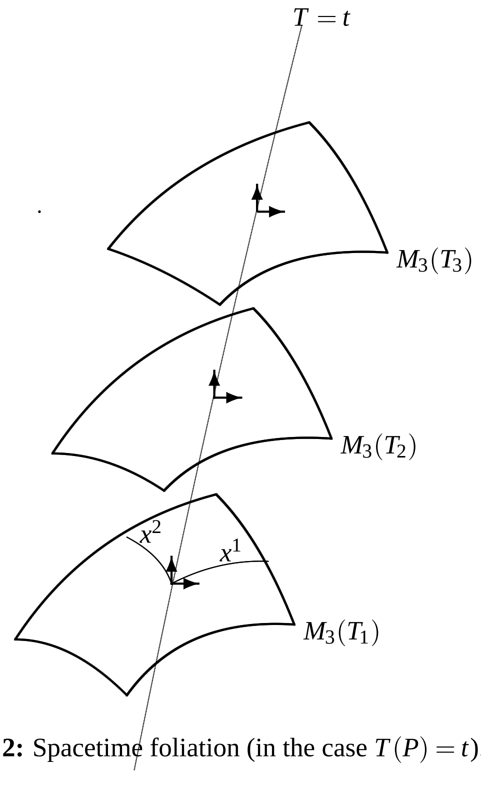

Figure 2: Spacetime foliation (in the case $T(P) = t$). Tangent frames and coordinate frames.

Before beginning the canonical construction, which consists in defining pairs of canonically conjugated variables and analyzing the constraints associated with the gauge invariances of the theory according to the Dirac-Bergmann procedure [22], let us proceed to a partial gauge fixing [23, 24], called the "temporal gauge", which simplifies the formalism and reduces the non-compact

Lorentz gauge symmetry group SL(2,ℂ) to its compact subgroup SU(2). This gauge is defined by setting to zero three of the vierbein form components[2]: $e^0{}_a = 0$, or, equivalently, $e^t{}_i = 0$. Moreover we choose space-time coordinates in such a way that the three coordinates x^a also parametrize the space slice M_3. (In other words, we choose the time function $T(x) = t$). Geometrically, the temporal gauge choice means that the basis vector $e_{I=0}$ at point P is tangent to the time coordinate line passing at this point: $e_0 = e^t{}_0(x)\partial_t$.

With the notations $N \equiv e^0{}_t$, $N^i \equiv e^i{}_t$ and $N^a \equiv e^a{}_i N^i$, the space-time metric takes the ADM form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(N^2 - h_{ab}N^aN^b\right)dt^2 + 2N_a dx^a dt + h_{ab}dx^a dx^b$, where $h_{ab} = \delta_{ij}e^i{}_a e^j{}_b$ is the metric of the space slice M_3 at coordinate time t. One recognizes in $N(x)$ and N^a the "lapse" and "shift" functions of the ADM formalism [17].

The Palatini-Holst action reads, in the temporal gauge:

$$S_{\rm PH} = \int dt \left(\int d^3x\, P^a{}_i \partial_t A^i{}_a - H \right),$$

with

$$A^i{}_a = \frac{1}{2}\varepsilon^{ijk}\omega_{ija} + \gamma\omega^{0i}{}_a\,, \quad P^a{}_i = \frac{1}{2k\gamma}\varepsilon^{abc}\varepsilon_{ijk}e^j{}_b e^k{}_c\,.$$

$A^i{}_a$ is the Ashtekar SU(2) connection in the real formalism of Barbero and Immirzi [21]. Its curvature is

$$F^i{}_{ab} = \partial_a A^i{}_b - \partial_b A^i{}_a - \varepsilon^i{}_{jk}A^j{}_a A^k{}_b\,.$$

$P^a{}_i$ is the conjugate momentum of $A^i{}_a$, and we have thus the Poisson brackets (only the nonzero ones are shown):

$$\left\{A^i{}_a(\mathbf{x},t),\, P^b{}_j(\mathbf{y},t)\right\} = \delta^i_j\delta^b_a\delta^3(\mathbf{x}-\mathbf{y})\,, \tag{2.1}$$

The Hamiltonian is given by

$$H = \int d^3x \left(\Lambda^i\mathscr{G}_i - N^a\mathscr{V}_a - N\mathscr{S}\right). \tag{2.2}$$

where

$$\mathscr{G}_i = D_a P^a{}_i = \partial_a P^a{}_i - \varepsilon_{ij}{}^k A^j{}_a P^a{}_k\,,$$

$$\mathscr{V}_a = P^b{}_i F^i{}_{ab} + \text{ terms proportional to } \mathscr{G}_i\,,$$

$$\mathscr{S} = \frac{k\gamma^2}{2\sqrt{\det h}}P^a{}_i P^b{}_j\left(\varepsilon_{ijk}F^k{}_{ab} - (\gamma^2+1)(K^i{}_aK^j{}_b - K^i{}_bK^j{}_a)\right) + \text{ terms proportional to } \mathscr{G}_i \text{ and } \mathscr{V}_a\,,$$

$\Lambda^i = \frac{1}{2}\varepsilon^{ijk}\omega_{ijt}$, and $K^i{}_a = \omega^{0i}{}_a$ is the extrinsic curvature of the space slice M_3. Observing that the fields Λ^i, N^a and N appear linearly in the Hamiltonian and have no conjugate momentum, we deduce that their coefficients are constraints:

$$\begin{aligned} \mathscr{G}[\Lambda] &= \int d^3x \Lambda^i(x)\mathscr{G}_i(x) \approx 0\,, \\ \mathscr{V}[\vec{N}] &= \int d^3x N^a(x)\mathscr{V}_a(x) \approx 0\,, \\ \mathscr{S}[N] &= \int d^3x N(x)\mathscr{S}(x) \approx 0\,. \end{aligned} \tag{2.3}$$

[2]The four coordinate indices μ are devided in a time index t and and a space index $a = 1,2,3$. The tangent space indices I are devided in $I = 0$ and $I = i$, $i = 1,2,3$

The weak equality sign $\approx$ means that the constraints are effectively fulfilled only once all necessary Poisson Bracket algebra calculations are made.

One checks that these constraints are first class according to Dirac's terminology, i.e., they form a closed Poisson bracket algebra and are the infinitesimal generators of the gauge invariances of the theory. More specifically, $\mathscr{G}$ generates the SU(2) gauge transformations (the residual symmetry group left from the time gauge fixing), $\mathscr{V}$ generates the space diffeomorphisms, whereas $\mathscr{S}$ generates the time diffeomorphisms up to SU(2) transformations, space diffeomorphisms and field equations.

3. Loop quantum quantization

3.1 Canonical quantization

"Canonical quantization" is intended here as the construction of a quantum theory along the following lines:

1. Construct a "kinematical Hilbert space" $\mathscr{H}_{\rm kin}$ whose elements are the wave functional $\Psi[A]$ ($\Psi[A] = \langle A|\Psi\rangle$ in Dirac's notation) which are functionals of the configuration variables, taken here as the components of the Ashtekar connection $A^i{}_a(x)$, elements of a configuration space $\mathscr{A}$, and where the conjugate momentum operators act as:
$$P^a{}_i(x)\Psi[A] = -i\hbar\frac{\delta\Psi[A]}{\delta A^i{}_a(x)}\,,$$
in order to fulfill the commutation rules
$$[A^i{}_a(\mathbf{x},t), P^b{}_j(\mathbf{y},t)] = i\hbar\delta^i_j\delta^b_a\delta^3(\mathbf{x}-\mathbf{y})\,,$$
corresponding to the classical Poisson brackets (2.1);

2. Define the configuration space $\mathscr{A}$ and an integration measure $\mathscr{D}A$ in it, in such a way that the inner product
$$\langle\Psi_1|\Psi_2\rangle = \int \mathscr{D}A\,\overline{\Psi_1[A]}\Psi_2[A]$$
be well defined;

3. Define the constraints (2.3) as well defined operators in $\mathscr{H}_{\rm kin}$;

4. Solve the constraint equations
$$\mathscr{G}[\Lambda]\,|\Psi\rangle = 0\,,\quad \mathscr{V}[\vec{N}]\,|\Psi\rangle = 0\,,\quad \mathscr{S}[N]\,|\Psi\rangle = 0\,, \tag{3.1}$$
and define the physical Hilbert space $\mathscr{H}_{\rm phys}$ as the space of the solutions of these constraint equations.

We note that, since the Hamiltonian (2.2) is a linear combination of the constraints, fulfilling the constraints amounts to solve the timeless Schrödinger equation,
$$H\,|\Psi\rangle = 0\,,$$

known as the Wheeler-DeWitt equation in the context of quantum gravity in the second order formalism, mentioned at the beginning of Section 2. This equation provides an apparently paradoxical "dynamics without time". In the present context, canonical quantization is based on a "time" parameter t which is just a coordinate. Invariance under the "time" diffeomorphisms thus means that the "time" evolution is a mere gauge transformation, which has no physical meaning. We refer to the literature [25] where this "problem of time" has been widely discussed.

3.2 Construction of the kinematical space $\mathscr{H}_{\rm kin}$

Instead of taking arbitrary functionals $\Psi[A]$ of the classical Ashtekar SU(2) connection $A(x)$ as our wave functionals, we first consider functions $f(U_1,\cdots,U_N)$, called "cylindrical functions", whose arguments are the holonomies U_n of A along a finite set of oriented curves e_n $(n=1,\cdots,N)$ in the space manifold M_3. Such a set of oriented lines – the "edges" – together with their intersection points – the "vertices" – form a graph Γ. Examples are shown in Fig. 3 and 4.

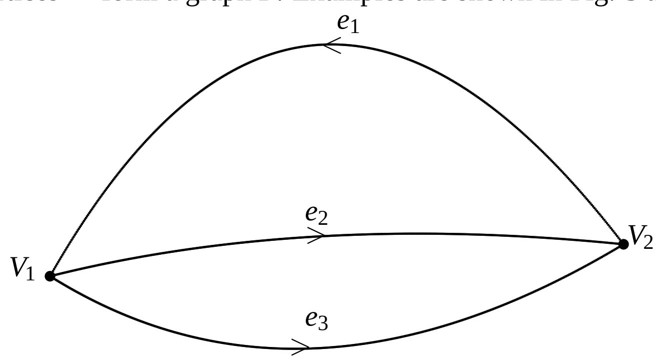

Figure 3: Graph with 3 edges and 2 vertices

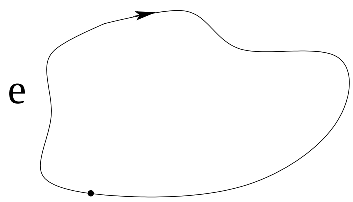

Figure 4: Closed loop graph.

The holonomy of the SU(2) connection A along the oriented path e from the points P_1 to P_2, parameterized by s, with $s_1 \leq s \leq s_2$, (see Fig. 5) is defined as

$$U \equiv U[A,e] = \mathrm{P}\exp\left[-\int_e A\right] = \mathrm{P}\exp\left[-\int_{s_1}^{s_2} \tau_i A^i{}_a(s)dx^a(s)\right],$$

where the $\tau_i = \frac{i}{2}\sigma_i$ are the generators of the gauge group. The symbol P ("path order") means that in the field products appearing in the expansion of the exponential function, factors are ordered with s increasing from the right to the left. The holonomy is an element of the gauge group SU(2). Under a gauge transformation $A'(x) = g^{-1}(x)dg(x) + g^{-1}(x)A(x)g(x)$, with $g(x) \in$ SU(2), it transforms as

$$U[A,e]' = g^{-1}(P_2)U[A,e]g(P_1)\,.$$

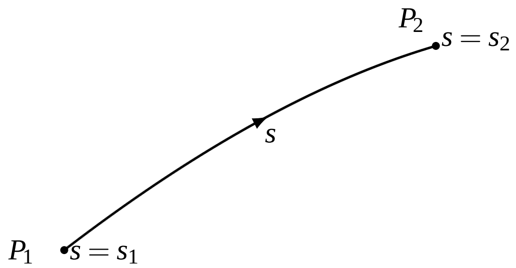

Figure 5: Parameterized oriented curve.

The vector space spanned by all the cylindrical functions, associated to arbitrary graphs, is denoted by Cyl. A vector $|\Gamma, f\rangle \in$ Cyl associated to a graph Γ of N edges and a function $f:\ \mathrm{SU}(2)\times \mathrm{SU}(2)\cdots\times \mathrm{SU}(2)\to\mathbb{C}$ is thus given by

$$\langle A|\Gamma, f\rangle = \Psi_{\Gamma,f}[A] = f\left(U[A,e_1],\cdots,U[A,e_L]\right) .$$

An hermitean scalar product can be defined in Cyl by the integral over the group

$$\langle\Gamma, f_1|\Gamma, f_2\rangle = \int dU_1\cdots dU_L\,\overline{f_1(U_1,\cdots,U_L)}\,f_2(U_1,\cdots,U_L)\,, \tag{3.2}$$

where $U_n = U[A,e_n]$ and dU is the (normalized) invariant Haar measure on the group SU(2). Note that it is the compactness of the group SU(2) which assures the normalizability of the measure – or, in other words, the convergence of the integral (3.2) for any locally integrable integrant.

This scalar product is invariant under the SU(2) gauge transformations due to the invariance of the Haar measure. It is also invariant under the space diffeomorphisms since it only depends on the number of the edges of the graph, and not on its location in space. It is however not invariant under the time diffeomorphisms.

We have defined here the scalar product between vectors both associated to a same graph. In order to define the scalar product $\langle\Gamma_1, f_1|\Gamma_2, f_2\rangle$, with $\Gamma_1\neq\Gamma_2$, one applies the definition (3.2), but with a graph $\widetilde{\Gamma}$ which is the union of the graphs Γ_1 and Γ_2.

The space Cyl being the set of all finite linear combinations of cylindrical vectors $|\Gamma, f\rangle$, we can complete it to a Hilbert space $\mathscr{H}_{\rm kin}$, the kinematical space, defining it as the set of the Cauchy sequences of Cyl, which are defined through the norm induced by the scalar product we have just defined.

An orthonormal basis of $\mathscr{H}_{\rm kin}$ can be defined using a generalization of the Peter-Weyl theorem, according to which any function on a compact Lie group can be expanded in a discrete orthonormal basis consisting of the matrix elements of all the unitary representations of the group. In the case of SU(2) these representations are labeled by half-integer spin j. In order to proceed, given a graph Γ, one associates to each edge e_n a spin value j_n, different of 0, and the holonomy in the representation of spin j_n, whose $(2j_n+1)\times(2j_n+1)$ matrix elements are denoted by $R^{(j_n)\,\alpha_n}{}_{\beta_n}\left(U[A,e_n]\right)$. The indices α_n, β_n correspond to the initial point, final point, respectively of the edge e_n. This yields a "colored" graph, an example of which is shown in Fig. 6.

The basis vectors are then given by the products of matrix elements

$$\begin{aligned}\langle A|\Gamma, j,\alpha,\beta\rangle \;&\equiv\; \langle A|\Gamma, j_1\cdots j_N,\alpha_1,\cdots,\alpha_N,\beta_1,\cdots,\beta_N\rangle \equiv \Psi_{\Gamma,j,\alpha,\beta}[A]\\ &=\; R^{(j_1)\,\alpha_1}{}_{\beta_1}\left(U[A,e_1]\right)\cdots R^{(j_N)\,\alpha_N}{}_{\beta_N}\left(U[A,e_N]\right)\end{aligned} \tag{3.3}$$

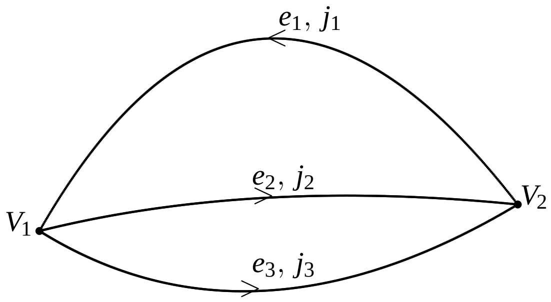

Figure 6: Colored graph with 3 edges and 2 vertices.

taken for all possible graphs, all possible spin values and all values of the matrix element indices α_n, β_n. Spin 0 is excluded in order to avoid over counting. Indeed, a colored graph with a spin 0 edge yields the same expression as the graph with this edge deleted.

The Peter-Weyl theorem assures that these vectors, called “spin networks”, form an orthonormal basis of $\mathscr{H}_{\rm kin}$, provided one includes the "null vector" $|\emptyset\rangle$, with $\Psi_\emptyset[A] = 1$, associated to the empty graph $\Gamma = \emptyset$. The orthonormality relations read

$$\left\langle \Gamma, j, \alpha, \beta | \Gamma', j', \alpha', \beta' \right\rangle = \delta_{\Gamma\Gamma'} \delta_{jj'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}\,.$$

In particular, basis vectors associated to two different graphs are orthogonal. It follows that the kinematical Hilbert space admits the orthogonal decomposition

$$\Rightarrow \quad \mathscr{H}_{\rm kin} = \bigoplus_{\Gamma} \mathscr{H}_\Gamma\,.$$

One sees, from the fact that the set of all graphs is not countable, that this Hilbert space is not separable. Separability will be attained after the diffeomorphism constraint is applied.

3.3 Implementing the constraints

3.3.1 The Gauss constraint

This is the first of Eqs.(3.1). It is implemented through the requirement of SU(2) gauge invariance. A gauge invariant basis is obtained by inserting invariant tensors, acting as generalized Clebsch-Gordan coefficients, at each of the vertices of a spin network graph (3.3), saturating the matrix indices α_n and β_n. More explicitly, at a vertex such as the one depicted in Fig. 7, one inserts an invariant tensor – and “intertwiner” – $v^{(i)}_{\alpha_1\cdots\alpha_m}{}^{\beta_{m+1}\cdots\beta_{m+n}}$, where the exponent (i) enumerates the various possible invariant tensors of the same ranks m and n, and contracts all the α and β indices. Due to the invariance property of these tensors:

$$\begin{aligned} v^{(i)}_{\alpha_1\cdots\alpha_m}{}^{\beta_{m+1}\cdots\beta_{m+n}} = \; & R^{(j_{m+1})\,\beta_{m+1}}{}_{\rho_{m+1}} \cdots R^{(j_{m+n})\,\beta_{m+n}}{}_{\rho_{m+n}} v^{(i)}_{\sigma_1\cdots\sigma_m}{}^{\rho_{m+1}\cdots\rho_{m+n}} \\ & R^{-1\,(j_1)\,\sigma_1}{}_{\alpha_1} \cdots R^{-1\,(j_m)\,\sigma_m}{}_{\alpha_m}\,, \end{aligned} \tag{3.4}$$

the result of these insertions is a gauge invariant expression. The vectors thus constructed form a basis of a SU(2) gauge invariant Hilbert space $\mathscr{H}_{\rm Gauss} \subset \mathscr{H}_{\rm kin}$.

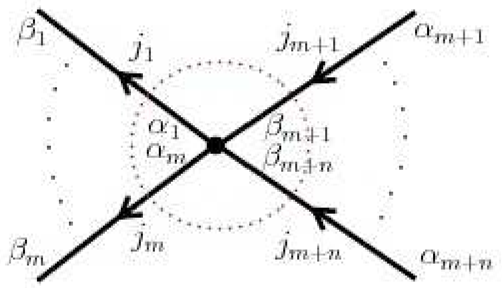

Figure 7: Vertex with *m* outgoing and *n* ingoing lines.

3.3.2 The vector constraint

The solution of the second of the constraints (3.1) using the group averaging procedure [8, 9] yields a Hilbert space $\mathscr{H}_{\text{diff}}$ whose elements are equivalence classes of vectors of $\mathscr{H}_{\text{Gauss}}$. Two vectors of $\mathscr{H}_{\text{Gauss}}$ are equivalent graphs if they are characterized by the same quantum numbers (spin and sets of intertwiners), their respective graphs being related by a space diffeomorphism (see Fig. 8). Through an adequate choice for the class of diffemorphisms under consideration [8],

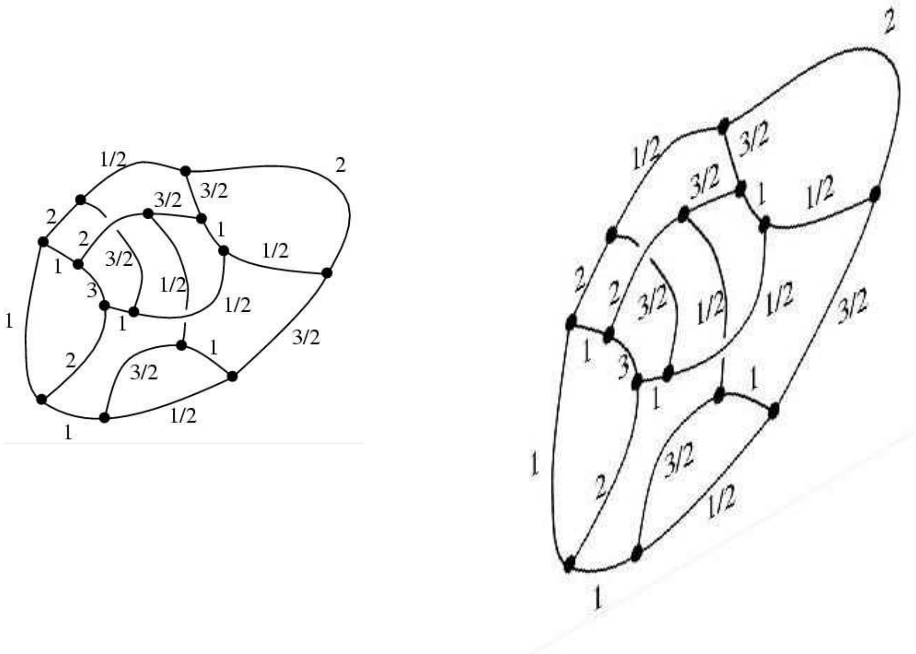

Figure 8: Two diffeomorphism-equivalent graphs.

the resulting Hilbert space $\mathscr{H}_{\text{diff}}$ is separable. One can understand this intuitionally observing that only the topological structure of the graphs is now relevant.

3.3.3 The scalar constraint

The solution of the third of the constraints (3.1), the scalar or Hamiltonian constraint, which corresponds to the invariance under the time diffeomorphisms, has to yield the physical Hilbert space. Whereas the first two previous constraints are pretty well understood, a complete solution of the last one is not yet known. It is this constraint which produces the dynamics of GR. Indeed, to the contrary of the first two ones, the scalar constraint operator – which may be rigorously defined [9] – makes one leave the space sheet M_3. Solving it amounts to calculate transition amplitudes between geometries of M_3. Let us only mention in this context the spin foam approach [8, 9, 26].

4. Some applications

4.1 Geometric observables

It turns out that the spin network basis vectors (3.3) are eigenstates of the area and volume operators $\mathscr{A}[\Sigma]$ and $\mathscr{V}[\Omega]$ for a superficies $\Sigma \in M_3$ and a region Ω of M_3.

In the case of a superficies Σ in a spin network state $|\Gamma, j_1, \cdots, j_N\rangle$ such that the intersection points $P_1, \cdots, P_N$ between Σ and Γ do not coincide with edge's endpoints (see Fig. 9) and such that each edge intersects Σ at most once, the eigenvalues of the area operator are given by:

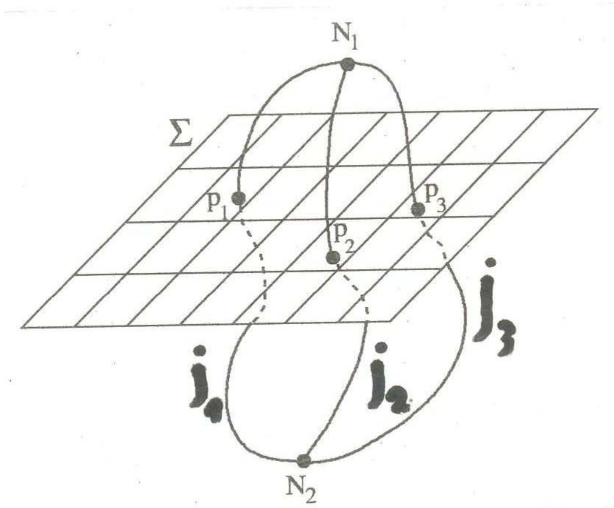

Figure 9: Intersection points P_1, P_2 and P_3 of a graph with a surface Σ.

$$\mathscr{A}[\Sigma]\,|\Gamma, j_1, \cdots, j_N\rangle = 8\pi\gamma l_{\mathrm{P}}^2 \sum \sqrt{j_n(j_n+1)}\,|\Gamma, j_1, \cdots, j_N\rangle\ ,$$

where $l_{\mathrm{P}} = \sqrt{\hbar G/c^3} \sim 10^{-34}$m is the Planck length, and the sum is performed on all edges of Γ which intersect Σ. One notes here the presence, as a numerical factor, of the Barbero-Immirzi parameter γ. Similar result hold in the general case.

Other geometrical operators are the volume and length operators. The latter however is not diagonal in the spin network basis, but both have discrete spectra as well.

4.2 Other applications

Very interesting applications of LQG ideas to the study of black holes physics and quantum cosmology have been made [13, 15, 14]. The black hole singularity of classical GR is removed, and its entropy can be calculated in terms of the fundamental constants – with the Barbero-Immirzi parameter as a factor. In cosmological, dynamics with bounce seem to be preferred – a dynamcics without the big-bang singularity of the classical theory [14].

Applications o LQG techniques to lower dimensional models, such as Chern-Simons theory, $2-D$ gravitational model of Jackiw-Teitelboiman $2-D$ supergravity, may be found in [27, 28, 29].

5. Conclusions

Much material which has not been covered in this seminar may be found in the quoted literature. I would only like to stress that important difficulties still need to be better faced.

In particular, we are still lacking of a precise definition of the scalar constraint operator and the solutions of the scalar constraint.

A serious issue is that of the existence of a semi-classical limit yielding the known classical RG.

The scheme outlined in this lecture relies crucially on the compactness of the gauge group, realized through the "non-covariant" time gauge fixing. Difficulties are expected in a "covariant formalism", since it implies the full Lorentz group, which is not compact (see [30] and references therein).

Finally, the very difficult question of experimental or observational tests of Quantum Gravity remains open (see [31] and references therein).

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