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On twisted symmetries and quantum mechanics with a magnetic field on noncommutative tori

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We study the twist-induced deformation procedure of a torus $\mathbb{T}^n$ and of quantum mechanics of a scalar charged quantum particle on $\mathbb{T}^n$ in the presence of a magnetic field B.
We first summarize our recent results regarding the equivalence of the undeformed theory on $\mathbb{T}^n$ to the analogous one on $\mathbb{R}^n$ subject to a *quasiperiodicity* constraint: we describe the sections of the associated hermitean line bundle on $\mathbb{T}^n$ as wavefunctions $\psi \in C^\infty(\mathbb{R}^n)$ periodic up to a suitable phase factor V depending on B and require the covariant derivative components ∇_a to map the space $\mathscr{X}^V$ of such ψ's into itself. The ∇_a corresponding to a constant B generate a Lie algebra $\mathbf{g}_Q$ and together with the periodic functions the algebra $\mathscr{O}_Q$ of observables. The non-abelian part of $\mathbf{g}_Q$ is a Heisenberg Lie algebra with the electric charge operator Q as the central generator; the corresponding Lie group G_Q acts on the Hilbert space as the translation group up to phase factors. The unitary irreducible representations of $\mathscr{O}_Q, Y_Q$ corresponding to integer charges are parametrized by a point in the reciprocal torus.
We then apply the $\star$-deformation procedure induced by a Drinfel'd twist $\mathscr{F} \in U\mathbf{g}_Q \otimes U\mathbf{g}_Q$, sticking for simplicity to abelian twists, to the symmetry Hopf algebra $U\mathbf{g}_Q$, to the algebra $\mathscr{X}$ of functions on $\mathbb{T}^n$ and to $\mathscr{O}_Q$ in a gauge-independent way, to $\mathscr{X}^V$ and to the action of $\mathscr{O}_Q$ on the latter in a specific gauge. $\mathscr{X}^V, \mathscr{O}_Q$ are 'rigid', i.e. isomorphic to $\mathscr{X}^V_\star, \mathscr{O}_{Q\star}$, although $\mathscr{X}$ and $\mathscr{X}_\star$ are *not* isomorphic and therefore $\mathscr{X}^V_\star$ as a $\mathscr{X}_\star$-bimodule is not isomorphic to the $\mathscr{X}$-bimodule $\mathscr{X}^V$.

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1. Introduction

The formulation on noncommutative spaces of quantum field theories, especially of the gauge type, is a major challenge in present research in mathematical and theoretical physics. A very powerful tool at hand is deformation quantization by *Drinfel'd twists* $\mathscr{F}$, which aims at building at the same time noncommutative deformations of a space(time) manifold X, of quantum theories on X and of their symmetries. Here we apply it to quantum mechanics of a single scalar particle on a manifold with nontrivial topology, a n-torus $\mathbb{T}^n$, in the presence of a $U(1)$-gauge field A with nonvanishing integral Chern numbers (i.e. fluxes of the associated field strength B). This can be considered a necessary preliminary step towards quantum field theory, independently of the approach we choose to reach the latter (path-integral as e.g. in [6], second quantization [10], etc.).

Calling λ the deformation parameter, deformation quantization [1] of an algebra $\mathscr{A}$ (over $\mathbb{C}$, say) into a new one $\mathscr{A}_\star$ means that the two have the same underlying vector space over the ring $\mathbb{C}[[\lambda]]$ of power series in λ, $V(\mathscr{A}_\star)=V(\mathscr{A})[[\lambda]]$, but the product $\star$ of $\mathscr{A}_\star$ is a deformation of the product $\cdot$ of $\mathscr{A}$. For instance, on the algebra $\mathscr{X}$ of smooth functions on a manifold X, as well as on the algebra of differential operators on $\mathscr{X}$, $f\star h$ can be defined by

$$f\star h := \cdot\circ\left[\overline{\mathscr{F}}(\triangleright\otimes\triangleright)(f\otimes h)\right], \tag{1.1}$$

where $\overline{\mathscr{F}}$ is a bi-pseudodifferential operator depending on the deformation parameter λ so that $\star$ is associative and reduces to $\cdot$ when $\lambda=0$. If one replaces all $\cdot$ by $\star$'s in an equation of motion, e.g. in the Schrödinger equation of a particle with electrical charge q

$$\mathsf{H}_\star\psi(x)=i\hbar\partial_t\psi(x), \qquad \mathsf{H}_\star := \left[\tfrac{1}{2m}\nabla^a\star\nabla_a+\mathrm{V}\right]\star, \qquad \nabla_a=-i\partial_a+qA_a, \tag{1.2}$$

one obtains a pseudodifferential equation and therefore introduces a (very special) non-locality in the interactions. [Interest in the latter can motivate the reader to study the effect of $\star$-products even if he/she is not ready to interpret noncommutative coordinates as physical observables of position]. Here and in the sequel we use natural units, so that $\hbar=1=c$, and absorb the positron charge e in the definition of A; then the quantization of charge reads $q\in\mathbb{Z}$. The undeformed differential equation $\mathsf{H}\psi=i\partial_t\psi$ is recovered for $\lambda=0$. One of the simplest examples is the Grönewold-Moyal-Weyl $\star$-product on $\mathbb{R}^n$, i.e. (1.1) with $f,h\in\mathscr{C}^\infty(\mathbb{R}^n)$ and

$$\overline{\mathscr{F}}\equiv\sum_I\overline{\mathscr{F}}_I^{(1)}\otimes\overline{\mathscr{F}}_I^{(2)} := \exp\left[\frac{i}{2}\theta^{ab}\partial_a\otimes\partial_b\right], \qquad \theta^{ab}:=\lambda\vartheta^{ab}, \tag{1.3}$$

where $a,b=1,...,n$, $\partial_a=\partial/\partial x^a$), and ϑ^{ab} is a fixed real antisymmetric matrix. Given a lattice $\Lambda\subset\mathbb{R}^n$ of rank n, (1.1) & (1.3) can be used also to deform the product in the algebra $\mathscr{X}=\mathscr{C}^\infty(\mathbb{T}^n)$ of smooth functions on the torus $\mathbb{T}^n=\mathbb{R}^n/\Lambda$, which can be identified with that of functions f on $\mathbb{R}^n$ periodic under translation by a $\lambda\in\Lambda$. For simplicity we shall assume $\Lambda=2\pi\mathbb{Z}^n$, i.e. $f(x+2\pi l)=f(x)$ for all $l:=(l_1,...,l_m)\in\mathbb{Z}^n$, or equivalently f is a function (Laurent series) of $u\equiv(u^1,...,u^n)\equiv(e^{ix^1},...,e^{ix^n})$ only; so the reciprocal lattice is $\mathbb{Z}^n$ [1]. Then (1.1) is Connes-Rieffel

[1] Since $\Lambda=2\pi\mathbb{Z}^n$ can be always be obtained by a linear transformation of $\mathbb{R}^n$ [$x\mapsto gx$, $g\in GL(n)$], this is no loss of generality, as we are not concerned with holomorphic algebra of functions, holomorphic line bundles, etc; in fact, if $n=2m$ and we regard $\mathbb{T}^n=\mathbb{R}^n/\Lambda$ as a complex m-torus then the holomorphic structure w.r.t. complex variables $z^j=x^j+ix^{m+j}$ is *not* invariant under $x\mapsto gx$ for generic $g\in GL(n)$. See also the end of section 2.2.

$\star$-product. Better definitions of $\star$ involve the Fourier transforms/series of f,h and will be recalled later. In either case the ∂_a generate translations and belong to the Lie algebra $\mathbf{g}_0$ of the group G_0 of symmetries of X. As the twist $\mathscr{F} \equiv \overline{\mathscr{F}}^{-1}$ belongs to $U\mathbf{g}_0 \otimes U\mathbf{g}_0[[\lambda]]$, it determines also a deformation $H \rightsquigarrow \hat{H}$ of the Hopf algebra $H = U\mathbf{g}_0$, so that $\widehat{\mathscr{X}} \equiv \mathscr{X}_\star$ is a $\hat{H}$-module algebra, as $\mathscr{X}$ was a H-module algebra: the space symmetries are preserved, although in a deformed form.

As known, if B has non-vanishing integral Chern numbers the (smooth) states of a charged (for simplicity scalar) particle on $\mathbb{T}^n$ have to be represented by wavefunctions in the space $\Gamma(\mathbb{T}^n, E)$ of sections of the associated hermitean line bundle $E \overset{\pi}{\mapsto} \mathbb{T}^n$, rather than in $\mathscr{X}$. But as the patches of any trivialization of E are not mapped into themselves by translations, $\Gamma(\mathbb{T}^n, E)$ and any isomorphic (by the Serre-Swan theorem [14]) finitely generated projective $\mathscr{X}$-module $e\mathscr{X}^m$ [here $m \in \mathbb{N}$, $e \in M_m(\mathscr{X})$ is a projector] are not $U\mathbf{g}_0$-modules. Therefore we cannot apply the standard $\star$-deformation $e\mathscr{X}^n \rightsquigarrow e_\star \mathscr{X}_\star^n$ choosing $H = U\mathbf{g}_0$. The way out is based on our recent results [11], which we summarize in section 2. Describing $\Gamma(\mathbb{T}^n, E)$ as a subspace $\mathscr{X}^V$ of $C^\infty(\mathbb{R}^n)$ whose elements are periodic up to a suitable phase factor V, we have shown that $\Gamma(\mathbb{T}^n, E)$ is a module of a *central extension* of G_0 that we call the *projective translation group* G_Q; the central generator in the Lie algebra $\mathbf{g}_Q$ is the electric charge operator Q. This is the analog in the smooth framework of well-known facts in the holomorphic one (see e.g. [2]). The ∇_a belongs to the *algebra of observables* $\mathscr{O}_Q \supset \mathscr{X}$ on $\mathscr{X}^V$; $\mathscr{O}_Q$ is a G_Q-module transforming under the adjoint action of G_Q. The gauge transformations of $\Gamma(\mathbb{T}^n, E)$ are described by those of $\mathscr{X}^V$. The irreducible unitary representations of $Y_Q, \mathscr{O}_Q$ with $Q = q \in \mathbb{Z}$ are parametrized by a point on the reciprocal torus $\mathbb{R}^n/\mathbb{Z}^n$.

In the remaining sections we deform $H = U\mathbf{g}_Q, \mathscr{X}, \mathscr{X}^V, \mathscr{O}_Q, \ldots$ by a twist $\mathscr{F} \in U\mathbf{g}_Q \otimes U\mathbf{g}_Q$. For simplicity we stick to twists of *Reshetikhin* (i.e. *abelian*) type; the corresponding deformations $\mathscr{X}_\star$ are only a subset of the possible Connes-Rieffel noncommutative tori. We describe the twist-induced deformations $H \rightsquigarrow \hat{H}$ of a cocommutative Hopf $*$-algebra in section 3.1, of its modules and module $*$-algebras in section 3.2. In section 4 we apply them to $\mathscr{X}, \mathscr{O}_Q, \mathscr{X}^V, \ldots$ and obtain $\hat{H}$-module $*$-algebras $\mathscr{X}_\star, \mathscr{O}_{Q\star}, \ldots$ and a $\hat{H}$-equivariant $\mathscr{X}_\star$-bimodule and left $\mathscr{O}_{Q\star}$-module $\mathscr{X}_\star^V$, which is completed into a Hilbert space. We also determine the *deforming map* $D_{\mathscr{F}} : \mathscr{O}_{Q\star} \leftrightarrow \mathscr{O}_Q[[\lambda]]$, a $\hat{H}$-module $*$-algebra isomorphism, which simplifies the study of the deformed representation theory: $\mathscr{X}^V, \mathscr{O}_Q$ are 'rigid' (their deformation boils down to a change of generators on the same representation space), i.e. there are isomorphisms $\mathscr{X}_\star^V \simeq \mathscr{X}^V$, $\mathscr{O}_{Q\star} \simeq \mathscr{O}_Q$, although $\mathscr{X}$ and $\mathscr{X}_\star$ are *not* isomorphic and therefore $\mathscr{X}_\star^V$ as a $\mathscr{X}_\star$-bimodule is not isomorphic to the $\mathscr{X}$-bimodule $\mathscr{X}^V$.

We shall use the following abbreviations. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $X \equiv \mathbb{T}^n$; M^t stands for the transpose of matrix M; elements $h, k \in \mathbb{C}^n$ are considered as columns; $h \cdot k := h^t k$ (at the rhs the product is row by column); $u^l := e^{il\cdot x}$; $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$; we denote as $V(\mathscr{A}), \mathscr{Z}(\mathscr{A})$ resp. the vector space underlying an algebra $\mathscr{A}$, the center of $\mathscr{A}$; $[a \overset{\star}{,} b] := a \star b - b \star a$; $a \wedge b := a \otimes b - b \otimes a$. We stick to linear spaces and algebras over $\mathbb{C}$ or the ring $\mathbb{C}[[\lambda]]$ of formal power series in λ with coefficients in $\mathbb{C}$. We shall often change notation: $\mathscr{A}_\star \mapsto \widehat{\mathscr{A}}$, $\mathscr{X}_\star \mapsto \hat{\mathscr{X}}$, $\mathscr{O}_{Q\star} \mapsto \hat{\mathscr{O}}_Q$, $\mathscr{X}_\star^V \mapsto \widehat{\mathscr{X}}^V$ $x^a \star \mapsto \hat{x}^a$, $u^a \star \mapsto \hat{u}^a$, $\partial_a \star \mapsto \hat{\partial}_a$, $p_a \star \mapsto \hat{p}_a$, $a_i^+ \star \mapsto \hat{a}_i^+$, etc (*'hat notation'*). In the new notation e.g. (1.2) becomes

$$\{\tfrac{1}{2m}[-i\hat{\partial}_a + q\hat{A}_a(\hat{x})][-i\hat{\partial}_a + q\hat{A}_a(\hat{x})] + \hat{\mathrm{V}}(\hat{u})\}\hat{\psi}(\hat{x}) = E\,\hat{\psi}(\hat{x});$$

here $\hat{V} = \wedge(V)$, $\hat{A}_a = \wedge(A_a)$, $\hat{\psi} = \wedge(\psi)$ and $\wedge$ is the *generalized Weyl map* (section 4). The pseudodifferential eq. (1.2) has thus become a noncommutative differential equation of second order (i.e. of second degree in $\hat{\partial}_a$). Solving the latter may be considerably simpler.

PoS(CNCFG2010)018

2. The undeformed theory

2.1 Quasiperiodic wavefunctions and related connections on $\mathbb{R}^n$

The particle probability density $|\psi|^2$ is periodic, i.e. invariant under discrete translations $\lambda \in \Lambda$, if ψ is quasiperiodic, i.e. invariant up to a phase factor V. A set of quasiperiodicity conditions of the form

$$\psi(x+2\pi l) = V(l,x)\,\psi(x) \qquad \forall x \in \mathbb{R}^n, \quad l \in \mathbb{Z}^n \tag{2.1}$$

relates the values of ψ in any two points $x, x+2\pi l$ of the lattice $x+2\pi\mathbb{Z}^n$ through a phase factor $V(l,x)$. Nontrivial solutions ψ of (2.1) may exist only if the factors relating three generic points $x, x+2\pi l, x+2\pi(l+l')$ of the lattice are consistent with each other, i.e.

$$V(l+l',x) = V(l,x+2\pi l')\,V(l',x), \qquad \forall\, l,l' \in \mathbb{Z}^n. \tag{2.2}$$

Note that this implies $V(0,x) \equiv 1$ and $[V(l,x)]^{-1} = V(-l,x+2\pi l)$. We introduce an auxiliary Hilbert space $\mathscr{H}_Q$ with an orthonormal basis $\{|q\rangle\}_{q\in\mathbb{Z}}$ and on $\mathscr{H}_Q$ a self-adjoint operator Q by $Q|q\rangle = q|q\rangle$. Given a smooth function $V : \mathbb{Z}^n \times \mathbb{R}^n \mapsto U(1)$ fulfilling (2.2) we introduce the space

$$\mathscr{X}^V := \{\psi \in C^\infty(\mathbb{R}^n)\otimes|q\rangle \quad | \quad \psi(x+2\pi l) = V(l,x)\,\psi(x) \qquad \forall x \in \mathbb{R}^n,\, l \in \mathbb{Z}^n\} \tag{2.3}$$

as the space of smooth wavefunctions of a particle with electric charge q (in e units), since it is an eigenspace with eigenvalue q of $\mathbf{1}\otimes Q$, which we adopt as the electric charge operator. We give the covariant derivative a form independent of q through $\nabla := (-i)d\otimes\mathbf{1}+A(x)\otimes Q$; here d stands for the exterior derivative. We shall abbreviate $\nabla = -id+A(x)Q$, $\psi\in C^\infty(\mathbb{R}^n)|q\rangle$, etc. The components of ∇ have to map $\mathscr{X}^V$ into itself,

$$\nabla_a : \mathscr{X}^V \mapsto \mathscr{X}^V. \tag{2.4}$$

Given such a ∇, also $QB_{ab}(x)\psi(x) = \{\frac{i}{2}[\nabla_a,\nabla_b]\psi\}(x)$ fulfills (2.1), implying that all the $B_{ab} = \frac{1}{2}(\partial_a A_b - \partial_b A_a)$ are periodic functions. From the Fourier expansions it follows

$$B_{ab}(x) = \beta^{A}_{ab} + \underbrace{\sum_{l\neq\mathbf{0}} \beta^l_{ab} e^{il\cdot x}}_{B'_{ab}(x)} \qquad \Rightarrow \qquad A_a(x) = x^b\beta^{A}_{ba} + \alpha_a + \underbrace{\sum_{l\neq\mathbf{0}} \alpha^l_a e^{il\cdot x}}_{A'_a(x)} + \text{gauge transf.}, \tag{2.5}$$

where $\mathbf{0} := (0,..,0) \in \mathbb{Z}^n$ and the periodic function $A'(x)$ is such that $B' = dA'$. We decompose the covariant derivative in a gauge-independent part $A'_a Q$ and a gauge-dependent part p_a:

$$\nabla_a := -i\partial_a + A_a Q = p_a + A'_a Q, \qquad A'_a \in \mathscr{X}. \tag{2.6}$$

Going back to (2.4), $\nabla_a\psi$ will fulfill (2.1) iff also $p_a\psi$ does, by the periodicity of $A'_a(x)$; up to a gauge transformation this implies the first formula in

$$V(l,x) \equiv V^{\beta^A}(l,x) := e^{-iq2\pi l^t\beta^A x}, \qquad p_a = -i\partial_a + x^b\beta^{A}_{ba}Q + \alpha_a Q, \tag{2.7}$$

which is consistent with (2.2) for all eigenvalues $q \in \mathbb{Z}$ of Q iff the quantization conditions

$$\nu_{ab} \in \mathbb{Z}, \qquad \nu_{ab} := 2\pi\beta^{A}_{ab} \tag{2.8}$$

for all a,b are satisfied. For $q\beta^{A}=0$ we find $V\equiv 1$ and $\mathscr{X}^1=\mathscr{X}\otimes|0\rangle\simeq\mathscr{X}$. Otherwise (2.1), $(2.7)_1$ do not admit solutions of the form $\psi(x)=e^{ik\cdot x}f(x),\ f\in\mathscr{X}$.

For all $f\in\mathscr{X}$ $Q,p_a,f\cdot,\nabla_a,\mathsf{H}$ map $\mathscr{X}^V$ into itself; they belong to the $*$-*algebra of observables* $\mathscr{O}_Q\equiv$ algebra of polynomials in $Q,p_1,...,p_m$ with coefficients f in $\mathscr{X}$, constrained by

$$\begin{aligned} &[p_a,p_b]=-i2\beta^{A}_{ab}Q, && [Q,\cdot\,]=0, && [p_a,f]=-i(\partial_a f),\\ &f^*(x)=\overline{f(x)}, && p_a^*=p_a, && Q^*=Q. \end{aligned} \tag{2.9}$$

These relations defining $\mathscr{O}_Q$ depend on the A_a only through the β^{A}_{ab} of (2.8), in particular are gauge-independent. Q,p_a generate the real Lie algebra $\mathbf{g}_Q$ of a Lie group G_Q. $\mathscr{O}_Q$ and $\mathscr{X}$ are $U\mathbf{g}_Q$-module $*$-algebras under the action

$$p_a\triangleright p_b=-i2\beta^{A}_{ab}Q,\qquad p_a\triangleright f=-i(\partial_a f),\qquad Q\triangleright f=0,\qquad Q\triangleright p_a=0, \tag{2.10}$$

for all $f\in\mathscr{X}$, and $\mathscr{X}^V$ is a left $U\mathbf{g}_Q$-equivariant $\mathscr{O}_Q$-module and $\mathscr{X}$-bimodule (but not an algebra, unless $V\equiv 1$); this means that all these structures are compatible with each other and the Leibniz rule[2]. The Weyl forms of $x^{a*}=x^a$, (2.9) and of their consequences $[p_a,x^b]=-i\delta^b_a$ are easily determined with the help of the Baker-Campbell-Hausdorff (BCH) formula and synthetically read

$$\begin{aligned} &e^{i(h\cdot x+p\cdot y+Qy^0)}e^{i(k\cdot x+p\cdot z+Qz^0)}=e^{i\left[(h+k)\cdot x+p\cdot(y+z)+Q(y^0+z^0)\right]}\,e^{-\frac{i}{2}\left[k\cdot y-h\cdot z+2Qy^t\beta^A z\right]}\\ &\left[e^{i(h\cdot x+p\cdot y+Qy^0)}\right]^*=e^{-i(h\cdot x+p\cdot y+Qy^0)} \end{aligned} \tag{2.11}$$

for any $h,k\in\mathbb{R}^n$ and $(y^0,y),(z^0,z)\in\mathbb{R}^{n+1}$. We define G_Q and other groups C_Q,R,Y_Q,T by

$$\begin{aligned} &G_Q:=\left\{g_{(z^0,z)}:=e^{i(p\cdot z+Qz^0)}\mid (z^0,z)\in\mathbb{R}^{n+1}\right\}, && \text{“projective translation group”}\\ &R:=\{e^{i(h^0+h\cdot x)}\mid (h^0,h),(z^0,z)\in\mathbb{R}^{n+1}\}, && T:=\{e^{i(h^0+l\cdot x)}\mid h^0\in\mathbb{R},\ l\in\mathbb{Z}^n\},\\ &Y_Q:=\{e^{i(h^0+l\cdot x+p\cdot z+Qz^0)}\mid h^0\in\mathbb{R},\ l\in\mathbb{Z}^n,\ (z^0,z)\in\mathbb{R}^{n+1}\}, && \text{“ observables' group”}\\ &C_Q:=\{e^{i(h^0+h\cdot x+p\cdot z+Qz^0)}\mid (h^0,h),(z^0,z)\in\mathbb{R}^{n+1}\}; \end{aligned} \tag{2.12}$$

the group law can be read off (2.11) and depends on A only through β^A. $c^*=c^{-1}$ for all $c\in Y_Q,R$. The inclusions $G_Q,T\subset Y_Q$ and $T\subset R$ hold as subgroup inclusions. R is isomorphic to $\mathbb{R}^n\times U(1)$. $T\sim\mathbb{Z}^n\times U(1)$ is a normal subgroup of Y_Q, and $Y_Q=G_Q\ltimes T$. Moreover, we shall call $\mathscr{Y}_Q$ the group algebra of Y_Q; it is a C^*-algebra. All $y\in Y_Q$ and $o\in\mathscr{O}_Q$ map $\mathscr{X}^V$ into itself. The $f\in T,\mathscr{X}$ act by multiplication, while in the gauge (2.7) $Q,p_a\in\mathbf{g}_Q$ and $g\in G_Q$ act as follows:

$$\begin{aligned} &Q\triangleright|q\rangle=q|q\rangle,\qquad Q\triangleright\psi=q\psi,\qquad p_a\triangleright|q\rangle=(qx^t\beta^A+q\alpha)_a|q\rangle,\qquad f\triangleright\psi=f\psi,\\ &p_a\triangleright\psi=(-i\partial+qx^t\beta^A+q\alpha)_a\psi,\qquad [g_{(z^0,z)}\triangleright\psi](x)=e^{iq\left[z^0+x^t\beta^A z+\alpha^t z\right]}\psi(x+z); \end{aligned} \tag{2.13}$$

$g_{\tilde z}$ acts shifting the argument by z and by multiplication by a phase factor, whence the name *projective translation group*. Let $r:=\frac{1}{2}\text{rank}(\beta^A)$; it is $r\in\mathbb{N}_0$. By the Frobenius theorem $\exists$ a matrix S with $S_{ab}\in\mathbb{Z}$, $\det S=\pm 1$ such that after the change of generators

$$p_a\mapsto(S^tp)_a,\qquad x^a\mapsto(S^{-1}x)^a,\qquad\Rightarrow\qquad u^l\mapsto u^{(S^{-1})^t l}, \tag{2.14}$$

[2]Namely, for all $c\in\mathscr{O}_Q$, $\psi\in\mathscr{X}^V$, $f\in\mathscr{X}$, $g\in\mathbf{g}_Q$ $g\triangleright(c\psi f)=(g\triangleright c)\psi f+c(g\triangleright\psi)f+c\psi(g\triangleright f)$.

resp. in $\mathbf{g}_Q$, $C^\infty(\mathbb{R}^n)$ and $\mathscr{X}$, the commutation relations $[x^a, p_b] = i\delta^a_b$ remain true, while $(2.9)_1$ become

$$[p_j, p_{r+j}] = ib_jQ \qquad j = 1,...,r, \qquad\qquad [p_a, p_b] = 0 \qquad \text{otherwise}, \tag{2.15}$$

where $\nu_j := 2\pi b_j \in \mathbb{Z}$ and fulfill $\nu_{j+1}/\nu_j \in \mathbb{N}$. This shows that

$$\mathbf{g}_Q \simeq \mathbf{h}_{Q2r+1} \oplus \mathbb{R}^{n-2r}, \qquad\qquad G_Q \simeq \mathbf{H}_{Q2r+1} \times \mathbb{R}^{n-2r}. \tag{2.16}$$

where $\mathbf{h}_{Qk}$,$\mathbf{H}_{Qk}$ denote the Heisenberg Lie algebra, group of dimension k and central generator Q.

Introducing fundamental k-dimensional cells $C^y_{a_1...a_k}$ for $k \le n$ and $a_1 < a_2 < .... < a_k$ by

$$C^y_{a_1...a_k} := \{x \in \mathbb{R}^n \mid x^{a_h} \in [y^{a_h}, y^{a_h} + 2\pi[,\ h = 1,...,k;\ x^a = y^a \text{ otherwise}\}, \tag{2.17}$$

one easily finds that the flux ϕ_{ab} of $B = B_{ab}dx^a dx^b$ through a plaquette C^y_{ab} equals that of $\tilde{\beta}^A = \beta^A_{ab}dx^a dx^b$

$$\phi_{ab} = \int_{C^y_{ab}} B = \int_{C^y_{ab}} \tilde{\beta}^A = 2\pi\nu_{ab} \tag{2.18}$$

and similarly for higher powers B^m. By (2.1) $\psi'^*\psi$ is periodic for all $\psi', \psi \in \mathscr{X}^V$, and the formula

$$(\psi', \psi) := \int_{C^y_{1...n}} d^n x\, \overline{\psi'(x)}\, \psi(x), \tag{2.19}$$

defines a hermitean structure in $\mathscr{X}^V$ making the latter a pre-Hilbert space. (The results are independent of y.) As $p_a \triangleright (\psi'^*\psi) \equiv p_a(\psi'^*\psi) = -i\partial_a(\psi'^*\psi)$, which has a vanishing integral, by the Leibniz rule the p_a are essentially self-adjoint. If some $\psi_0 \in \mathscr{X}^V$ vanishes nowhere, then $\psi\psi_0^{-1}$ is well-defined and periodic, i.e. in $\mathscr{X}$, for all $\psi \in \mathscr{X}^V$, whence the decomposition $\mathscr{X}^V = \mathscr{X}\,\psi_0$. We shall call $\mathscr{H}^V$ the Hilbert space completion of $\mathscr{X}^V$. Y_Q extends as a group of unitary transformations of $\mathscr{H}^V$; $f \in T$ still act by multiplication, $g_{(z^0 \underline{z})} \in G_Q$ in the above gauge still acts as in $(2.13)_6$. We shall call $\left(\rho^{\beta^A}(\mathscr{O}_Q), \mathscr{X}^{\beta^A}\right)$ and $\left(\rho^{\beta^A}(Y_Q), \mathscr{H}^{\beta^A}\right)$ the representations that we have used so far, determined by $\rho^{\beta^A}(o)\psi := o \triangleright \psi$ with action $\triangleright$ defined by (2.7-2.13).

Given a representation $\left(\rho(\mathscr{O}_Q), \mathscr{X}^V\right)$ of $\mathscr{O}_Q$ as a $*$-algebra of operators on $\mathscr{X}^V$, a unitary equivalent one is obtained through a smooth gauge transformation $U = e^{iq\varphi}$, $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$, acting as a unitary transformation $\left(\rho(\mathscr{O}_Q), \mathscr{X}^V\right) \mapsto \left(\rho^U(\mathscr{O}_Q), \mathscr{X}^{V^U}\right)$, with

$$\rho^U(o) = U\rho(o)U^{-1}, \qquad \psi^U = U\psi, \qquad V^U(l,x) = U(x + 2\pi l)V(l,x)U^{-1}(x). \tag{2.20}$$

U is a unitary transformation $\left(\rho(Y_Q), \mathscr{H}^V\right) \mapsto \left(\rho^U(Y_Q), \mathscr{H}^{V^U}\right)$ also for the associated representatrion of Y_Q as a group of unitary operators on the Hilbert space completion. All the relations (2.1-2.6), (2.8-2.12), (2.14-2.19) remain valid. Starting from $\left(\rho^{\beta^A}(\mathscr{O}_Q), \mathscr{X}^{\beta^A}\right)$, choosing $U(x) = e^{i\frac{q}{2}x^t\beta^S x}$ and setting $\beta := \beta^A + \beta^S$, we find an equivalent representation $\left(\rho^U(\mathscr{O}_Q), \mathscr{X}^{V^U}\right)$ characterized by

$$V^U(l,x) = e^{-iq2\pi l^t\beta(x + l\pi)}, \qquad\qquad p_a = -i\partial_a + x^b\beta_{ba}Q + \alpha_a Q \tag{2.21}$$

[for $U(x) \equiv 1$, i.e. $\beta = \beta^A$, we recover the original gauge (2.7)]. We shall adopt the shorter notations $\mathscr{X}^\beta \equiv \mathscr{X}^{\beta^{A^U}}$, $\mathscr{H}^\beta \equiv \mathscr{H}^{\beta^{A^U}}$, etc. for the spaces of complex functions fulfilling (2.1) with V given by (2.21). Performing a change (2.14) and choosing β^S so that β becomes lower-triangular we find

$$\beta^A \overset{(2.14)}{\mapsto} \overline{\beta}^A := \begin{pmatrix} & -b & \\ b & & \\ & & 0_{n-2r} \end{pmatrix} \qquad \Rightarrow \qquad \beta \overset{(2.14)}{\mapsto} \overline{\beta} = \begin{pmatrix} & 0_r & \\ 2b & & \\ & & 0_{n-2r} \end{pmatrix} \tag{2.22}$$

(0_k is the $k\times k$ zero matrix; the missing blocks are zero matrices of the appropriate sizes), and (2.1) becomes

$$\psi(x+2\pi l)=e^{-i2q\sum_{j=1}^r \nu_j l_{r+j}x^j}\,\psi(x) \qquad \forall x\in\mathbb{R}^n,\, l\in\mathbb{Z}^n. \tag{2.23}$$

The most general solution of (2.23) reads [11]

$$\psi(x)=\sum_{k\in K}\sum_{l\in\mathbb{Z}^r} e^{i\sum_{j=1}^r(k_j+2q\nu_j l_j)x^j}\,\psi_k(x^{r+1}+2\pi l_1,\dots,x^{2r}+2\pi l_r,x^{2r+1},\dots,x^n), \tag{2.24}$$

where all ψ_k belong resp. to $\mathscr{S}\left(\mathbb{R}^r\times\mathbb{T}^{n-2r}\right)$, $\mathscr{L}^2\left(\mathbb{R}^r\times\mathbb{T}^{n-2r}\right)$ if $\psi\in\mathscr{X}^\beta,\mathscr{H}^\beta$, and

$$K:=\{0,1,\dots,|2q\nu_1|-1\}\times\dots\times\{0,1,\dots,|2q\nu_r|-1\}\subset\mathbb{Z}^r. \tag{2.25}$$

The subspaces $\mathscr{X}_k\subset\mathscr{X}^\beta$, $\mathscr{H}_k\subset\mathscr{H}^\beta$ characterized by $\psi_s\equiv 0$ for $s\in K\setminus\{k\}$ are orthogonal to each other. In next subsection we present bases of $\mathscr{X}_k,\mathscr{X}^\beta$.

2.2 Physical representations of $Y_Q,\mathscr{O}_Q$

The physical representations of $Y_Q,\mathscr{O}_Q$ are characterized by integer eigenvalues of Q; so we consider an irreducible one with $Q=q\in\mathbb{Z}$ and drop the subscript $_Q$: $C,Y,G,g,\mathscr{O},\mathbf{h}_k,\mathbf{H}_k$. Let $\mathscr{C},\mathscr{Y}$ be the group C^*-algebras of C,Y. We abbreviate $\tilde\alpha:=q\alpha$, $\tilde\beta^A:=q\beta^A$, $\tilde\nu_j:=q\nu_j\in\mathbb{N}$, etc. All commutation relations depend only on $\tilde\beta^A$. After the Frobenius transformation $(x,p)\mapsto(S^{-1}x,S^tp)$ we let

$$m_{r+j}:=\exp[i(x^j+\pi p_{r+j}/\tilde\nu_j)], \qquad m_j:=\exp[i(x^{r+j}-\pi p_j/\tilde\nu_j)]. \tag{2.26}$$

Proposition 1. *[11] Y decomposes into a product of commuting subgroups as follows*

$$Y\;=\;M^1\dots M^r\,\mathbf{H}_3^1\dots\mathbf{H}_3^r\,Y_{2r+1}\dots Y_n, \tag{2.27}$$

- *M^j is discrete, generated by $m_j,m_{r+j},e^{\frac{i\pi}{\tilde\nu_j}},m_j^{-1},m_{r+j}^{-1}$, that fulfill $m_jm_{r+j}=m_{r+j}m_je^{\frac{i\pi}{\tilde\nu_j}}$;*
- *$\mathbf{H}_3^j:=\{e^{i(h+wp_j+zp_{r+j})}\mid(h,w,z)\in\mathbb{R}^3\}$ is isomorphic to the 3-dim Heisenberg Lie group $\mathbf{H}_3$*
- *$Y_a:=\{e^{i(lx^a+h+zp_a)}\mid l\in\mathbb{Z},\,(h,z)\in\mathbb{R}^2\}$ is isomorphic to the observables' group on a circle.*

$\zeta_j:=(m_j)^{2\tilde\nu_j}$, $\zeta_{r+j}:=(m_{r+j})^{2\tilde\nu_j}$ $(j=1,\dots,r)$ and their inverses are central;
with $e^{ih}\in U(1)$ they generate the subgroup $\mathscr{Z}(Y)\subset Y$ and the subalgebra $\mathscr{Z}(\mathscr{Y})\subset\mathscr{Y}$.

$M:=M^1..M^r$ commutes with $\mathsf{H}_0=\sum_{a=1}^n p_a^2$, so is the *magnetic translation group* in the sense of [16].

By Proposition 1 the irreducible unitary representations (briefly *irreps*) of $Y,\mathscr{O}$ for $n\geq 3$ are obtained from tensor products of those for $n=1,2$. The irreps of the C^*-algebra $\mathscr{Y}$ are those of Y.

$\mathbf{n=1}$ (quantum mechanics on a circle S^1). The Casimir eigenvalue $\zeta=e^{i2\pi\tilde\alpha}$ $(\tilde\alpha\in\mathbb{R}/\mathbb{Z})$ identifies the inequivalent irreps of $Y,\mathscr{Y}$ $(\rho_{\tilde\alpha},\mathscr{L}^2(S^1))$, with

$$\rho_{\tilde\alpha}\left[e^{ilx}\right]\psi(x)=e^{ilx}\psi(x), \qquad \rho_{\tilde\alpha}(e^{izp})\psi(x)=e^{i\tilde\alpha z}\psi(x+z). \tag{2.28}$$

The associated irrep $(\rho_{\tilde\alpha},\mathscr{C}^\infty(S^1))$ of $\mathscr{O}$ is defined by $(2.28)_1$ and $\rho_{\tilde\alpha}(p)\psi=(\tilde\alpha-i\partial_x)\psi$.

$\{\frac{e^{ilx}}{\sqrt{2\pi}}\}_{l\in\mathbb{Z}}$ is an orthonormal basis consisting of eigenvectors of p: $\rho_{\tilde{\alpha}}(p)e^{ilx}=(l+\tilde{\alpha})e^{ilx}$.

$\mathbf{n}=\mathbf{2}=\mathbf{2r}$. This implies $Y=M\,\mathbf{H}_3$. The Casimir eigenvalues $\zeta_a=e^{i2\pi\tilde{\alpha}_a}$ $(\tilde{\alpha}\in\mathbb{R}^2/\mathbb{Z}^2)$ identify the inequivalent irreps of $Y,\mathscr{Y}$ $(\rho_{\tilde{\alpha}},\mathscr{H})$, with

$$\mathscr{H}=\bigoplus_{k=0}^{2\tilde{\nu}-1}\mathscr{H}_h,\qquad \begin{array}{l}\rho_{\tilde{\alpha}}(m_2)\,\mathscr{H}_k=e^{i\frac{\pi}{\tilde{\nu}}(\tilde{\alpha}_2-k)}\mathscr{H}_k,\qquad \rho_{\tilde{\alpha}}(m_1)\mathscr{H}_k=\mathscr{H}_{k'},\ k'=k+1 \bmod 2\tilde{\nu}\\ \rho_{\tilde{\alpha}}(\mathbf{H}_3)\ \text{is Schrödinger representation of } \mathbf{H}_3 \text{ on } \mathscr{H}_k\simeq\mathscr{L}^2(\mathbb{R})\end{array}\tag{2.29}$$

Setting $a:=\frac{p_1+ip_2}{\sqrt{2\tilde{b}}}$, $a^*:=\frac{p_1-ip_2}{\sqrt{2\tilde{b}}}$, $\mathbf{n}:=a^*a$, we find $[a,a^*]=\mathbf{1}$. Defining

$$\begin{array}{l}\psi_{0,0}(x;\tilde{\alpha})=N\sum\limits_{k\in\mathbb{Z}}e^{ikx^1-\frac{1}{2\tilde{b}}\left(\tilde{b}x^2+k+\tilde{\alpha}_1+i\tilde{\alpha}_2\right)^2},\qquad \psi_{\mathrm{n},k}=\rho_{\tilde{\alpha}}\left[\frac{(a^*)^{\mathrm{n}}}{\sqrt{\mathrm{n}!}}(m_1)^k\right]\psi_{0,0},\\ \rho_{\tilde{\alpha}}(a^*)=\frac{-\partial_2-i\partial_1+\tilde{b}x^2+\tilde{\alpha}_1-i\tilde{\alpha}_2}{\sqrt{2\tilde{b}}},\qquad m_1=e^{\frac{1}{\tilde{b}}(i\tilde{\alpha}_1+\partial_1)},\qquad m_2=e^{ix^1+\frac{1}{\tilde{b}}(i\tilde{\alpha}_2+\partial_2)}.\end{array}\tag{2.30}$$

(N is a normalization factor) one finds that $\{\psi_{\mathrm{n},k}\}_{\mathrm{n}\in\mathbb{N}_0}$ is an orthonormal basis of $\mathscr{H}_k$ and $\{\psi_{\mathrm{n},k}\}_{(\mathrm{n},k)\in\mathbb{N}_0\times K}$ an orthonormal basis of $\mathscr{H}$, consisting of eigenvectors of $\mathbf{n},m_2$: $\mathbf{n}\,\psi_{\mathrm{n},k}=\mathrm{n}\,\psi_{\mathrm{n},k}$, $m_2\,\psi_{\mathrm{n},k}=e^{i\frac{\pi}{\tilde{\nu}}(\tilde{\alpha}_2-k)}\,\psi_{\mathrm{n},k}$. It is $a\,\psi_{0,k}=0$. Up to a gaussian factor, the $\psi_{\mathrm{n},k}$ are Jacobi Theta functions or their derivatives and are analytic in $z=x^1+ix^2$ [11].

2.3 The line bundle E as a quotient and the isomorphism $\mathscr{X}^V\simeq\Gamma(\mathbb{T}^n,E)$

As known, the formula $T_l: x\mapsto x+2\pi l$ ($l\in\mathbb{Z}^n$) defines a free action of the abelian group $\mathbb{Z}^n$ on $\mathbb{R}^n$, and setting "$x\sim y$ iff $y=T_l(x)$ for some $l\in\mathbb{Z}^n$" defines an equivalence relation in $\mathbb{R}^n$. The elements of the quotient $\mathbb{T}^n=\mathbb{R}^n/2\pi\mathbb{Z}^n$ are the corresponding equivalence classes, i.e. $[x]=\{T_l(x),\ l\in\mathbb{Z}^n\}$. The universal cover map is $P: x\in\mathbb{R}^n\mapsto[x]\in\mathbb{T}^n$. Similarly, given a smooth phase factor $V:\mathbb{Z}^n\times\mathbb{R}^n\mapsto U(1)$ fulfilling (2.2) we define [11] a free action of the abelian group $\mathbb{Z}^n$ on $\mathbb{R}^n\times\mathbb{C}$ by

$$\chi_l^V:(x,w)\in\mathbb{R}^n\times\mathbb{C}\mapsto\Big(x+2\pi l\,,V(l,x)\,w\Big),\qquad l\in\mathbb{Z}^n,\tag{2.31}$$

an equivalence relation $\sim_V$ in $\mathbb{R}^n\times\mathbb{C}$ by setting "$(x,w)\sim_V(x',w')$ iff $(x',w')=\chi_l^V[(x,w)]$ for some $l\in\mathbb{Z}^n$", and E by

$$E=(\mathbb{R}^n\times\mathbb{C})/\sim_V;\tag{2.32}$$

in other words, an element of E is an equivalence class $[(x,w)]=\{\chi_l^V[(x,w)],\ l\in\mathbb{Z}^n\}$. E is trivial (i.e. $E=\mathbb{T}^n\times\mathbb{C}$) if V is [i.e. $V(l,x)\equiv1$]. Given a smooth function $\psi:\mathbb{R}^n\mapsto\mathbb{C}$ fulfilling (2.1) we can define a $\tilde{\psi}\in\Gamma(\mathbb{T}^n,E)$, i.e. a smooth global section of E, by

$$\tilde{\psi}:[x]\in\mathbb{T}^n\mapsto\Big[\Big(x,\psi(x)\Big)\Big]=\Big\{\chi_l^V\Big[\Big(x,\psi(x)\Big)\Big],\ l\in\mathbb{Z}^n\Big\}\overset{(2.1)}{=}\Big\{\Big(x+2\pi l,\psi(x+2\pi l)\Big),\ l\in\mathbb{Z}^n\Big\}\in E.$$

The correspondence $\psi\in\mathscr{X}^V\mapsto\tilde{\psi}\in\Gamma(\mathbb{T}^n,E)$ is one-to-one and allows us to lift the hermitean structure $(\ ,\ )$, the covariant derivative ∇, the actions of $\mathscr{O},\mathbf{g}\,,Y,G$, the gauge transformations from $\mathscr{X}^V$ to $\Gamma(\mathbb{T}^n,E)$. Therefore we can and shall identify $\Gamma(\mathbb{T}^n,E)$ with $\mathscr{X}^V$.

The above data determine also trivializations of $E,\Gamma(\mathbb{T}^n,E),\tilde{\nabla}$. For each set X_i of a (finite) open cover $\{X_i\}_{i\in\mathscr{I}}$ of $\mathbb{T}^n$ let $W_i\subset\mathbb{R}^n$ be such that the restriction $P_i\equiv P: W_i\mapsto X_i$ is invertible. Let

$$\tilde{\psi}_i(u):=\psi[P_i^{-1}(u)],\qquad A_{ia}(u):=A_a[P_i^{-1}(u)],\qquad \nabla_i:=-id+qA_i\tag{2.33}$$

for $u\in X_i$. In $X_i\cap X_j$ (2.1) implies[3]

$$\tilde{\psi}_i = t_{ij}\tilde{\psi}_j, \qquad \nabla_i = t_{ij}\nabla_j t_{ji}, \qquad t_{ij}(u) := V\left\{\tfrac{1}{2\pi}\left[P_i^{-1}(u) - P_j^{-1}(u)\right], P_j^{-1}(u)\right\} \tag{2.34}$$

Condition (2.2) becomes the (Čech cohomology) cocycle condition for the transition functions t_{ij}:

$$t_{ik} = t_{ij}t_{jk}, \qquad \text{in } X_i\cap X_j\cap X_k. \tag{2.35}$$

The set $\{(X_i,U_i)\}_{i\in\mathscr{I}}$, with $U_i(u) := U[P_i^{-1}(u)]$, defines the trivialization of a gauge transformation:

$$\tilde{\psi}_i \mapsto \tilde{\psi}_i^U = U_i\tilde{\psi}_i, \qquad t_{ij}^U = U_i t_{ij} U_j^{-1}, \qquad \nabla_i \mapsto \nabla_i^U = U_i\nabla_i U_i^{-1}. \tag{2.36}$$

3. Twist-induced deformations

3.1 Twisted $H = U\mathbf{g}$ to a noncocommutative Hopf algebra $\hat{H}$

The Universal Enveloping $*$-Algebra (UEA) $H := U\mathbf{g}$ of the Lie algebra $\mathbf{g}$ of any Lie group G is a Hopf $*$-algebra. We briefly recall what this means. Let

$$\begin{aligned}
&\varepsilon(\mathbf{1}) = 1, &&\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, &&S(\mathbf{1}) = \mathbf{1}, &&\\
&\varepsilon(g) = 0, &&\Delta(g) = g\otimes\mathbf{1} + \mathbf{1}\otimes g, &&S(g) = -g, &&\text{if } g\in\mathbf{g};
\end{aligned}$$

ε,Δ are extended to all of H as $*$-algebra maps, S as a $*$-antialgebra map:

$$\begin{aligned}
&\varepsilon : H \mapsto \mathbb{C}, &&\varepsilon(ab) = \varepsilon(a)\varepsilon(b), &&\varepsilon(a^*) = [\varepsilon(a)]^*,\\
&\Delta : H \mapsto H\otimes H, &&\Delta(ab) = \Delta(a)\Delta(b), &&\Delta(a^*) = [\Delta(a)]^{*\otimes *},\\
&S : H \mapsto H, &&S(ab) = S(b)S(a), &&S\{[S(a^*)]^*\} = a.
\end{aligned} \tag{3.1}$$

The extensions of ε,Δ,S are unambiguous, as $\varepsilon(g)=0$, $\Delta([g,g']) = [\Delta(g),\Delta(g')]$, $S([g,g']) = [S(g'),S(g)]$ if $g,g'\in\mathbf{g}$. The maps ε,Δ,S are the abstract operations by which one constructs the trivial representation, the tensor product of any two representations and the contragredient of any representation, respectively. $H = U\mathbf{g}$ endowed with $*,\varepsilon,\Delta,S$ is a Hopf $*$-algebra.

One can deform $(H,*,\varepsilon,\Delta,S)$ into a new Hopf algebra $(\hat{H},*,\varepsilon,\hat{\Delta},\hat{S})$ using a *twist* [7]:

1. $\hat{H}$ is the ring $H[[\lambda]]$ of formal power series in a real deformation parameter λ with coefficients in H, endowed with the same $*$-algebra structure (over $\mathbb{C}[[\lambda]]$) and counit ε as H;
2. the coproduct $\hat{\Delta}$ is related to $\Delta(g) \equiv \sum_I g^I_{(1)}\otimes g^I_{(2)}$ by $\hat{\Delta}(g) = \mathscr{F}\Delta(g)\mathscr{F}^{-1} \equiv \sum_I g^I_{(\hat{1})}\otimes g^I_{(\hat{2})}$;
3. the antipodes $S,\hat{S}$ are related by $\hat{S}(g) = \gamma S(g)\gamma^{-1}$, with $\gamma = \sum_I \mathscr{F}_I^{(1)} S\left(\mathscr{F}_I^{(2)}\right)$,

where the *twist* [7](see also [15, 3]) is for our purposes a <u>unitary</u> element $\mathscr{F}\in(H\otimes H)[[\lambda]]$ fulfilling

$$\mathscr{F} = \mathbf{1}\otimes\mathbf{1} + O(\lambda), \qquad (\varepsilon\otimes\mathrm{id})\mathscr{F} = (\mathrm{id}\otimes\varepsilon)\mathscr{F} = \mathbf{1},$$

$$(\mathscr{F}\otimes\mathbf{1})[(\Delta\otimes\mathrm{id})(\mathscr{F})] = (\mathbf{1}\otimes\mathscr{F})[(\mathrm{id}\otimes\Delta)(\mathscr{F})] =: \mathscr{F}_3. \tag{3.2}$$

[3] The points $x\in W_j$, $x'\in W_i$ such that $u = P_j x = P_i x'$ are related by $x' = x + 2\pi l$, with some $l\in\mathbb{Z}^n$. One has just to replace the arguments l,x of V in (2.1) resp. by $P_i^{-1}(u) - P_j^{-1}(u)$, $P_j^{-1}(u)$.

$*, \varepsilon, \hat{\Delta}, \hat{S}$ fulfill the analogs of conditions (3.1). While H is cocommutative, $\hat{H}$ is noncocommutative with a unitary triangular structure $\mathscr{R} = \mathscr{F}_{21}\mathscr{F}^{-1}$, i.e. $\tau \circ \hat{\Delta}(g) = \mathscr{R}\Delta(g)\mathscr{R}^{-1}$ and $\mathscr{R}^{-1} = \mathscr{R}_{21} = \mathscr{R}^{*\otimes *}$, where τ is the flip operator $[\tau(a \otimes b) = b \otimes a]$. $\hat{\Delta}, \hat{S}$ replace Δ, S in the construction of the tensor product of any two representations and the contragredient of any representation, respectively.

In this work we take $H = U\mathbf{g}_Q$, $\mathscr{F} \in (U\mathbf{g}_Q \otimes U\mathbf{g}_Q)[[\lambda]]$ and for simplicity use only abelian twists, i.e. of the form $\mathscr{F} = e^{i\lambda h^{(2)}}$, where $h^{(2)} \in \bigwedge^2(\mathbf{h})$ and $\mathbf{h}$ is a real Cartan subalgebra $\mathbf{h} \subset \mathbf{g}$. This leads to $\gamma = \mathbf{1}$, $\hat{S} = S$. We can always choose the change (2.14) so that $\mathbf{h}$ is spanned by the transformed $p_{r+1}, \ldots, p_n$ and by Q. $\mathscr{F}$ will be of the form

$$\mathscr{F} = e^{\frac{i}{2}(p_a \otimes \theta^{ab} p_b + \Xi^a p_a \wedge Q)}, \qquad \theta = \lambda \begin{pmatrix} 0_r & \\ & \theta' \end{pmatrix}, \qquad \Xi = \lambda \begin{pmatrix} \\ \xi' \end{pmatrix} \tag{3.3}$$

here θ' is a real antisymmetric of size $(n-r)$, $\xi' \in \mathbb{R}^{n-r}$, and the missing blocks are zero matrices of the appropriate sizes. Note that (3.3) implies $\theta\beta^A\theta = 0$. Incidentally, considering Q as a primitive element, i.e. $\Delta(Q) = Q \otimes \mathbf{1} + \mathbf{1} \otimes Q$, and not just $\mathbf{1}$ times a constant, will be essential to extend the 1-particle results to multi-particle systems and QFT as done in [10]: the previous formula formalizes the additivity of the electric charge in composite systems. Here are examples for $n = 2, 3, 4$:

$$\beta^A = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}, \qquad \theta = 0_2, \qquad \Xi = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \qquad \Rightarrow \qquad \mathscr{F} = e^{\frac{i}{2}\xi p_2 \wedge Q}, \quad \hat{\Delta} = \Delta,$$

$$\beta^A = \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & -\eta & 0 \end{pmatrix}, \quad \Rightarrow \quad \begin{array}{l} \mathscr{F} = e^{\frac{i}{2}\eta p_2 \wedge p_3}, \qquad \hat{\Delta}(Q) = \Delta(Q), \\ \hat{\Delta}(p_a) = \Delta(p_a) + \delta_{a1}\frac{\eta b}{2} p_3 \wedge Q, \end{array}$$

$$\beta^A = \begin{pmatrix} 0_2 & -b \\ b & 0_2 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}, \quad \Rightarrow \quad \begin{array}{l} \mathscr{F} = e^{\frac{i}{2}\eta p_3 \wedge p_4}, \qquad \hat{\Delta}(Q) = \Delta(Q), \\ \hat{\Delta}(p_a) = \Delta(p_a) + \delta_a^1 \frac{\eta b_1}{2} p_3 \wedge Q + \delta_a^4 \frac{\eta b_2}{2} p_2 \wedge Q. \end{array}$$

where in the last line $b := \mathrm{diag}(b_1, b_2)$. Note that for $n = 2$ H is not deformed: $\hat{H} = H$.

3.2 Twisted H-modules and H-module algebras

A left H-module $(\mathscr{M}, \triangleright)$ is defined to be a vector space $\mathscr{M}$ over $\mathbb{C}$ equipped with a left action, i.e. a $\mathbb{C}$-bilinear map $(g, a) \in H \times \mathscr{M} \mapsto g \triangleright a \in \mathscr{M}$ such that $(3.4)_1$ holds. Equipped also with an antilinear involution $*$ fulfilling $(3.4)_2$ $(\mathscr{M}, \triangleright, *)$ is a left H-$*$-module. A left H-module $*$-algebra $\mathscr{A}$ is a $*$-algebra over $\mathbb{C}$ equipped with a left H-module structure $\big(V(\mathscr{A}), \triangleright\big)$ such that

$$(gg') \triangleright a = g \triangleright (g' \triangleright a), \qquad (g \triangleright a)^* = [S(g)]^* \triangleright a^*, \qquad g \triangleright (ab) = \sum_I \left(g^I_{(1)} \triangleright a\right)\left(g^I_{(2)} \triangleright b\right). \tag{3.4}$$

Given such an $\mathscr{A}$, $\big(V(\mathscr{A})[[\lambda]], \triangleright\big)$ endowed with the new product and $*$-structure

$$a \star a' := \sum_I \left(\overline{\mathscr{F}}_I^{(1)} \triangleright a\right)\left(\overline{\mathscr{F}}_I^{(2)} \triangleright a'\right), \qquad a^{*_\star} := S(\gamma) \triangleright a^* \tag{3.5}$$

gets a $\hat{H}$-module $*$-algebra $\mathscr{A}_\star$: in fact, $\star$ is associative by (3.2), fulfills $(a\star a')^{*_\star} = a'^{*_\star}\star a^{*_\star}$ and

$$g \triangleright (a\star a') = \sum_I \left[g^I_{(\hat{1})} \triangleright a\right] \star \left[g^I_{(\hat{2})} \triangleright a'\right]. \tag{3.6}$$

Finally, given a left H-module $*$-algebra $\mathscr{A}$ and a left H-equivariant $\mathscr{A}$-$*$-bimodule $\mathscr{M}$, i.e. a left H-$*$-module and $\mathscr{A}$-bimodule $\mathscr{M}$ such that $(3.4)_3$ holds for all $a \in \mathscr{A}$, $b \in \mathscr{M}$ and for all $a \in \mathscr{M}$, $b \in \mathscr{A}$, then $V(\mathscr{M})[[\lambda]]$ gets a left $\hat{H}$-equivariant $\mathscr{A}_\star$-$*$-bimodule $\mathscr{M}_\star$ when endowed with the $*$-structure and the left, right $\mathscr{A}_\star$-multiplications (3.5) for all $a \in \mathscr{A}_\star$, $a' \in \mathscr{M}_\star$ and $a \in \mathscr{M}_\star$, $a' \in \mathscr{A}_\star$.

If $\mathscr{A}$ is defined by H-equivariant generators a_i and polynomial relations (most interesting $\mathscr{A}$ are), **then also $\mathscr{A}_\star$ is**, with the same Poincaré-Birkhoff-Witt series and related polynomial relations. One can define a *linear map* $\wedge : f \in \mathscr{A} \mapsto \hat{f} \in \mathscr{A}_\star$ (*generalized Weyl map*) by the equation

$$f(a_1, a_2, \dots) = \hat{f}(a_1\star, a_2\star, \dots) \qquad \text{in } V(\mathscr{A})[[\lambda]] = V(\mathscr{A}_\star). \tag{3.7}$$

Using (3.2) it is easy to show that $\wedge, \wedge^{-1}$ fulfill

$$\begin{aligned} &\wedge(ff') = \sum_I \wedge\left[\mathscr{F}_I^{(1)} \triangleright f\right] \star \wedge\left[\mathscr{F}_I^{(2)} \triangleright f'\right], \\ &\wedge^{-1}(\hat{f}\star\hat{f}') = \sum_I \left[\overline{\mathscr{F}}_I^{(1)} \triangleright \wedge^{-1}(\hat{f})\right]\left[\overline{\mathscr{F}}_I^{(2)} \triangleright \wedge^{-1}(\hat{f}')\right] = \left[\wedge^{-1}(\hat{f})\right] \star \left[\wedge^{-1}(\hat{f}')\right]. \end{aligned} \tag{3.8}$$

If one can express the H-action on $\mathscr{A}$ in the (cocommutative) left "adjoint-like" form

$$g \triangleright a = \sum_I \sigma\left(g^I_{(1)}\right) a\, \sigma\left(Sg^I_{(2)}\right), \tag{3.9}$$

through a ($*$)-algebra map $\sigma : H \mapsto \mathscr{A}$, then we can make $\mathscr{A}[[\lambda]]$ into a $\hat{H}$-module $*$-algebra by defining the corresponding action $\hat{\triangleright}$ in the (noncocommutative) "adjoint-like" form:

$$g \,\hat{\triangleright}\, a := \sum_I \sigma\left(g^I_{(\hat{1})}\right) a\, \sigma\left(\hat{S}g^I_{(\hat{2})}\right) \tag{3.10}$$

(here the linear extension $\sigma : \hat{H} = H[[\lambda]] \mapsto \mathscr{A}[[\lambda]]$ is used). Formula

$$D^\sigma_{\mathscr{F}}(a) := \sum_I \left(\overline{\mathscr{F}}_I^{(1)} \triangleright a\right) \sigma\left(\overline{\mathscr{F}}_I^{(2)}\right) \tag{3.11}$$

defines a **$\hat{H}$-module $*$-algebra isomorphism** $D^\sigma_{\mathscr{F}} : \mathscr{A}_\star \leftrightarrow \mathscr{A}[[\lambda]]$ (a *deforming map*, in the language of [9, 10]), i.e. a map intertwining between $\triangleright$ and $\hat{\triangleright}$, $*$ and $*_\star$, the original product and $\star$:

$$g \,\hat{\triangleright}[D^\sigma_{\mathscr{F}}(a)] = D^\sigma_{\mathscr{F}}(g \triangleright a), \qquad [D^\sigma_{\mathscr{F}}(a)]^* = D^\sigma_{\mathscr{F}}\left[a^{*_\star}\right], \qquad D^\sigma_{\mathscr{F}}(a\star a') = D^\sigma_{\mathscr{F}}(a) D^\sigma_{\mathscr{F}}(a'). \tag{3.12}$$

If $\mathscr{A}$ can be defined by a set of H-equivariant generators a_i and polynomial relations we find that the $\check{a}_i := D^\sigma_{\mathscr{F}}(a_i) \equiv \hat{D}^\sigma_{\mathscr{F}}(\hat{a}_i)$, which make up an alternative set of generators of $\mathscr{A}[[\lambda]]$, in fact span a $\hat{H}$-submodule and fulfill the same deformed polynomial relations as $\hat{a}_i$, so they provide an explicit realization of $\widehat{\mathscr{A}} \sim \mathscr{A}_\star$ within $\mathscr{A}[[\lambda]]$. Therefore $D^\sigma_{\mathscr{F}}$ can be seen as a change from a set of H-equivariant to a set of $\hat{H}$-equivariant generators of $\mathscr{A}[[\lambda]]$. If, as in next section, $\mathscr{A} \supseteq H$ then one can adopt as σ the inclusion map $\mathrm{id} : H \mapsto \mathscr{A}$, then (3.9) becomes the adjoint action of H, and the action defined by (3.10) makes $H[[\lambda]]$ itself into a $\hat{H}$-module $*$-algebra. In general, in the 'hat notation' the deforming map is a map $\widehat{D}^\sigma_{\mathscr{F}} : \widehat{\mathscr{A}} \leftrightarrow \mathscr{A}[[\lambda]]$.

Finally, one can try to extend the above definitions also to suitable completions (e.g. Hilbert space) of $*$-modules $\mathscr{M}[[\lambda]]$ and of $*$-algebras $\mathscr{A}[[\lambda]]$ (as algebras of operators on $\mathscr{M}[[\lambda]]$), as we will do below.

PoS(CNCFG2010)018

4. Twisted deformations of $\mathscr{X}, \mathscr{X}^{\mathrm{V}}, \mathscr{O}_Q, \ldots$

Let $\mathscr{D}_Q$ be the H-module $*$-algebra of polynomials in $Q, p_1, \ldots, p_m$ with (left, say) coefficients $f \in C^\infty(\mathbb{R}^n)$ fulfilling again (2.9). We adopt (3.3) as a (formal) twist and tentatively define the $\star$-product by (1.1) for any $f, h \in \mathscr{D}_Q$; the λ-power (i.e. θ-power) series involved in (1.1) is termwise well-defined and reduces to a finite sum if either f or g is a polynomial in x^a, p_a, in particular

$$\begin{aligned}
&(h{\cdot}x+p{\cdot}y)\star(k{\cdot}x+p{\cdot}z) = (h{\cdot}x+p{\cdot}y)(k{\cdot}x+p{\cdot}z) + \tfrac{i}{2}(h+2Q\beta^A y)^t\theta(k+2Q\beta^A z),\\
&(h{\cdot}x+p{\cdot}y)\star e^{ik\cdot x} = e^{ik\cdot x}\left[h{\cdot}x+p{\cdot}y+k{\cdot}y-(\tfrac{h}{2}+Q\beta^A y)^t\theta k\right],\\
&e^{ik\cdot x}\star(h{\cdot}x+p{\cdot}y) = e^{ik\cdot x}\left[h{\cdot}x+p{\cdot}y+(\tfrac{h}{2}+Q\beta^A y)^t\theta k\right], \qquad Q\star o = Qo = o\star Q,\\
&(h{\cdot}x+p{\cdot}y)\star e^{ik\cdot x}|q\rangle = e^{ik\cdot x}\left[h{\cdot}x+p{\cdot}y+k{\cdot}y-(\tfrac{h}{2}+Q\beta^A y)^t\theta k\right],
\end{aligned} \tag{4.1}$$

for all $h,k,y,z \in \mathbb{R}^n$ and $o \in \mathscr{D}_Q$. In deriving these relations we have used the fomula $p_a \triangleright e^{i(k\cdot x+p\cdot z)} = e^{i(k\cdot x+p\cdot z)}(k+2Q\beta^A z)_a$. The $*$-structure is undeformed, as $\gamma = \mathbf{1}$. Eq. $(4.1)_1$ entails in particular the basic Moyal $\star$-product $x^a \star x^b = x^a x^b + \mathbf{1}\frac{i}{2}\theta^{ab}$. The θ-power series involved in (1.1) is infinite but convergent if both f,h are exponentials:

$$\begin{aligned}
e^{i(h\cdot x+p\cdot y)}\star e^{i(k\cdot x+p\cdot z)} &= e^{i(h\cdot x+p\cdot y)}e^{i(k\cdot x+p\cdot z)}e^{-\frac{i}{2}(h+Q\beta^A y)^t\theta(k+Q\beta^A z)}\\
&\overset{(2.11)}{=} e^{i[(h+k)\cdot x+p\cdot(y+z)]}e^{-\frac{i}{2}\left[h\cdot z-k\cdot y-Qy\beta^A z+(h+Q\beta^A y)^t\theta(k+Q\beta^A z)\right]}
\end{aligned} \tag{4.2}$$

for all $h,k,y,z \in \mathbb{R}^n$. All the $\star$-products are associative as a consequence of the cocycle condition (3.2). We also stress that they are *gauge-independent*, since $\mathscr{F}$ is expressed in terms of p_a (rather than ∂_a, x^a) and so are (2.10). Moreover, from the antisymmetry of θ it easily follows $(h{\cdot}x+p{\cdot}y)^k \star (h{\cdot}x+p{\cdot}y) = (h{\cdot}x+p{\cdot}y)^{k+1}$ for all $k \in \mathbb{N}$, whence by iteration $[(h{\cdot}x+p{\cdot}y)\star]^k = (h{\cdot}x+p{\cdot}y)^k\star$ and $\exp[i(h{\cdot}x+p{\cdot}y)]\star = \exp[i(h{\cdot}x+p{\cdot}y)\star]$; in particular, $\exp[ih{\cdot}x]\star = \exp[ih{\cdot}x\star]$, which is a series converging for all $x \in \mathbb{R}^n$. Therefore we can replace $(h{\cdot}x+p{\cdot}y)$ by $(h{\cdot}x+p{\cdot}y)\star$ as argument in the exponentials in (4.1-4.2), etc. Going to the 'hat notation', we find as consequences

$$\begin{aligned}
&[h{\cdot}\hat{x}+\hat{p}{\cdot}y, k{\cdot}\hat{x}+\hat{p}{\cdot}z] = i\left[h{\cdot}z-k{\cdot}y-2\hat{Q}y^t\beta^A z+(h+2\hat{Q}\beta^A y)^t\theta(k+2\hat{Q}\beta^A z)\right]\\
&(h{\cdot}\hat{x}+\hat{p}{\cdot}y)e^{ik\cdot\hat{x}} = e^{ik\cdot\hat{x}}[h{\cdot}\hat{x}+\hat{p}{\cdot}y+k{\cdot}y-(h+2\hat{Q}\beta^A y)^t\theta k], \qquad [\hat{Q},\hat{o}] = 0,\\
&e^{i(h\cdot\hat{x}+\hat{p}\cdot y+\hat{Q}y^0))}e^{i(k\cdot\hat{x}+\hat{p}\cdot z+\hat{Q}z^0)} = e^{i[(h+k)\cdot\hat{x}+\hat{p}\cdot(y+z)]}e^{-\frac{i}{2}\left[h\cdot z-k\cdot y+(h+2\hat{Q}\beta^A y)^t\theta(k+2\hat{Q}\beta^A z)+2\hat{Q}(y^0+z^0)\right]}\\
&\hat{Q}^{\hat{*}} = \hat{Q}, \qquad \hat{p}_a^{\hat{*}} = \hat{p}_a, \qquad \hat{x}^{a\hat{*}} = \hat{x}^a, \qquad \left[e^{i(k\cdot\hat{x}+\hat{p}\cdot y+\hat{Q}y^0)}\right]^{\hat{*}} = e^{-i(k\cdot\hat{x}+\hat{p}\cdot y+\hat{Q}y^0)}
\end{aligned} \tag{4.3}$$

(here $h,k,y,z \in \mathbb{R}^n$, $y^0, z^0 \in \mathbb{R}$, $\hat{o} \in \widehat{\mathscr{D}}_Q$). The fourth is the Weyl form of the first and third [it can be formally derived also by the BCH formula]; for $y = z = 0$ it becomes

$$e^{ih\cdot\hat{x}}e^{ik\cdot\hat{x}} = e^{i(h+k)\cdot\hat{x}}e^{-\frac{i}{2}h^t\theta k}, \tag{4.4}$$

i.e. the relation defining the Grönewold-Moyal-Weyl spaces, if $h,k \in \mathbb{R}^n$, and Connes-Rieffel non-commutative tori, if $h,k \in \mathbb{Z}^n$ [however, they are not the most general ones due to the particular form $(3.3)_2$ for θ]. Up to isomorphisms, the latter product depends only on the group $\mathsf{H}^2(\mathbb{Z}^n, U(1))$ cohomology class of the $U(1)$-valued two-cocycle $\Theta(h,k) := e^{-\frac{i}{2}h^t\theta k}$. As the replacement $\theta \to \theta + \theta'$

with $\theta' \in M_n(2\pi\mathbb{Z})$ leaves the algebras unchanged, one may restrict to $0 \le \theta^{ab} < 2\pi$. In all the previous relations the deformation parameters Ξ of (3.3) have given no contribution.

Motivated by the previous arguments we shall postulate (4.4) as defining relations for the (uncountable) set of generators (parametrized by the continuous indices $h,k,y,z \in \mathbb{R}^n$, $y^0,z^0 \in \mathbb{R}$) of the various algebras and linear spaces we introduce below. The functions f on $\mathbb{R}^n$ that one needs for QM and QFT [test functions f in Schwarz space $\mathscr{S} \equiv \mathscr{S}(\mathbb{R}^n)$, $f \in \mathscr{L}^2 \equiv \mathscr{L}^2(\mathbb{R}^n)$, distributions $f \in \mathscr{S}'$, etc.] all admit suitably generalized notions of Fourier transform $\tilde{f}$ (Fourier, Fourier-Plancherel, Fourier for distributions), so that f can be expressed in terms of the anti-Fourier transform $f(x) = \int d^nk\, e^{ik\cdot x}\tilde{f}(k)$; the symbol $\tilde{f}$ respectively belongs to $\widetilde{\mathscr{S}} = \mathscr{S}$, $\widetilde{\mathscr{L}^2} = \mathscr{L}^2$, $\widetilde{\mathscr{S}'}$. The previous arguments suggest that we correspondingly define $\widehat{\mathscr{S}}, \widehat{\mathscr{L}^2}, \widehat{\mathscr{S}'}$ as the spaces (and $\hat{H}$-$*$-modules) of objects of the form

$$\hat{f}(\hat{x}) = \int_{\mathbb{R}^n} d^nk\, e^{ik\cdot\hat{x}}\tilde{f}(k). \tag{4.5}$$

The (Connes-Rieffel) deformation of $\mathscr{X} = C^\infty(\mathbb{T}^n)$ is the $\hat{*}$-algebra

$$\widehat{\mathscr{X}} = \left\{ \hat{f}(\hat{x}) = \textstyle\sum_{m\in\mathbb{Z}^n} f_m\, \hat{u}^m \quad \middle| \quad \{f_m\}_{m\in\mathbb{Z}^n} \in \mathscr{S}(\mathbb{Z}^n) \right\}, \qquad \hat{u}^m := e^{im\cdot\hat{x}} \tag{4.6}$$

where $\mathscr{S}(\mathbb{Z}^n)$ is the space of sequences of complex numbers rapidly decreasing at "infinity", i.e. fulfilling the inequalities $\sup_{m\in\mathbb{Z}^n} |f_m| \left(1+|m|\right)^h < \infty$ for all $h \in \mathbb{N}_0$. $\widehat{\mathscr{X}}$ is the subspace of $\widehat{\mathscr{S}'}$ characterized by $\tilde{f}(k) = \sum_{m\in\mathbb{Z}^n} f_m\, \delta^{(n)}(k-m)$, with $\{f_m\}_{m\in\mathbb{Z}^n} \in \mathscr{S}(\mathbb{Z}^n)$. We denote as $\widehat{\mathscr{O}_Q}$ the $\hat{H}$-module $*$-algebra of polynomials in $\hat{Q}, \hat{p}_1, ..., \hat{p}_m$ with coefficients in $\widehat{\mathscr{X}}$, constrained by (4.3).

We can define a *generalized Weyl map* $\wedge$ on various domains by setting on the generators

$$\wedge(Q) = \hat{Q}, \qquad \wedge(x^a) = \hat{x}^a, \qquad \wedge(p_a) = \hat{p}_a, \qquad \wedge\left(e^{i(h\cdot x+p\cdot y)}\right) = e^{i(h\cdot\hat{x}+\hat{p}\cdot y)} \tag{4.7}$$

for all $h,y \in \mathbb{R}^n$, and extending it using linearity and (3.8) (formulated in 'hat notation'). Eq. $(4.7)_4$ is formally consistent with (3.8) and (4.3); essentially, we have already proved this when showing $\exp[i(h\cdot x+p\cdot y)]\star = \exp[i(h\cdot x+p\cdot y)\star]$. Restricting to $y=0$, $h=l\in\mathbb{Z}^m$ one finds the (invertible) *Weyl* map $\wedge : \mathscr{X}[[\lambda]] \mapsto \widehat{\mathscr{X}}$ and its extensions to $\mathscr{L}^2, \mathscr{X}'$:

$$f = \sum_{l\in\mathbb{Z}^m} f_l\, u^l \qquad \Rightarrow \qquad \wedge(f) = \sum_{l\in\mathbb{Z}^m} f_l\, \hat{u}^l = \hat{f}. \tag{4.8}$$

We will also use (4.7) to define maps $\wedge : \mathscr{O}_Q[[\lambda]] \mapsto \widehat{\mathscr{O}_Q}$ and $\wedge : Y_Q[[\lambda]] \mapsto \widehat{Y_Q}$. Also the inverses (or *generalized Wigner maps*) $\wedge^{-1}$ are immediately determined from (4.7).

As $H \subset \mathscr{D}_Q$ we can use (3.11) with $\sigma = \mathrm{id}$ to construct *deforming maps* (i.e. $\hat{H}$-module $*$-algebra isomorphism) on various domains. On the generators of $\mathscr{D}_Q$ we find

$$\check{x}^a \equiv \hat{D}_{\mathscr{F}}(\hat{x}^a) = (x - \tfrac{\theta}{2}p - \Xi\tfrac{Q}{2})^a, \qquad \check{p}_a \equiv \hat{D}_{\mathscr{F}}(\hat{p}_a) = (p+\beta^A\theta pQ+\beta^A\Xi Q)_a, \qquad \check{Q} \equiv \hat{D}_{\mathscr{F}}(\hat{Q}) = Q; \tag{4.9}$$

using the BCH relation and the one $p_a \triangleright e^{i(h\cdot x+p\cdot y)} = e^{i(h\cdot x+p\cdot y)}(h+2Q\beta^A y)_a$ we find

$$\hat{D}_{\mathscr{F}}\left[e^{i(h\cdot\hat{x}+\hat{p}\cdot y+y^0\frac{\hat{Q}}{2})}\right] = e^{i\left\{h\cdot x+p^t\left[y+\theta(\frac{h}{2}+\beta^A yQ)\right]+Q\left[\frac{y^0}{2}-\Xi^t(\frac{h}{2}+\beta^A yQ)\right]\right\}}. \tag{4.10}$$

Choosing $y=0$, $h^b=\delta^b_a$ in (4.10) one finds in particular $\check{u}^a \equiv \hat{D}_{\mathscr{F}}(\hat{u}^a) = u^a \exp\left[-\frac{i}{2}(\theta p - \Xi Q)^a\right]$. Eq. $(4.9)_{2,3}$ and (4.10) allow to define deforming maps $\hat{D}_{\mathscr{F}} : \widehat{\mathscr{O}_Q} \mapsto \mathscr{O}_Q[[\lambda]]$ and $\hat{D}_{\mathscr{F}} : \widehat{Y_Q} \mapsto Y_Q[[\lambda]]$; as $\hat{D}_{\mathscr{F}}^{-1}$

maps the unitaries in Y_Q into unitaries in $\widehat{Y_Q}$ obtained by a (linear) redefinition of the parameters h, y, y^0, and viceversa, $\hat{D}_{\mathscr{F}}$ extends as a map of C^*-algebras $\hat{D}_{\mathscr{F}}:\widehat{\mathscr{Y}_Q}\mapsto\mathscr{Y}_Q[[\lambda]]$ The existence of such an algebra map means that the deformation $\mathscr{Y}_Q \rightsquigarrow \widehat{\mathscr{Y}_Q}$ of the algebra structure of $\mathscr{Y}_Q$ is "trivial", i.e. amounts just to a change of generators in $\mathscr{Y}_Q[[\lambda]]$ (whereas the deformation $\mathscr{X} \rightsquigarrow \widehat{\mathscr{X}}$ of the subalgebra $\mathscr{X}$ is not "trivial" at all, at least for θ generic). Similarly for the deformation $\mathscr{O}_Q \rightsquigarrow \widehat{\mathscr{O}_Q}$. At the level of formal deformations (i.e. of power series in λ, i.e. in θ) this result was actually expected by cohomological reasons [8] (in fact, the first and second Hochschild cohomology groups of $U\mathbf{g}_Q$ vanish). This is to be contrasted with the nontrivial deformation $\mathscr{X} \rightsquigarrow \widehat{\mathscr{X}}$. Replacing $Q\mapsto q\in\mathbb{Z}$ and applying $\hat{D}_{\mathscr{F}}^{-1}$ one determines the analog of M and $\mathscr{Z}(\mathscr{Y})$ [cf. (2.12)]:

$$\widehat{M} := \{\hat{m}\in\widehat{Y} \mid [\hat{m},\hat{H}]=0\} = \hat{D}_{\mathscr{F}}^{-1}(M), \qquad \mathscr{Z}\left(\widehat{\mathscr{Y}}\right) = \hat{D}_{\mathscr{F}}^{-1}\left[\mathscr{Z}(\mathscr{Y})\right]. \tag{4.11}$$

We now look for the deformed analogs of $\mathscr{X}^V, \mathscr{H}^V$. As in subsection 2.2 we set $\tilde{\beta}^A = q\beta^A$, etc. We start with the gauge (2.21-2.22). From (3.3) it follows

$$\beta\theta = 0, \qquad \beta\Xi = 0, \qquad \Rightarrow \qquad \tilde{\beta}\theta = 0, \qquad \tilde{\beta}\Xi = 0. \tag{4.12}$$

In such a gauge we define the space $\widehat{\mathscr{X}^\beta}\subset\widehat{\mathscr{S}'}|q\rangle$ by

$$\widehat{\mathscr{X}^\beta} := \left\{\hat{\psi}\in\widehat{\mathscr{S}'}|q\rangle \;\middle|\; \tilde{\psi}(k+2\pi\tilde{\beta}^t l) = e^{i2\pi l^t(k+\pi\tilde{\beta}l)}\tilde{\psi}(k),\ \int_{\mathbb{R}^n}\!d^n k\,|\tilde{\psi}(k)|\left[1+|k|\right]^h<\infty\ \ \forall\,(k,l,h)\in\mathbb{R}^n\times\mathbb{Z}^n\times\mathbb{N}\right\} \tag{4.13}$$

This ensures[4] for all $\hat{\psi}\in\widehat{\mathscr{X}^\beta}$ the noncommutative quasiperiodicity property in the first line of

$$\begin{aligned} &\hat{\psi}(\hat{x}+2\pi l) = \hat{V}(l,\hat{x})\,\hat{\psi}(\hat{x}), && l\in\mathbb{Z}^n,\\ &\hat{\nabla}_a = -i\hat{\partial}_a + \hat{A}_a\hat{Q} = \hat{p}_a + \hat{A}'_a\hat{Q} && \hat{A}'_a\in\widehat{\mathscr{X}}, \end{aligned} \tag{4.15}$$

where $\hat{V}, \hat{p}_a$ are defined by

$$\hat{V}(l,\hat{x}) \equiv \hat{V}^\beta(l,\hat{x}) := e^{-i\hat{Q}2\pi l^t\beta(\hat{x}+l\pi)}, \qquad \hat{p}_a \equiv \hat{p}_a^\beta := -i\hat{\partial}_a + \hat{x}^b\beta_{ba}\hat{Q} + \alpha_a\hat{Q}, \tag{4.16}$$

in complete analogy with (2.21). The second line is our definition of the deformed covariant derivative. Using (4.12), (4.4) and the relation $e^{-iq4\pi^2 l^t\beta^A l'} = 1$, consequence of (2.8), it is easy to check that $\hat{V}^\beta$ fulfills (2.2). For $\hat{V}\equiv\mathbf{1}$, or equivalently $\tilde{\beta} = \tilde{\beta}^A = 0$, $\widehat{\mathscr{X}^\beta}$ reduces to $\widehat{\mathscr{X}}|q\rangle$. By $(4.15)_1$ $\widehat{\mathscr{X}^\beta}$ is mapped into itself by multiplication by all $\hat{f}\in\widehat{\mathscr{X}}$, the action of $\hat{p}_a$[5] and therefore also by

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[4]
$$\begin{aligned} \hat{V}^\beta(l,\hat{x})\,\hat{\psi}(\hat{x}) &\overset{(4.4)}{=} \int_{\mathbb{R}^n}\! d^n k\, e^{-i2\pi l^t\tilde{\beta}(\hat{x}+l\pi)+ik^t\hat{x}} e^{i\pi l^t\tilde{\beta}\theta k}\tilde{\psi}(k)|q\rangle \overset{(4.12)}{=} \int_{\mathbb{R}^n}\! d^n k\, e^{i(k^t-2\pi l^t\tilde{\beta})\hat{x}} e^{-i2\pi^2 l^t\tilde{\beta} l}\,\tilde{\psi}(k)|q\rangle \\ &= \int_{\mathbb{R}^n}\! d^n k\, e^{ik\cdot\hat{x}} e^{-i2\pi^2 l^t\tilde{\beta} l}\,\tilde{\psi}(k+2\pi\tilde{\beta}^t l) \overset{(4.13)}{=} \int_{\mathbb{R}^n}\! d^n k\, e^{ik\cdot(\hat{x}+2\pi l)}\,\tilde{\psi}(k) = \hat{\psi}(\hat{x}+2\pi l) \qquad \square \end{aligned} \tag{4.14}$$

(in the third equality we have performed the shift $k\mapsto k+2\pi q\tilde{\beta}^t l$ of the integration variable).

[5] In fact, if $\hat{\psi}$ fulfills the quasiperiodicity condition $(4.15)_1$, also $\hat{u}^m\hat{\psi}$ and $\hat{p}_a\hat{\psi}$ do:

$$[\hat{p}_a\hat{\psi}](\hat{x}+2\pi l) \overset{(4.15-4.16)}{=} (\hat{p}+\pi l^t\tilde{\beta})_a e^{-i\pi l^t\tilde{\beta}(\hat{x}+l\pi)}\,\psi(\hat{x}) \overset{(4.3)_3}{=} e^{-i\pi l^t\tilde{\beta}(\hat{x}+l\pi)}(\hat{p}-\pi l^t\tilde{\beta}\theta\tilde{\beta}^A)_a\,\psi(\hat{x}) \overset{(4.12)}{=} e^{-i\pi l^t\tilde{\beta}(\hat{x}+l\pi)}[\hat{p}_a\hat{\psi}](\hat{x}).$$

The quasiperiodicity is unambiguously defined, since for all $q\in\mathbb{Z}$, $l,l'\in\mathbb{Z}^n$

$$\hat{\psi}[\hat{x}+2\pi(l+l')] = e^{-i\pi l^t\tilde{\beta}(\hat{x}+l'2\pi+l\pi)}\,\hat{\psi}(\hat{x}+2\pi l') \qquad \Leftrightarrow \qquad e^{-i\pi(l+l')^t\tilde{\beta}[\hat{x}+(l+l')\pi]} = e^{-i\pi l^t\tilde{\beta}(\hat{x}+l'2\pi+l\pi)}e^{-i\pi l'^t\tilde{\beta}(\hat{x}+l'\pi)};$$

that the last equality holds follows from (2.8), the BCH formula and the fact that the commutator between the last two exponents is proportional to $l^t\tilde{\beta}\theta\tilde{\beta}l'$, and thus vanishes by (4.12).

the action of $\hat{\nabla}_a$. In other words, $\widehat{\mathscr{X}}^{\beta}$ is a ($\hat{H}$-equivariant) $\widehat{\mathscr{O}_Q}$-module. As an internal consistency check, one can verify that the decomposition of $\hat{p}_a$ in the second line of (4.15) indeed fulfills (4.3). Moreover, eq. (4.15) guarantees that $\hat{\psi}'^{\hat{*}}\hat{\psi} \in \widehat{\mathscr{X}}$. The magnetic field $\hat{B}_{ab}$ is defined by

$$-2i\hat{Q}\hat{B}_{ab} := [\hat{\nabla}_a, \hat{\nabla}_b] = [\hat{p}_a + \hat{Q}\hat{A}'_a(\hat{u}), \hat{p}_b + \hat{Q}\hat{A}'_b(\hat{u})] \qquad \overset{(4.3)}{\Rightarrow} \qquad \hat{B}_{ab} \in \widehat{\mathscr{X}}; \tag{4.17}$$

the constant part in the Laurent series expansion of $\hat{B}_{ab}$ in the $\hat{u}^a$ is $[\beta^A + 2\hat{Q}\beta^A\theta\beta^A]_{ab}$. On the other hand, since the conditions on $\tilde{\psi}$ in (4.13) characterize also the Fourier transform of $\psi \in \mathscr{X}^{\beta}$ (an easy check), then we can extend $\wedge$ to $\mathscr{X}^{\beta}$ so that $\wedge(\mathscr{X}^{\beta}) = \widehat{\mathscr{X}}^{\beta}$, but only in the gauge (4.12)[6].

In the notation (4.5-4.6) we define integration over the noncommutative torus $\int_{\hat{X}} : \hat{f} \in \widehat{\mathscr{X}} \mapsto \int_{\hat{X}} \hat{f} \in \mathbb{C}$ in one of the equivalent ways

$$\int_{\hat{X}} \hat{f} := \int_X \left[\wedge^{-1}(\hat{f})\right](x) = (2\pi)^n f_{\mathbf{0}} \,. \tag{4.18}$$

This is just Connes-Rieffel integration [4, 13]. It fulfills linearity, reality, the trace property and invariance under the action of $H, \hat{H}$; the latter means $\int_{\hat{X}} g \hat{\triangleright} \hat{f} = \varepsilon(g) \int_{\hat{X}} \hat{f}$, in particular $\int_{\hat{X}} \hat{p}_a \hat{\triangleright} \hat{f} = -i\int_{\hat{X}} \hat{\partial}_a \hat{f} = 0$ for any $\hat{f} \in \widehat{\mathscr{X}}$ (as $\hat{Q}\hat{\triangleright}\hat{f} = 0$). $\int_{\hat{X}}$ reduces to the ordinary translation invariant integration over $\mathbb{T}^n$ if $\theta = 0$. For all $\hat{\psi}', \hat{\psi} \in \widehat{\mathscr{X}}^{\beta}$ it is $\hat{\psi}'^{\hat{*}}\hat{\psi} \in \widehat{\mathscr{X}}$. In the appendix we show the first equality in

$$\int_{\hat{X}} \hat{\psi}^{\hat{*}}\hat{\psi}' = \int_X \psi^* \psi' \overset{(2.19)}{=} (\psi, \psi') =: (\hat{\psi}, \hat{\psi}'); \tag{4.19}$$

the second is the definition of the Hermitean structure in $\mathscr{X}^{\beta}$. It follows that one can use it also to define an Hermitean structure (,) in $\widehat{\mathscr{X}}^{\beta}$ (last equality); we shall call $\widehat{\mathscr{H}}^{\beta}$ the Hilbert space completion of the latter in the Hilbert norm $\|\hat{\psi}\| := (\hat{\psi}, \hat{\psi})^{1/2}$. The map $\wedge : \mathscr{X}^{\beta} \mapsto \widehat{\mathscr{X}}^{\beta}$ [with β fulfilling (4.12)] extends to a unitary H-equivariant transformation $\wedge : \mathscr{H}^{\beta} \mapsto \widehat{\mathscr{H}}^{\beta}$. On $\widehat{\mathscr{X}}$ (i.e. for $\hat{V} \equiv \mathbf{1}$) formula (4.19) reduces to $(\hat{f}', \hat{f}) = \int_{\hat{X}} \hat{f}'^{\hat{*}} \hat{f} = \sum_{l \in \mathbb{Z}^n} \overline{f'_l} f_l$, implying that $\int_{\hat{X}} : \hat{f} \in \widehat{\mathscr{X}} \mapsto \int_{\hat{X}} \hat{f} \in \mathbb{C}$ is a normalized positive-definite trace[7].

Next goal would be to extend the previous construction to generic gauges. The gauge-transformed magnetic field should still belong to $\widehat{\mathscr{X}}$. As we have not determined the most general gauge transformation, we stop here the discussion. We hope to report soon on this point elsewhere.

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[6] In fact, using (2.21), (4.12) computation we find for any $f \in \mathscr{S}'$

$$e^{ik\cdot x}\star|q\rangle = e^{ik\cdot x}e^{\frac{-i}{2}k\cdot(\theta p+\Xi q)}\triangleright|q\rangle = e^{ik\cdot x}e^{\frac{-i}{2}k\cdot[\theta(\tilde{\beta}^t x+\tilde{\alpha})+\Xi q]}\triangleright|q\rangle = e^{ik\cdot\left[x-\frac{\tilde{\alpha}}{2}-\frac{q}{2}\Xi\right]}|q\rangle \qquad \Rightarrow \qquad f(x)|q\rangle = f'(x)\star|q\rangle$$

where $f'(x) := f\left[x + \frac{\tilde{\alpha}}{2} + \frac{q}{2}\Xi\right]$. If $\psi(x) \equiv \psi_0(x)|q\rangle \in \mathscr{X}^{\beta}$ then by $(2.13)_6$ also $\psi'(x) := \psi'_0(x)|q\rangle$ belongs to $\mathscr{X}^{\beta}$. Setting $\wedge(|q\rangle) = |q\rangle$, we thus find $\wedge(\psi) = \wedge(\psi_0|q\rangle) = \wedge(\psi'_0)\star\wedge(|q\rangle) = \widehat{\psi'_0}|q\rangle = \widehat{\psi'}$.

[7] Actually, $\int_{\hat{X}}$ is the only normalized positive-definite trace and the C^*-algebra $\overline{\widehat{\mathscr{X}}}$ is simple if θ is *quite irrational*, i.e. if the lattice Λ_θ generated by its columns is such that $\Lambda_\theta + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$ (see e.g. [12], p. 537-538). The C^*-algebra $\overline{\widehat{\mathscr{X}}}$ admits a faithful representation $\rho^{\beta} : \widehat{\mathscr{X}} \mapsto \mathscr{B}(\widehat{\mathscr{H}}^{\beta})$ in terms of bounded operators acting on $\widehat{\mathscr{H}}^{\beta}$, defined by $\rho^{\beta}(\hat{f})\hat{\psi} = \hat{f}\hat{\psi}$ for any $\hat{f} \in \widehat{\mathscr{X}}$, $\hat{\psi} \in \widehat{\mathscr{H}}^{\beta}$. If $\hat{\psi}_0 \in \widehat{\mathscr{X}}^{\beta}$ is cyclic and separating then the Tomita involution is just the extension of $\hat{*}$ to $\widehat{\mathscr{H}}^{\beta}$. $\widehat{\mathscr{H}}^{\beta}$ can be recovered also by the GNS construction with state $\omega^{\beta}(\hat{f}) := (\hat{\psi}_0, \hat{f}\hat{\psi}_0) = \int_X (f\star\psi_0)\psi_0^*$; that the integrand is a periodic function follows from $\widehat{\mathscr{X}}^{\beta}$ being a $\widehat{\mathscr{X}}$-bimodule, $\wedge^{-1}(\hat{f}\hat{\psi}_0) = f\star\psi_0$ and (4.19). More explicitly, one easily finds $\omega^{\beta}(\hat{u}^m) = \int_X e^{im\cdot x}\mu_m(x)$, where $\mu_m(x) := \psi_0\left(x - \frac{1}{2}\theta m\right)\psi_0^*(x)e^{-i\frac{q}{2}m^t\theta\alpha}$. When $\hat{V} = 1$, $\psi_0 \equiv \frac{1}{\sqrt{(2\pi)^n}} \in \widehat{\mathscr{X}}$, $\rho^1(\hat{f}) = \frac{1}{(2\pi)^n}\int_X f = f_{\mathbf{0}}$, and this reduces to the GNS construction of the Hilbert space completion of $\widehat{\mathscr{X}}$.

Appendix: Proof of the first equality in (4.19)

$$
\begin{aligned}
\hat{f}_\theta(\hat{x}) &:= [\hat{\psi}'^{\hat{*}}\hat{\psi}](\hat{x}) = \int_{\mathbb{R}^n} d^n k \int_{\mathbb{R}^n} d^n h \, \overline{\tilde{\psi}}(h)\,\tilde{\psi}(k) e^{-ih\cdot\hat{x}} e^{ik\cdot\hat{x}} \stackrel{(4.4)}{=} \int_{\mathbb{R}^n} d^n k \int_{\mathbb{R}^n} d^n h \, \overline{\tilde{\psi}}(h)\,\tilde{\psi}(k) e^{i(k-h)\cdot\hat{x} - \frac{i}{2}k^t\theta h} \\
&= \int_{\mathbb{R}^n} d^n k \, \tilde{f}_\theta(k) e^{ik\cdot\hat{x}} \qquad\qquad \tilde{f}_\theta(k) := \int_{\mathbb{R}^n} d^n h \, \overline{\tilde{\psi}}(h)\,\tilde{\psi}(k+h) e^{-\frac{i}{2}k^t\theta h}.
\end{aligned}
$$

In the third equality we have shifted the integration variable and used the antisymmetry of θ. We choose a ε-dependent ($\varepsilon \in]0,1[$) family of functions $\chi_\varepsilon \in \mathscr{S}$ with the property that $\chi_\varepsilon(k) = 1$ for $|k| \leq \varepsilon/2$ and $\chi_\varepsilon(k) = 0$ for $|k| \geq 0$. As $\hat{f}_\theta \in \widehat{\mathscr{X}}$ is such that $\tilde{f}(k) = \sum_{m\in\mathbb{Z}^n} f_m \, \delta^{(m)}(k-m)$, we find

$$
f_{\theta\mathbf{0}} = \int_{\mathbb{R}^n} d^n k \, \tilde{f}_\theta(k) \chi_\varepsilon(k),
$$

for all $\varepsilon \in]0,1[$. Hence $\quad f_{\theta\mathbf{0}} = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} d^n k \, \tilde{f}_\theta(k) \chi_\varepsilon(k) = \int_{\mathbb{R}^n} d^n k \, \tilde{f}(k) \chi_\varepsilon(k) = f_{\mathbf{0}},$

where $\tilde{f}(k) := \tilde{f}_{\theta=0}(k) = \int_{\mathbb{R}^n} d^n h \, \overline{\tilde{\psi}}(h)\,\tilde{\psi}(k+h) = \sum_l f_l \, \delta^{(m)}(k-l)$. This and (4.18) imply (4.19).

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