PROCEEDINGS OF SCIENCE

Enthalpy and the first law of black hole thermodynamics

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In black hole thermodynamics a cosmological constant contributes a pressure to the equation of state whose the conjugate variable is a 'volume'. It is shown that, for a negative cosmological constant, this 'thermodynamic volume' is the volume of space excluded by the event horizon, when quantum gravity corrections are ignored. This applies to black holes in any dimension with any event horizon geometry compatible with Einstein's equations, such as spheres, tori, spaces of constant negative curvature or, in general, any Einstein space with constant Ricci scalar. Quantum corrections to the volume are calculate for the BTZ black hole.

Corfu Summer Institute on Elementary Particles and Physics - Workshop on Non Commutative Field Theory and Gravity,
September 8-12, 2010
Corfu Greece

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PoS(CNCFG2010)017

1. Review of black hole thermodynamics

A black hole has surface temperature $T = \frac{\kappa\hbar}{2\pi}$, where κ is the surface gravity, and entropy, $S = \frac{1}{4}\frac{A}{\ell_{Pl}^2}$, where A is the event horizon area ($\ell_{Pl}^2 = G_N\hbar$, with G_N Newton's constant and $c = 1$), [1]. In the description of black hole thermodynamics, the black hole mass M is usually identified with the internal energy E which, in the absence of rotation or electric charges, should be considered to be a function of the entropy and the volume [2].

The first law of black hole thermodynamics is usually written

$$dM = \frac{\kappa}{8\pi G_N}\,dA \;= T\,dS, \qquad (J = Q = 0),$$

at zero pressure. The (Helmholtz) free energy, $F(T,V)$, is the Legendre transform of $E(S,V)$

$$F = E - TS = M - \frac{A\,\kappa}{8\pi G_N}.$$

In a Euclidean path integral formulation the action, S_E, is related to the free energy F through the bridge equation

$$F = -T\ln Z,$$

where $Z = e^{-S_E}$.

Hawking has also shown that the heat capacity of a Schwarzschild black hole is negative, hence it is thermodynamically unstable. For the Schwarzschild metric $r_h = 2G_N M$, $T = \frac{\hbar}{8\pi G_N M}$, and $A = 16\pi G_N^2 M^2$. The free energy evaluates to

$$F = \frac{M}{2} = \frac{\hbar}{16\pi G_N T},$$

and the heat capacity is

$$C = \frac{\partial E}{\partial T} = -T\frac{\partial^2 F}{\partial T^2} = -\frac{8\pi G_N M^2}{\hbar} < 0.$$

The instability is due to thermal radiation from the black hole: in a vacuum it t radiates with power $\mathscr{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{G_n^2 M^2}$. Since the energy available is $E = M$ the lifetime will be $\tau \sim \frac{M}{\mathscr{P}} \sim \frac{G_N^2 M^3}{\hbar} \sim \frac{\hbar^3}{G_N T^3}$.

As the black hole loses energy through radiation it shrinks and this suggests that the heat capacity calculated above cannot be interpreted as the heat capacity at constant volume.

2. Enthalpy

When pressure is included in any thermodynamic system the first law reads

$$dE = T\,dS \;- P\,dV$$

Which raises the question, where is the $P\,dV$ term in black hole thermodynamics? To answer this include a cosmological constant Λ. This contributes pressure P and energy density $\varepsilon = -P = \frac{\Lambda}{8\pi G_N}$, [3] see also [4]. This modifies the thermal energy

$$E = M + \varepsilon V = M - PV \quad\Rightarrow\quad M = E + PV$$

which suggests that black hole mass should be identified with enthalpy rather than internal energy, [3, 5]:

$$M = H(S,P).$$

With a cosmological constant included the Schwarzschild line element is

$$d^2s = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d^2\,\Omega, \qquad (d^2\Omega = d\theta^2 + \sin^2\theta d\phi^2),$$

with

$$f(r) = 1 - \frac{2G_N M}{r} - \frac{\Lambda}{3}r^2.$$

For $\Lambda > 0$ there are both inner and outer event horizons, making the definition of a single temperature ambiguous in general, so to avoid such ambiguities we restrict to $\Lambda < 0$. Then $P = -\frac{\Lambda}{8\pi G_N} > 0$. The event horizon is at $r = r_h$ where

$$f(r_h) = 0 \quad \Rightarrow \quad M = \frac{r_h}{2G_N}\left(1 - \frac{\Lambda}{3}r_h^2\right).$$

Now identifying M with the enthalpy, and using $S = \frac{1}{4}\frac{A}{G_N \ell_{Pl}^2}$, gives

$$H(S,P) = \frac{1}{2G_N}\left(\frac{\ell_{Pl}^2 S}{\pi}\right)^{\frac{1}{2}}\left(1 + \frac{8G_N \ell_{Pl}^2 SP}{3}\right).$$

The temperature is then determined by the standard thermodynamic relation

$$T = \left(\frac{\partial H}{\partial S}\right)_P \quad \Rightarrow \quad T = \frac{\hbar(1 - \Lambda r_h^2)}{4\pi r_h}. \tag{2.1}$$

We can now ask, what is the thermodynamic volume, V? The thermodynamic definition is

$$V = \left(\frac{\partial H}{\partial P}\right)_S \quad \Rightarrow \quad V = \frac{4}{3}\frac{(\ell_{Pl}^2 S)^{\frac{3}{2}}}{\sqrt{\pi}} = \frac{4\pi r_h^3}{3},$$

which is the three dimensional volume excluded by the black hole, as suggested for other reasons in [3].

The Gibbs free energy is Legendre transform of $H(S,P)$

$$G(T,P) = H - TS = -T\ln Z.$$

Information is lost if one interprets $\ln Z$ as a function of T and V, [2], since $V = \left.\frac{\partial G}{\partial P}\right|_T$, and so

$$G(T,P) = -T\ln Z\left(T, \left.\frac{\partial G}{\partial P}\right|_T\right),$$

results in a differential equation with an undetermined integration constant.

This distinction between the Helmholtz free energy and the Gibbs free energy will almost certainly be important in the AdS/CFT approach to condensed matter systems [6].

The heat capacity is now clearly the heat capacity at constant pressure,

$$C_P = \left(\frac{\partial H}{\partial T}\right)_P \qquad \Rightarrow \qquad C_P = \frac{T}{\left.\frac{\partial T}{\partial S}\right|_P} = 2S\left(\frac{8G_N P \ell_{Pl}^2 S + 1}{8G_N P \ell_{Pl}^2 S - 1}\right).$$

This diverges for $-\Lambda r_h^2 = 1$: the black hole is unstable if $|\Lambda|$ too small, stability requires $T > T_{H-P}$ with T_{H-P} the temperature of the Hawking-Page phase transition, [7],

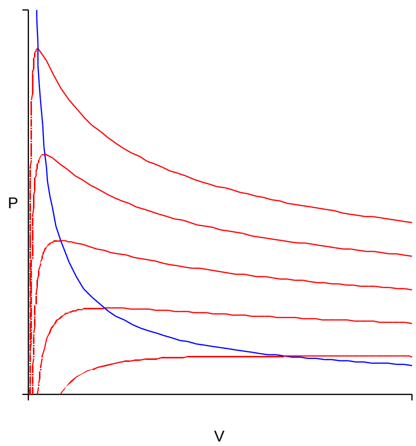
$$T_{H-P} = \hbar\sqrt{\frac{2G_N P}{\pi}}.$$

2.1 Equation of state

From (2.1) we have the equation of state

$$T(V,P) = \frac{\hbar}{4\pi}\left\{\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}} + 8\pi G_N P\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}\right\}$$

and the $P-V$ diagram is shown below, with red lines being curves of constant T. The region below and left of the blue line is unstable.



3. BTZ black hole

It is instructive to consider the special case of $2+1$-dimensions, since more is known about quantum corrections in this case than in other dimensions [8, 9]. The line element can be expressed as

$$d^2s = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\phi^2,$$

with

$$f(r) = -8G_N M + \frac{r^2}{L^2}. \qquad (\Lambda = -\tfrac{1}{L^2}),$$

giving horizon radius, $r_h = \sqrt{8G_N M}L$. The Hawking temperature is $T = \frac{\hbar\sqrt{2G_N M}}{\pi L}$ and the Beckenstein-Hawking entropy is $S = \frac{\pi r_h}{2\ell_{Pl}}$ (with $\ell_{Pl} = \hbar G_N$ in three dimensions).

Identifying the Schwarzschild mass with the enthalpy, $H = M$, the above results give

$$H(S,P) = \frac{4\ell_{Pl}^2}{\pi}S^2 P,$$

where again $P = -\frac{\Lambda}{8\pi G_N}$.

The equation of state is

$$PV^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4\ell_{Pl}}T,$$

and the thermodynamic volume,

$$V = \pi r_h^2,$$

again agrees with ones naïve geometrical intuition.

The Gibbs free energy evaluates to minus the enthalpy,

$$G = H - TS = -M,$$

and the heat capacity is strictly positive for $M > 0$,

$$C_P = \frac{T}{\left.\frac{\partial T}{\partial S}\right|_P} = S > 0,$$

which never diverges at finite temperature.

These results should be compared pure AdS_3 with

$$f(r) = 1 + \frac{r^2}{L^2}$$

(equivalent to setting $M = -\frac{1}{8G_N}$ in BTZ). In this case the enthalpy $H = -\frac{1}{8G_N}$ is constant and $T = 0$, implying that the Gibbs free energy is

$$G = H = -\frac{1}{8G_N}.$$

If $M < \frac{1}{8G_N}$ in BTZ, pure AdS$_3$ has lower free energy, implying a phase transition at $T = \hbar\sqrt{\frac{2G_NP}{\pi}}$, but this is of a different nature to the Hawking-Page phase transition in four (or more) dimensions.

3.1 Quantum corrections to entropy and volume

To understand quantum corrections to the BTZ thermodynamics, [8], [9], it is useful to first expand the discussion to include rotating black-holes. For a rotating BTZ black hole with angular momentum J,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\phi - \frac{4G_NJ}{r^2}dt\right)^2$$

where

$$f(r) = \left(-8G_NM + \frac{r^2}{L^2} + \frac{16G_N^2J^2}{r^2}\right).$$

There are now both inner and outer event horizons at

$$r_\pm^2 = 4G_NML^2\left\{1 \pm \left[1 - \left(\frac{J}{ML}\right)^2\right]^{\frac{1}{2}}\right\}.$$

In the region exterior to r_+ the Hawking temperature is

$$T = \frac{f'(r_+)}{4\pi} = \frac{(r_+^2 - r_-^2)\hbar}{2\pi L^2 r_+}.$$

Now Wick rotate to Euclidean time $t \to -it_E$, $J \to iJ_E$ while $r_+ \to r_{E,+}$ and $r_- \to ir_{E,-}$ with

$$r_{E,\pm}^2 = 4G_N M L^2 \left\{ \left[1 + \left(\frac{J_E}{ML}\right)^2\right]^{\frac{1}{2}} \pm 1 \right\}.$$

Then it is useful to define the complex variable

$$\tau = \frac{r_{E,-} + i r_{E,+}}{L}$$

parameterising the upper-half complex plane, since $Im(\tau) > 0$, in terms of which the inverse Hawking temperature can be written

$$\frac{1}{2\pi T} = \frac{r_{E,+}}{(r_{E,+}^2 + r_{E,-}^2)} \frac{L^2}{\hbar} = \frac{L}{\hbar} Im\left(-\frac{1}{\tau}\right).$$

The partition function, including quantum corrections to all orders in perturbation theory (but not including non-perturbative corrections), was obtained in [9] and is most easily written in terms of $q = e^{2\pi i\tau}$ as

$$Z_{BTZ} = (q\bar{q})^{-\frac{L}{16\hbar G_N}} \prod_{n=2}^{\infty} |1 - q^n|^{-2}.$$

Having described the more general case we now restrict again to $J = 0$. Then $T = \frac{r_+}{2\pi}\frac{\hbar}{L^2}$, $\tau = i\frac{r_+}{L}$ and $q = e^{-4\pi^2 \frac{LT}{\hbar}}$. The $J = 0$ partition function then reads

$$Z_{BTZ} = e^{\frac{\pi^2 T L^2}{2\hbar^2 G_N}} \prod_{n=2}^{\infty} \left(1 - e^{-4\pi^2 n \frac{TL}{\hbar}}\right)^{-2}.$$

In terms of $x = \frac{TL}{\hbar} = \frac{r_+}{2\pi L}$, the Gibbs free energy is

$$G(T,P) = -T \ln Z_{BTZ} = -\frac{\pi^2 x^2}{2G_N} + 2T \sum_{n=2}^{\infty} \ln\left(1 - e^{-4\pi^2 nx}\right).$$

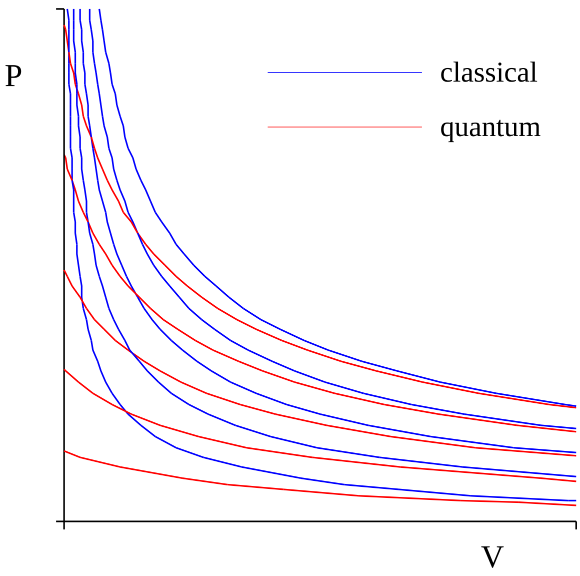
One finds that quantum corrections reduce the entropy below the Bekenstein-Hawking value,

$$S < \frac{1}{4} \times \text{ area}.$$

From the equation of state, expressed as $V(T,P)$, we obtain the “quantum volume”

$$V(T,P) = \left.\frac{\partial G}{\partial P}\right|_T = \pi r_h^2 \left[1 - 8\pi G_N \frac{\hbar}{L} \sum_{n=2}^{\infty} \frac{n}{e^{4\pi^2 nx} - 1}\right],$$

which is lower than the naïve geometrical volume.



4. Higher dimensional black holes

The case of higher dimensional black-holes proceeds as in section §2, with minor modifications. The line element is

$$d^2s = -f(r)dt^2 + f^{-1}(r)dr^2 + r^d d^2\Omega_{(d)},$$

where $\Omega_{(d)} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the volume of a d-dimensional unit sphere. In fact this can be generalised to let the event horizon be any d-dimensional Einstein space

$$R_{ij} = \frac{R}{d} g_{ij} \qquad\qquad i,j = 1,\dots,d$$

with constant Ricci curvature R (positive, negative or zero) and unit radius volume $\Omega_{(d)}$ (e.g. flat torus, or $CP^{\frac{d}{2}}$ for even d). Then

$$f(r) = \frac{R}{d(d-1)} - \frac{16\pi G_N}{\Omega_{(d)}\,d}\frac{M}{r^{d-1}} - \frac{2\Lambda}{d(d+1)} r^2$$

is a solution of the $(d+2)$-dimensional Einstein equations with Cosmological constant Λ. In terms of the Planck length, $\ell_{Pl}^d = \hbar G_N$, the Bekenstein-Hawking entropy is

$$S = \frac{\Omega_{(d)}}{4}\frac{r_h^d}{\ell_{Pl}^d}, \qquad P = -\frac{\Lambda}{8\pi G_N}.$$

Now identifying the mass with the enthalpy, $H(S,P) = M$, leads to

$$H(S,P) = \frac{\hbar S}{4\pi}\left\{ \frac{R}{d-1}\left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{-\frac{1}{d}} + \frac{16\pi G_N P}{d+1}\left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{\frac{1}{d}} \right\},$$

giving thermodynamic volume

$$V = \left(\frac{\partial H}{\partial P}\right)_S = \frac{\Omega_{(d)} r_h^{d+1}}{d+1},$$

which is the naïve geometrical result The equation of state is

$$T = \frac{\hbar}{4\pi d}\left\{R\left(\frac{(d+1)V}{\Omega_{(d)}}\right)^{-\frac{1}{d+1}} + 16\pi G_N P\left(\frac{(d+1)V}{\Omega_{(d)}}\right)^{\frac{1}{d+1}}\right\},$$

and the heat capacity is

$$C_P = Sd\left\{\frac{16\pi G_N P\left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{\frac{2}{d}} + R}{16\pi G_N P\left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{\frac{2}{d}} - R}\right\}.$$

For $R > 0$ there is a Hawking-Page phase transition at

$$T = \frac{2\hbar}{d}\sqrt{\frac{R_{(d)} G_N P}{\pi}}.$$

For flat a event horizon, $R = 0$, the heat capacity is the particularly simple expression

$$C_P = Sd.$$

5. Conclusions

It has been argued that a Cosmological constant induces a $P\,dV$ term in the first law of black hole thermodynamics. The black hole mass is then most naturally identified with the **enthalpy**, $H(S,P)$:

$$dM = dH = T\,dS + V\,dP,$$

where $P = -\frac{\Lambda}{8\pi G_N}$. The Gibbs free energy is $G(T,P) = -T\ln Z$.

This identification gives the correct Hawking temperature from the standard thermodynamic relation $T = \left(\frac{\partial H}{\partial S}\right)_P$ and allows one to define a "thermodynamic volume" $V = \left(\frac{\partial H}{\partial P}\right)_T$, which, classically, agrees with the naïve geometrical result, $\frac{4}{3}\pi r_h^3$ in $(3+1)$-dimensions. From the specific example of the BTZ black-hole in $(2+1)$-dimensions one expects quantum gravity corrections to the Bekenstein-Hawking entropy and the thermodynamic volume.

The heat capacity at constant pressure is $C_P = \frac{T}{\left.\frac{\partial T}{\partial S}\right|_P}$ but this, and the equation of state $V(T,P)$, will also get quantum corrections.

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