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Non-Commutative Gravitational Quantum Well: Determination of the Energy Spectrum Through Self-Adjoint Extensions

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We study the free-fall of a quantum particle in the context of Non-Commutative Quantum Mechanics (NCQM). Assuming non-commutativity of the canonical type between the coordinates of a two dimensional configuration space, we consider a neutral particle trapped in a gravitational well and exactly solve the energy eigenvalue problem. This is achieved by means of a thorough study of the self-adjoint extentions of the Hamiltonian operator. By resorting to experimental data from the GRANIT experiment [1], in which the first energy levels of freely falling quantum ultracold neutrons were determined, we impose an upper-bound on the non-commutativity parameter.

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1. Non-Commutative Quantum Mechanics

The idea that space-time would have a non-commutative structure has attracted a lot of attention since last decade, although it was proposed when Quantum Field Theory (QFT) was still being born, plagued by ultra-violet divergences. In fact, the idea that non-commuting space-time coordinates (for reviews, see [2–4]) would cure those divergences was proposed by Heisenberg, being later formalized by Snyder (for a historical introduction, see [2, 5]). Before the revival in non-commutative was boosted by achievements in String Theory in the late 1990s (see [2–4]), the issue of space-time non-commutativity was considered in an earlier work by Doplicher et al., who constructed a unitary QFT based on a non-commutative space-time [6], motivated by the issue of the description of the quantum nature of the space-time.

Since quantum mechanics can be interpreted as the one-particle sector of quantum field theory, it is interesting to investigate the quantum mechanics defined on non-commutative spaces. Lately, this Non-Commutative Quantum Mechanics (NCQM) has attracted interest (see, for instance, Ref.s [7–11]). Specially, there have been studies about the energy levels of a quantum particle under the action of the newtonian gravitational potential of the Earth, with the aim of obtaining an upper-bound on the non-commutative parameter (see Ref.s [12, 15]). All this has been motivated by the experimental determination of the first energy levels of freely falling quantum ultracold neutrons in a gravitational well (the GRANIT experiment), performed by Nesvizhevsky et al. [1, 16].

In this communication, we review the determination of the energy levels of a non-commutative quantum particle in a gravitational well and the imposition of an upper-bound on the non-commutative parameter, reported by the authors recently in Ref. [17]. This was achieved by means of the study of the self-adjoint extensions [18] of the Hamiltonian operator that describes the system. It should be noted that the first study to treat the non-commutative gravitational quantum well was done by Bertolami et al. [12]. Nonetheless, it was achieved in a non-commutative model completely different from the one we have considered. For details and a comparative discussion with the considerations in [12], as well as with other studies, we refer the reader to [17]. We remark that we have assumed only spatial non-commutativity, in contrast to those works, which considered non-commutativity of both configuration and momentum spaces (see [12]-[14]) or time-space non-commutativity (see [15]). The Ref.s [12]-[14] treated the non-commutative gravitational quantum well and used data from the GRANIT experiment to find upper-bounds on the value of the *momentum-momentum* non-commutativity parameter, while in the Ref. [15] an upper bound on the *time-space* component of the non-commutative matrix was found, by means of second quantization techniques.

2. Non-commutative quantum mechanics

When space-space non-commutativity is considered, the extended phase space commutation relations read

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}\,, \qquad [\hat{p}_i, \hat{p}_j] = 0\,, \qquad [\hat{p}_i, \hat{x}_j] = -i\hbar\delta_{ij}\,. \tag{2.1}$$

The corresponding non-commutative Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V \star \Psi = i\hbar\frac{\partial\Psi}{\partial t}\,, \tag{2.2}$$

where $\star$ denotes the so-called Moyal product, defined by

$$(\psi\star\phi)(x_1,x_2,x_3)=\psi(x_1,x_2,x_3)\,e^{\frac{i}{2}\overleftarrow{\partial}_i\,\theta_{ij}\,\overrightarrow{\partial}_j}\,\phi(x_1,x_2,x_3)\,. \tag{2.3}$$

The energy eigenvalue equation of NCQM can be obtained by means of the usual stationary problem *ansatz*, namely

$$\Psi(\vec{r},t)=\psi(\vec{r})\,e^{-\frac{i}{\hbar}Et} \quad ; \quad -\frac{\hbar^2}{2m}\bigtriangledown^2\psi+V\,e^{\frac{i}{2}\overleftarrow{\partial}_i\,\theta_{ij}\,\overrightarrow{\partial}_j}\,\psi=E\psi\,. \tag{2.4}$$

The gravitational well potential energy is

$$V(x,y,z)=\begin{cases} mgy\,, & y>0\,, \quad \forall x,z\,,\\ \infty\,, & y=0\,, \quad \forall x,z\,.\end{cases} \tag{2.5}$$

As argued in [17], in order one can handle with a well-definite problem a local approximation for the Moyal product is needed. It is easily achieved and is given by

$$V\,e^{\frac{i}{2}\overleftarrow{\partial}_i\,\theta_{ij}\,\overrightarrow{\partial}_j}\,\psi\simeq V\,\psi+\frac{i}{2}\theta_{ij}\frac{\partial V}{\partial x_i}\frac{\partial}{\partial x_j}\psi\,. \tag{2.6}$$

It follows that the time-independent Schrödinger equation reads

$$\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}-\frac{mgi\theta}{2}\frac{\partial\psi}{\partial x}}_{H_x\psi}\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2}+mgy\psi}_{H_y\psi}=E\psi\,. \tag{2.7}$$

3. Determination of the Energy Levels by Self-Adjoint Extensions

In this section, we consider the non-commutative version of the gravitational quantum well and determine the energy spectrum of a particle trapped in it. We remark that usually the correct definition of an operator is not addressed in most applications in physics, but rather an operator is defined only by means of its law of action (the so-called formal operator), without any mention about its domain of definition. Operators having the same formal expression but acting in different domains can lead to different physics and this is a crucial question specially in quantum theory [18]. Thus, in order to determine the spectrum of the non-commutative gravitational well, we have carefully examined the domain of the differential operator we have to handle with. This is closed related to the self-adjointness of the operator and will ultimately lead us to consider its self-adjoint extensions. We refer the reader to [17] and references therein.

Note that we have $H=H_x+H_y$, where

$$H_x=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\frac{mgi\theta}{2}\frac{\partial}{\partial x} \quad \text{and} \quad H_y=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}+mgy\,, \tag{3.1}$$

with domains $D(H_x)=C_0^\infty(\mathbb{R})$ and $D(H_y)=C_0^\infty(\mathbb{R}_+^*)$. In spite of being Hermitian, neither H_x nor H_y are self-adjoint operators, since the domains $D(H_x^*)$ and $D(H_y^*)$ are lager then $D(H_x)$ and $D(H_y)$, respectively.

The usual strategy to solve the above problem is based on the fact that the larger the domain of an operator T, the smaller the domain of its adjoint T^*. So we basically have to extend T to an operator T_e so that $D(T_e) = D(T_e^*)$. The construction of such extensions makes use of some important mathematical results, which are described at some detail in the sequel.

Let $\mathscr{K}_\pm(T)$ be the vector spaces of the square integrable solutions of the equations $T\psi = \pm i\lambda\psi$. The deficiency index, $n_\pm$, are defined as $n_\pm = \dim \mathscr{K}_\pm(T)$. Now we quote the theorem by von Neumann which fully classifies the self-adjoint extensions of the Hermitian operators.

Theorem 1. *Let T be a Hermitian operator on a Hilbert space $\mathscr{H}$ and let $\overline{T}$ be its closure. Then:*
(*a*) *T is essentially self-adjoint if and only if $n_+ = n_- = 0$;*
(*b*) *T has self-adjoint extensions if and only if $n_+ = n_-$;*
(*c*) *There is a one-to-one correspondence between the unitary maps $U : \mathscr{K}_+(T) \to \mathscr{K}_-(T)$ and the self-adjoint extensions of T (which we shall denote by T_U);*
(*d*) *The domain of T_U is $D(T_U) = \{\psi + \psi_+ + U\psi_+ \in D(T^*) : \psi \in D(\overline{T}),\, \psi_+ \in D(U)\}$.*
(*e*) *The action of T_U in $D(T_U)$ is given by $T^*(\psi + \psi_+ + U\psi_+) = \overline{T}\psi + i\psi_+ - iU\psi_+$.*

Let us firstly apply the von Neumann theorem to the case of H_x:

$$-\frac{\hbar^2}{2m}\psi''_\pm(x) - \frac{mgi\theta}{2}\psi'_\pm(x) = \pm i\lambda\,\psi_\pm(z)\ ,\ \ \psi_\pm(x) = e^{\omega_\pm x}\,. \tag{3.2}$$

In this case $n_+ = n_- = 0$, so that $\overline{H}_x$ is the unique self-adjoint extension of H_x. Formally $\overline{H}_x$ acts just like H_x, but its domain is

$$D(\overline{H}_x) = D(H_x^*) = \left\{\psi \in L^2(\mathbb{R}) \cap C^0(\mathbb{R}) : \psi' \in AC(\mathbb{R})\ ,\ H_x\psi \in L^2(\mathbb{R})\right\}. \tag{3.3}$$

Now, we consider the case of H_y:

$$\psi''_\pm(z_\pm) - \left(\frac{2m^2 g}{\hbar^2}\right) z_\pm\,\psi_\pm(z_\pm) = 0\ ,\ z_\pm = y \mp i\lambda/(mg)\,. \tag{3.4}$$

In this case $n_+ = n_- = 1$. Formally the extensions of H_y acts just like H_y itself, but their domains are given by

$$D(H_{y,\alpha}) = \left\{\psi \in L^2(\mathbb{R}^*_+) : \psi'' \in L^2_{loc}(\mathbb{R}^*_+)\,,\, H_y^*\psi \in L^2(\mathbb{R}^*_+)\,,\, \psi(0) = \alpha\,\psi'(0)\right\}. \tag{3.5}$$

The range of α can be extended so as to comprehend the case $\alpha = \infty$, which corresponds to $\psi'(0) = 0$.

Among all the self-adjoint extensions of H_y, the only one which correctly describe the experimental setup of the GRANIT experiment corresponds to the choice $\alpha = 0$ in Eq. (3.5).

Now we proceed to the determination of the spectrum of H. We make use the *ansatz* $\psi(x,y) = e^{ikx}\phi(y)$ to get

$$\frac{d^2\phi}{dy^2} + \frac{2m^2 g}{\hbar^2}\left(\frac{\mathscr{E}_\theta}{mg} - y\right)\phi(y) = 0\,, \tag{3.6}$$

where

$$\mathscr{E}_\theta = E - \frac{\hbar^2 k^2}{2m} - \frac{mgk\theta}{2}\,. \tag{3.7}$$

Now, by setting $\xi = \frac{y-b_\theta}{a}$, $a = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}$ and $b_\theta = \frac{\mathscr{E}_\theta}{mg}$, we can put (3.6) in the form of an Airy equation (see Ref. [19], for example),

$$\frac{d^2\phi(\xi)}{d\xi^2} - \xi\phi(\xi) = 0, \tag{3.8}$$

whose general solution is

$$\phi(y) = A\,Ai\left(\frac{y-b_\theta}{a}\right) + B\,Bi\left(\frac{y-b_\theta}{a}\right). \tag{3.9}$$

The boundary conditions for $\phi(y)$ imply that $Ai\left(-\frac{b_\theta}{a}\right) = 0$, so that $-b_\theta/a$ are the roots of the Airy function Ai, i.e.,

$$b_{\theta,n} = -a\,\alpha_n, \tag{3.10}$$

where α_n denotes the n-th zero of Ai. The result (3.10) combined with the definition of b_θ gives the spectrum of the Hamiltonian of a non-relativistic quantum particle trapped in the non-commutative gravitational quantum well:

$$E_{k,n,\theta} = \frac{\hbar^2 k^2}{2m} + \left(\frac{mg^2\hbar^2}{2}\right)^{1/3}.(-\alpha_n) + \frac{mgk\theta}{2}. \tag{3.11}$$

For $\theta \to 0$, we recover the text-book result of ordinary quantum mechanics. For discussions about other characteristics of the above spectrum, we refer the reader to our paper in Ref. [17].

4. Conclusions

We have studied the free-fall of a particle under the action of a uniform gravitational field in NCQM. Assuming noncommutativity only on configuration space and carefully studying the self-adjointness of the Hamiltonian operator involved, as well as determining its self-adjoint extensions, we have exactly solved the noncommutative Schrödinger equation and determined the energy eigenvalues. Instead of simply imposing by hand the usual boundary condition associated with the reflecting mirror at the bottom of the gravitational well, we concluded that this condition is among those permitted by the theory of self-adjoint extensions when applied to the original operator we started from. Obtaining the energy spectrum is specially important, since from the data of the gravitational quantum well experiment with freely falling neutrons - the GRANIT experiment [1], [16] - we can then set an upper-bound on the value of the spatial noncommutative parameter, θ. Applying Eq. (3.11) to a neutron under the conditions (its horizontal velocity) of the GRANIT experiment, we arrived at the bound discussed in Ref. [17]. We remark that our result can be improved in the future, when more accurate experimental data will be available [16]. We note that the works that considered the noncommutative gravitational quantum well have not established an upper-bound on θ, but rather on the momentum-momentum noncommutativity parameter [12]-[14] or on the time-space component of the noncommutative matrix [15].

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