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Two Aspects of M-(brane) theory

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The Reconstruction Algebra of [3] is quantized, and a novel approach to Quantum M-branes presented.

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The attempt to quantize Relativistic M(em)branes (M-dimensional extended objects in D-dimensional space-time) is intimately related to non-commutative field theory and gravity. The fuzzy sphere was invented in this context (cp. [1]), and the hope of including gravity is reflected from many points of view (e.g. [2]). Here I would like to report on 2 topics relevant to this endavour, namely

A) Quantum Reconstruction Algebras

In a recent paper [3] it was found that the reconstruction [4] of the coordinate which disappears in the light cone description of relativistic extended objects,

$$\zeta(\varphi) = \zeta_0 + \frac{1}{\eta}\int G(\varphi,\widetilde{\varphi})\widetilde{\nabla}^a(\frac{\overrightarrow{p}}{\rho}\widetilde{\nabla}_a\overrightarrow{x})(\widetilde{\varphi})\rho(\widetilde{\varphi})d^M\widetilde{\varphi} \tag{1}$$
$$\int G(\varphi,\widetilde{\varphi})\rho(\varphi)d^M\varphi = 0, \quad \widetilde{\triangle}G(\varphi,\widetilde{\varphi}) = \frac{\delta(\varphi,\widetilde{\varphi})}{\rho} - 1,$$

leads to higher-dimensional generalisations of the Witt-Virasoro algebra, when considering the (classical or quantum) commutation relations of the field ζ at different points φ of the parameter manifold Σ_M (modulo volume-preserving diffeomorphisms);

Namely

$$[L_\alpha, L_{\alpha'}] = e_{[\alpha,\alpha']\varepsilon}L_\varepsilon \tag{2}$$

where

$$e_{\alpha\beta\gamma} := \frac{\mu_\beta - \mu_\gamma}{\mu_\alpha}\int_{\Sigma_M} Y_\alpha Y_\beta Y_\gamma \rho d^M\varphi =: \frac{\mu_\beta - \mu_\gamma}{\mu_\alpha} d_{\alpha\beta\gamma} \tag{3}$$

with Y_α, resp. $-\mu_\alpha$ ($\alpha = 1,2,\ldots$), being the (non-constant) eigenfunctions, resp (negative) eigenvalues of the Laplacian on Σ_M, $\triangle = \frac{1}{\rho}\partial_a\rho h^{ab}\partial_b$, $\sqrt{\det h^{ab}} = \rho$.

In [3] it was shown that (modulo volume-preserving diffeomorphisms)

$$\eta\zeta_\alpha := \eta\int Y_\alpha\zeta(\varphi)\rho d^M\varphi, \quad \alpha = 1,2,\ldots \tag{4}$$

form a representation of (2), with [,] being the classical Poisson bracket. Rewriting (1) as

$$\zeta = \zeta_0 + \frac{1}{2\eta}(\frac{\overrightarrow{p}}{\rho}\overrightarrow{x} - \int\overrightarrow{p}\overrightarrow{x}) + \frac{1}{2\eta}\int G(\varphi,\widetilde{\varphi})(\frac{\overrightarrow{p}}{\rho}\triangle\overrightarrow{x} - \overrightarrow{x}\triangle\frac{\overrightarrow{p}}{\rho})(\widetilde{\varphi})\rho(\widetilde{\varphi})d^M\varphi \tag{5}$$

$$\widetilde{\zeta}_\alpha := 2\eta\zeta_\alpha - 2\overrightarrow{P}\overrightarrow{x}_\alpha \tag{6}$$

one finds that

$$[\widetilde{\zeta}_\alpha, \widetilde{\zeta}_{\alpha'}] = 2e_{[\alpha,\alpha]\varepsilon}\widetilde{\zeta}_\varepsilon = (e_{\alpha\alpha'\varepsilon} - e_{\alpha'\alpha\varepsilon})\widetilde{\zeta}_\varepsilon \tag{7}$$

and that $\widetilde{\zeta}_\alpha$ can be written as

$$\widetilde{\zeta}_\alpha = \widetilde{D}_\alpha + \widetilde{E}_\alpha := (d_{\alpha\beta\gamma} + e_{\alpha\beta\gamma})\vec{x}_\beta \vec{p}_\gamma \tag{8}$$

Promoting the classical variables, $x_{j\beta}$ and $p_{k\gamma}$, to quantum operators satisfying

$$[x_{j\beta}, p_{k\gamma}] = i\delta_{jk}\delta_{\beta\gamma}, \tag{9}$$

it is not difficult to see that

$$\hat{\zeta}_\alpha := \frac{i}{2}(D_\alpha + E_\alpha), \;\; 2D_\alpha = d_{\alpha\beta\gamma}(\vec{x_\beta}\vec{p}_\gamma + \vec{p_\gamma}\vec{x}_\beta), \;\; E_\alpha = e_{\alpha\beta_\gamma}\vec{x_\beta}\vec{p}_\gamma \tag{10}$$

will form a representation of (7), with [,] the ordinary commutator of operators. Rather than lifting the derivation given in [3] to one involving quantummechanical operators one can also formally verify (7) by showing that

$$[D_\alpha, D_{\alpha'}] = i(\vec{x}_\alpha \vec{p}_{\alpha'} - \vec{x}_{\alpha'}\vec{p}_\alpha) \tag{11}$$

$$[D_\alpha, E_{\alpha'}] + [E_\alpha, D_{\alpha'}] = 2ie_{[\alpha,\alpha']\varepsilon}D_\varepsilon \tag{12}$$

$$[E_\alpha, E_{\alpha'}] \approx 2ie_{[\alpha,\alpha']\varepsilon}E_\varepsilon - i(\vec{x_\alpha}\vec{p}_{\alpha'} - \vec{x_{\alpha'}}\vec{p}_\alpha). \tag{13}$$

To obtain (11) one simply notes that

$$d_{\alpha\varepsilon\gamma}d_{\alpha'\beta\varepsilon} - d_{\alpha\beta\varepsilon}d_{\alpha'\varepsilon\gamma} = -\delta_{\alpha\gamma}\delta_{\alpha'\beta} + \delta_{\alpha\beta}\delta_{\alpha'\gamma} \tag{14}$$

due to the completeness-relation

$$\sum_{\alpha=1}^{\infty} Y_\alpha(\varphi)Y_\alpha(\widetilde{\varphi}) = \frac{\delta(\varphi,\widetilde{\varphi})}{\rho(\varphi)} - 1. \tag{15}$$

To obtain (12) one first obtains

$$2[D_\alpha, E_{\alpha'}] = i(d_{\alpha\varepsilon\gamma}e_{\alpha'\beta\varepsilon} - d_{\alpha\beta\varepsilon}e_{\alpha'\varepsilon\gamma})(\vec{x}_\beta\vec{p}_\gamma + \vec{p}_\gamma\vec{x}_\beta)$$

and then proves that

$$\begin{aligned} &e_{\alpha\varepsilon\beta}d_{\alpha'\varepsilon\gamma} + e_{\alpha\varepsilon\gamma}d_{\alpha'\varepsilon\beta} - e_{\alpha'\varepsilon\beta}d_{\alpha\varepsilon\gamma} - e_{\alpha'\varepsilon\gamma}d_{\alpha\varepsilon\beta} \\ &= (e_{\alpha\alpha'\varepsilon} - e_{\alpha'\alpha\varepsilon})d_{\varepsilon\beta\gamma}, \end{aligned} \tag{16}$$

using

$$e_{\alpha\beta\gamma} = \frac{1}{\mu_\alpha}\int Y_\alpha(Y_\beta \triangle Y_\gamma - Y_\gamma \triangle Y_\beta)\rho d^M\varphi \tag{17}$$

and (15).

The calculation leading to (2) in [3] then implies that (modulo volume-preserving diffeomorphisms/topological terms)

$$e_{\alpha\varepsilon\beta}e_{\alpha'\varepsilon\gamma} - e_{\alpha'\varepsilon\beta}e_{\alpha\varepsilon\gamma} + \delta_{\alpha\beta}\delta_{\alpha'\gamma} - \delta_{\alpha\gamma}\delta_{\alpha'\beta} \approx (e_{\alpha\alpha'\varepsilon} - e_{\alpha'\alpha\varepsilon})e_{\varepsilon\beta\gamma} \tag{18}$$

which proves (13).

For $M=2$, an identity related to (18) has been derived in [5], using a completeness relation that allows to write problematic terms involving

$$\sum_{\varepsilon} \frac{1}{\mu_\varepsilon} \partial_a Y_\varepsilon(\varphi) \widetilde{\partial}_b Y_\varepsilon(\widetilde{\varphi}) \tag{19}$$

(appearing also on the r.h.s. of (18)) in terms of harmonic vectorfields, and $\varepsilon_{aa'}\partial^{a'}Y_\varepsilon \varepsilon_{bb'}\partial^{b'}Y_\varepsilon$ (leading to terms proportional to the areapreserving diffeomorphism constraints), and $\delta(\varphi,\widetilde{\varphi})$. Otherwise, the 'trick' is again to use (17) and (15), resp. to write

$$\begin{aligned}
&(e_{\alpha\varepsilon\beta}e_{\alpha'\varepsilon\gamma} - e_{\alpha'\varepsilon\beta}e_{\alpha\varepsilon\gamma})\mu_\alpha\mu_{\alpha'} \\
&= (\mu_\varepsilon^2 + \mu_\beta\mu_\gamma - \mu_\varepsilon(\mu_\beta+\mu_\gamma))d_{\alpha\varepsilon\beta}d_{\alpha'\varepsilon\gamma} - (\alpha \leftrightarrow \alpha') \\
&= \int Y_\alpha \triangle Y_\varepsilon Y_\beta \int Y_{\alpha'} \triangle Y_\varepsilon Y_\gamma + \int Y_\alpha Y_\varepsilon \triangle Y_\beta \int Y_{\alpha'} Y_\varepsilon \triangle Y_\gamma \\
&- \int Y_\alpha \triangle Y_\varepsilon Y_\beta \int Y_{\alpha'} Y_\varepsilon \triangle Y_\gamma - \int Y_\alpha Y_\varepsilon \triangle Y_\beta \int Y_{\alpha'} \triangle Y_\varepsilon Y_\gamma - (\alpha \leftrightarrow \alpha')
\end{aligned} \tag{20}$$

and use (15), after integrating by parts in order to have no derivative acting on Y_ε. Note that $\int(\nabla Y_\alpha \nabla Y_\beta)(\nabla Y_{\alpha'}\nabla Y_\gamma) - (\alpha \leftrightarrow \alpha')$ is equal to $\int\{Y_\alpha, Y_{\alpha'}\}\{Y_\beta, Y_\gamma\}$ when $M=2$, which has also been observed in [5] (and probably in [6] as well).

B) Codimension 2 Quantum M-branes:

M-branes are known to have special descriptions and properties when the world volume swept out has codimension 1 (cp. [7-12]). Here I would like to propose a route to quantizing M-branes when the codimension is 2.

The internal (Mass)^2 of membranes in D-dimensional Minkowski-space is known [1,13] to be, in orthonormal light-cone gauge, equal to

$$\mathbb{M}^2 = \sum_{i=1}^{D-2}\sum_{\alpha=1}^{\infty} p_{i\alpha}p_{i\alpha} + \frac{1}{2} g_{\alpha\beta\gamma}g_{\alpha\beta'\gamma'}\vec{x}_\beta\cdot\vec{x}_{\beta'}\vec{x}_\gamma\cdot\vec{x}_{\gamma'} \tag{21}$$

where

$$g_{\alpha\beta\gamma} := \int Y_\alpha \left(\frac{\partial Y_\beta}{\partial\varphi^1}\frac{\partial Y_\gamma}{\partial\varphi^2} - \frac{\partial Y_\beta}{\partial\varphi^2}\frac{\partial Y_\gamma}{\partial\varphi^1}\right) d^2\varphi$$

are totally antisymmetric structure constants of the Lie-algebra of "area-preserving" (i.e. unit Jacobian) diffeomorphisms. When $D=5$, (21) can be written as

$$\mathbb{M}^2 = \sum_{\alpha=1}^{\infty} \vec{a}_\alpha^\dagger \vec{a}_\alpha \tag{22}$$

where

$$a_{j\alpha} = ip_{j\alpha} + \frac{1}{2} g_{\alpha\beta\gamma}\varepsilon_{jkl}x_{k\beta}x_{l\gamma} \tag{23}$$

Motivated in parts by some classical structures observed in [7], Moncrief [8] (though for the codimension one case) argued that (22) may be a good take-off for quantization.

Note that, at least formally,

$$[\hat{\zeta}_\alpha, a_{j\beta}] = i(d_{\alpha\beta\gamma} + e_{\alpha\beta\gamma})a_{j\gamma} \tag{24}$$

holds, and that

$$\Psi_0(x) := e^{-\frac{1}{3}\varepsilon_{jkl}g_{\alpha\beta\gamma}x_{j\alpha}x_{k\beta}x_{l\gamma}} \tag{25}$$

is (formally) annihilated by the quantization of (23) as well as $\hat{\zeta}_\alpha$. While the exponent in (25) for the corresponding Matrix-theory is conventionally considered to take all real values - as pointed out by V. Moncrief (years ago, in a discussion at the Albert Einstein Institute) - it is, with the geometric interpretation of enclosed volume at hand, in the continuum theory extremely natural [8] to restrict to strictly negative exponents by choosing a definite orientation.

Leaving for the moment unanswered the very interesting question whether (25) (resp. its supersymmetric analogue) may actually be Lorentz-invariant, in particular annihilated by the crucial mixed generator $\mathbb{M}_{k-}$ (cp. [14]), let me note that by diagonalizing the real-symmetric matrix $S(= R\Lambda R^T)$ appearing in

$$[a_{j\alpha}, a^\dagger_{j'\alpha'}] = 2\varepsilon_{jj'k}g_{\alpha\alpha'\gamma}x_{k\gamma} =: 2S_{j\alpha,j'\alpha'}, \tag{26}$$

$$a_J = \partial_J + \frac{1}{2}S_{JL}x_L,$$

one has

$$A_K := R^T_{KJ}a_J = R^T_{KJ}\partial_J + \frac{1}{2}\lambda_K R^T_{KJ}x_J \tag{27}$$

For $M = 1$ (string in 4 Dimensions), S (hence R) are independent of x, so that

$$A_J = \tilde{\partial}_J + \lambda_J\tilde{x}_J,$$

and J can be taken as (j,n) while $j = 1,2$, $n \in \mathbb{Z} - \{0\}$. The explicit formulae for $D = 4$ ($M = 1$) are:

$$a_j = ip_j + \varepsilon_{jk}x'_k \tag{28}$$

$$\hat{a}_{j\alpha} := \int Y_\alpha(\varphi)\hat{a}_j = \partial_{j\alpha} + \varepsilon_{jk}r_{\alpha\beta}x_{k\beta} \tag{29}$$

with

$$r_{\alpha\beta} = \int_0^{2\pi} Y_\alpha Y'_\beta d\varphi \tag{30}$$

$$[a_{j\alpha}, a^\dagger_{j'\alpha'}] = 2\varepsilon_{jj'}r_{\alpha\alpha'} =: S_{j\alpha,j'\alpha'} \tag{31}$$

$$\mathbb{M}^2 = a^\dagger_J a_J$$

and $a_{j\alpha}\Psi_0 = 0$ would give

$$\Psi_0 \sim e^{-\frac{1}{2}\varepsilon_{jj'}r_{\alpha\alpha'}x_{j\alpha}x_{j'\alpha'}} \tag{32}$$

where the exponent, $-\frac{1}{2}x_J S_{JJ'}x_{J'} = -\frac{1}{2}\tilde{x}_J\lambda_J\tilde{x}_J$, is proportional to the area enclosed by the curve.

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References

[1] J.Hoppe Quantum theory of a massless relativistic surface and a two-dimensional bound state problem PhD Thesis MIT 1982 (http://dspace.mit.edu/handle/1721.1/15717)

[2] E. Witten; Nucl.Phys. B 460 (1996) 335.
T. Banks, W. Fischler, S. Shenker, L. Susskind; Phys.Rev. D 55 (1997) 5112.

[3] J. Hoppe; Phys.Lett.B 695 (2011) 384, doi:10.1016/j.physletb.2010.11.038.

[4] J. Goldstone, unpublished notes.

[5] K. Ezawa, Y. Matsuo, K. Murakami; Prog.Theor.Phys. 98 (1997) 485

[6] S.Melosch, Diplomarbeit, Hamburg University 1990.

[7] J.Hoppe; Canonical 3+1 Description of Relativistic Membranes arXiv:hep-th/9407103

[8] V. Moncrief; Gen. Rel. Grav. (2006) 38(4):561

[9] M. Bordemann, J. Hoppe; Phys. Lett. B317 (1993) 315

[10] M. Bordemann, J. Hoppe; Phys. Lett. B325 (1994) 359

[11] M.Bordemann, J.Hoppe; J.Math.Phys. 39 (1998) 683

[12] J.Hoppe, T.Ratiu; Class.Quant.Grav.14:L45,1997

[13] J.Goldstone, unpublished

[14] J.Hoppe; M-brane dynamical symmetry and quantization arXiv:1101.4334