

## NC GUTS: A Status Report

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I review the main results that have been obtained so far on the construction of noncommutative GUTs

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## 1. Introduction

It is already ten years since the publishing of refs. [1–3], where it was put forward a formalism –called the enveloping-algebra formalism– which led to refs. [4, 5], where the Noncommutative Standard Model and noncommutative GUTs were formulated. An excellent introduction to noncommutative gauge theories defined within the enveloping-algebra formalism can be found in ref. [6].

Let us recall that in the enveloping-algebra formalism the noncommutative fields are functions of the ordinary fields –ie, no change in the number of degrees of freedom as we move from ordinary to noncommutative space-time– such that ordinary gauge orbits are mapped into noncommutative gauge orbits:

$$\begin{aligned}
A_\mu[a_\mu, \psi, \theta] + s_{NC} A_\mu[a_\mu, \psi, \theta]A &= A_\mu[a_\mu + s a_\mu, \psi + s \psi, \theta], \\
\Psi[a_\mu, \psi, \theta] + s_{NC} \Psi[a_\mu, \psi, \theta] &= \Psi[a_\mu + s a_\mu, \psi + s \psi, \theta], \\
s_{NC} \Lambda[\lambda, \lambda, \psi, \theta] &= s \Lambda[\lambda, \lambda, \psi, \theta], \\
A_\mu[a_\mu, \psi, \theta = 0] &= a_\mu, \Psi[a_\mu, \psi, \theta = 0] = \psi, \Lambda[\lambda, \lambda, \psi, \theta = 0] = \lambda \\
s_{NC} A_\mu &= \partial_\mu \Lambda - i[A_\mu, \Lambda]_\star, s_{NC} \Psi = i\Lambda \star \Psi, s_{NC} \Lambda = i\Lambda \star \Lambda, \\
s a_\mu &= \partial_\mu \lambda - i[a_\mu, \lambda], s \psi = i\lambda \psi, s \lambda = i\lambda \lambda,
\end{aligned} \tag{1.1}$$

I shall call these equations standard Seiberg-Witten map equations since  $\Lambda$  acts from the left on the matter fields  $\Psi$ . The solution to these equations which match the corresponding ordinary field when the noncommutativity matrix,  $\theta^{\mu\nu}$ , vanishes shall be called standard Seiberg-Witten map. Now, since  $a_\mu$  and  $\lambda$  take values on the Lie algebra,  $\mathfrak{g}$ , of a compact Lie group,  $G$ , then, the noncommutative fields  $A_\mu$  and  $\Lambda$  take values on the universal enveloping algebra of  $\mathfrak{g}$ . This is a characteristic feature of noncommutative gauge fields defined in the enveloping-algebra formalism.

Having defined the noncommutative gauge and matter fields in terms of the ordinary fields using the solution to eq. (1.1), we now introduce de classical (nonSUSY) noncommutative GUT(-inspired) theory for a compact Lie group,  $G$ , by giving its action  $S$ :

$$\begin{aligned}
S &= S_{gauge} + S_{fermionic} + S_{Higgs} + S_{Yukawa}, \\
S_{gauge} &= \int d^4x - \frac{1}{2} \sum_{\mathcal{R}} c_{\mathcal{R}} \text{Tr}_{\mathcal{R}} F_{\mu\nu}[\mathcal{R}(A)] \star F^{\mu\nu}[\mathcal{R}(A)], \\
S_{fermionic} &= \int d^4x \bar{\Psi}_L i \not{D}[\rho_\psi(A)] \Psi_L, \\
S_{Higgs} \quad \text{and} \quad S_{Yukawa} & \text{ give to,} \\
F_{\mu\nu}[\mathcal{R}(A)] &= \partial_\mu \mathcal{R}(A)_\nu - \partial_\nu \mathcal{R}(A)_\mu - i[\mathcal{R}(A)_\mu, \mathcal{R}(A)_\nu]_\star, \\
D_\mu[\rho_\psi(A)] \Psi_L &= \partial_\mu \Psi_L - i\rho_\psi(A_\mu) \star \Psi_L.
\end{aligned} \tag{1.2}$$

$S_{Higgs}$  and  $S_{Yukawa}$  yield, respectively, the Higgs and Yukawa sectors of the GUT theory and are dropped to define what we call noncommutative GUT-inspired theories. We shall see later on that the construction of  $S_{Yukawa}$  is far from trivial and it demands the use of the so-called Hybrid Seiberg-Witten maps [7] –needed to define noncommutative gauge transformation acting from the left and from right. In eq. (1.2),  $\Psi_L[\theta^{\mu\nu}, \rho_\psi(a), \psi_L]$  is the noncommutative left-handed spinor multiplet which is the noncommutative counterpart of the ordinary left-handed spinor multiplet  $\psi_L$ .  $\psi_L$  carries an arbitrary unitary representation,  $\rho_\psi$ , of  $\mathfrak{g}$ .  $\mathcal{R}$  labels the unitary irreps –typically the adjoint and matter irreps– of  $\mathfrak{g}$  and  $\sum_{\mathcal{R}} c_{\mathcal{R}} \text{Tr}_{\mathcal{R}} \mathcal{R}(T_1^a) \mathcal{R}(T_1^a) = 1/g_1^2$ .

Next, the quantum version of the classical field theory defined above is obtained by integrating over the ordinary fields in the path-integral with Boltzmann factor  $e^{iS}$ .  $S$  is the action above, which we shall understand as a formal power series in  $\theta^{\mu\nu}$ . I believe that this expansion in  $\theta$  will not yield the right Physics at Energies  $> 1/\sqrt{\theta}$ .

After those ten years, it is advisable that we pause to look back and assess what has been achieved as regards the quantum properties of those GUT(-inspired) theories. I will not cover all that has been done so far, but I will focus on

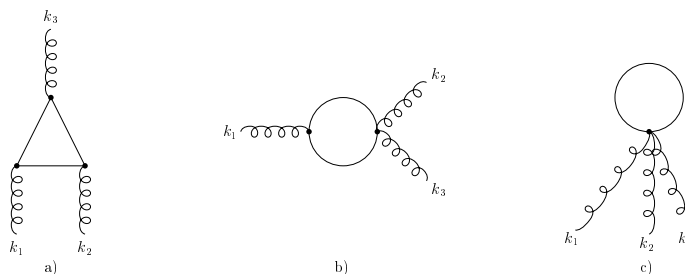
- Gauge anomalies.
- Renormalisability (when there are no Higgs and no Yukawa sectors),
- Construction of Yukawa terms.
- Existence of Supersymmetric versions.

## 2. Gauge Anomalies

When quantising a chiral gauge theory the first problem one has to face is that of gauge anomalies, for if the latter exist the theory will not make sense at the quantum level. The chiral vertices in the classical action acquire  $\theta$ -dependent terms, which can give rise to new  $\theta$ -dependent anomalous contributions to the famous ordinary triangle diagrams:

$$S_{fermionic} = \int d^4x \bar{\psi} i \not{\partial} \psi + \bar{\psi} \{ \not{A} - \theta^{\alpha\beta} [ \frac{1}{2} f_{\alpha\beta} i \not{D}(a) + \gamma^p f_{p\alpha} i D_\beta(a) ] \} P_L \psi + o(\theta^2).$$

Thus, I started long ago the computation of the three types of one-loop three-point diagrams in Figure 1 giving would-be  $\theta$ -dependent anomalies.



**Figure 1:** Types of  $\theta$ -dependent would-be anomalous three-point diagrams.

Actually, I was completely sure that these diagrams would give rise to new  $\theta$ -dependent anomalous terms, which would lead to extra anomaly cancelation conditions, which in turn would make most –eg., the Noncommutative Standard Model, noncommutative GUTs,...– of these theories meaningless at the quantum level. I could not be more mistaken! I was very surprised to find that the

$\theta$ -dependent anomalous contributions to the 1PI functional,  $\Gamma$ , were BRS-exact. ie, they were not truly anomalous terms:

$$\begin{aligned} s\Gamma[A[a, \theta], \theta] &= -\frac{i}{24\pi^2} \int d^4x \varepsilon^{\mu_1\mu_2\mu_3\mu_4} \text{Tr}(\partial_{\mu_1} \lambda a_{\mu_2} \partial_{\mu_3} a_{\mu_4}) \\ &+ s \left[ \frac{1}{48\pi^2} \int d^4x \varepsilon^{\mu_1\mu_2\mu_3\mu_4} \theta^{\alpha\beta} \text{Tr}(\partial_\alpha \partial_{\mu_1} a_{\mu_2} \partial_{\mu_3} a_{\mu_4} a_\beta) \right] + o(a^3) + o(\theta^2). \end{aligned} \quad (2.1)$$

The computations that led to the previous results were carried out by using dimensional regularization with a nonanticommuting  $\gamma_5$ . More details can be found un ref. [8]. I would like to point out now that when I did the computations back in 2002, I was completely unaware of the results –obtained using cohomological techniques– by Barnich, Henneaux and Brandt on the lack on nonBardeen anomalies for semisimple Lie algebras. The result in eq. (2.1) holds, though, for nonsemisimple Lie algebras as well.

The next challenge was to show, at one-loop, that there were no  $\theta$ -dependent gauge anomalies at any order in  $\theta$  and for any number of  $a_\mu$ 's. We did so –see ref. [10], for details– by using a mixture of explicit Dimensional Regularization computations, brute force of BRS equations and cohomological BRS techniques. Indeed, by taking advantage of the fact that in Dimensional Regularization the Jacobian of  $\mathcal{J} = \mathbb{I} + M$  –an operator which enters the Seiberg-Witten map for fermions  $\Psi_{\alpha I} = (\delta_{IJ} \delta_{\alpha\beta} + M[a, \partial, \gamma, \gamma_5; \theta]_{\alpha\beta IJ}) \psi_{\beta J}$ – is trivial, we were able to obtain the complete gauge anomaly candidate:

$$\begin{aligned} \mathcal{A}[A, \Lambda, \theta] &= -\frac{i}{24\pi^2} \int d^4x \varepsilon^{\mu_1\mu_2\mu_3\mu_4} \text{Tr} \Lambda \star \partial_{\mu_1} \left( A_{\mu_2} \star \partial_{\mu_3} A_{\mu_4} + \frac{1}{2} A_{\mu_2} \star A_{\mu_3} \star A_{\mu_4} \right), \\ A_\mu &= A[a, \theta]_\mu, \quad \Lambda = \Lambda[\lambda, \theta]. \end{aligned}$$

Then, by carrying out brute force computations and by using cohomological techniques, we obtained  $\mathcal{B}[A^{(a,t\theta)}, t\theta]$  such that

$$t \frac{d}{dt} \mathcal{A}[A(a, t\theta), \Lambda(\lambda, t\theta), t\theta] = s_{NC} \mathcal{B}[A^{(a,t\theta)}, t\theta],$$

and, hence,

$$\mathcal{A}[A(a, \theta), \Lambda(\lambda, \theta), \theta] = \mathcal{A}^{\text{Bardeen}} - s \int_0^1 \frac{dt}{t} \mathcal{B}[A(a, t\theta), t\theta].$$

We thus concluded that the  $\theta$ -dependent contributions to  $\mathcal{A}[A(a, \theta), \Lambda(\lambda, \theta), \theta]$  are cohomologically trivial: they are not anomalous contributions!

Since the previous result partially relies on the use a dimensionally regularised Feynman integrals involving  $\gamma_5$ , it would be advisable to check whether that result still holds for other regularization methods. Another way to obtain the gauge anomaly is Fujikawa's method: the gauge anomaly signals that the fermionic measure is not invariant under chiral gauge transformations. Fujikawa's method helps establish a connection with index theorems. As yet, we lack a derivation of the absence of  $\theta$ -dependent anomalous terms by using Fujikawa's method.

Within Fujikawa's formalism, the ordinary gauge anomaly comes in two guises, related by local redefinitions of the corresponding currents: the consistent form,  $\mathcal{A}_{con}$ , and the covariant form,  $\mathcal{A}_{cov}$ :

- $\mathcal{A}_{con}$  verifies the Wess-Zumino consistency conditions and involves lengthy and tedious algebra. It is not gauge covariant.

- $\mathcal{A}_{cov}$  does not verify the Wess-Zumino conditions, it is gauge covariant and, as a result, the algebraic computations that lead to it are simpler than in the "consistent" case.

As I was preparing a preliminary version of this talk, I decided to work out the covariant form, up to first order in  $\theta$ , of the gauge anomaly in the U(1) case. Let me point out that the absence of  $\theta$ -dependent contributions to the U(1) gauge anomaly is nontrivial from the cohomological point of view of Barnich, Brandt and Henneaux. The results that I obtained are displayed next.

Let me begin with the following partition function

$$\begin{aligned} Z[a, \theta] &\equiv \int d\bar{\psi} d\psi e^{-\int d^4x \bar{\psi} i\mathcal{D}\psi} \\ \mathcal{D} &= \hat{\mathcal{D}} + \hat{\mathcal{R}}, \quad \hat{\mathcal{D}} = \not{\partial} - i\not{a}P_L \\ \hat{\mathcal{R}} &= -[\frac{1}{4}\theta^{\alpha\beta} f_{\alpha\beta}\gamma^\mu D_\mu + \frac{1}{2}\theta^{\alpha\beta} \gamma^\rho f_{\rho\alpha} D_\beta]P_L \end{aligned}$$

Then, following Fujikawa, one introduces two bases of orthonormal eigenfunctions  $\{\varphi_m\}$  and  $\{\phi_m\}$ ,

$$\left(i\mathcal{D}(a)\right)^\dagger i\mathcal{D}(a)\varphi_m = \lambda_m^2 \varphi_m, \quad i\mathcal{D}(a)\left(i\mathcal{D}(a)\right)^\dagger \phi_m = \lambda_m^2 \phi_m,$$

and expands

$$\psi = \sum_m a_m \varphi_m, \quad \bar{\psi} = \sum_m \bar{b}_m \phi_m^\dagger,$$

which leads to the following definition of the fermionic measure:

$$d\bar{\psi} d\psi \equiv \prod_m d\bar{b}_m da_m.$$

It is not difficult to show that the gauge anomaly equation in covariant disguise reads

$$\int d^4x \text{Tr} \omega(x) (D^\mu [a] \mathcal{J}_\mu^{(cov)})(x) = -\delta J \equiv \mathcal{A}[\omega, a, \theta]_{cov},$$

where

$$\begin{aligned} \delta J &= d\bar{\psi}' d\psi' - d\bar{\psi} d\psi \quad \psi' = \psi + i\omega P_L \psi, \quad \bar{\psi}' = \bar{\psi} - i\bar{\psi} P_R \omega \\ \delta J &= \lim_{\Lambda \rightarrow \infty} \int d^4x \sum_m \{ \phi_m^\dagger \omega e^{-\lambda_m^2/\Lambda^2} P_R \phi_m - \phi_m^\dagger \omega e^{-\lambda_m^2/\Lambda^2} P_L \phi_m \} \\ \mathcal{J}_\mu^{a, (cov)}(x) &= \frac{1}{Z[a, \theta]} \int d\bar{\psi} d\psi \frac{\delta S_{fermionic}}{\delta a_\mu^a(x)} e^{-S_{fermionic}}, \quad S_{fermionic} = \int d^4x \bar{\psi} i\mathcal{D}\psi. \end{aligned}$$

By changing to a plane wave basis, one gets

$$\begin{aligned} \mathcal{A}[\omega, a, \theta]_{cov} &= \lim_{\Lambda \rightarrow \infty} - \int d^4x \text{Tr} \omega(x) \int \frac{d^4p}{(2\pi)^4} \text{tr} \left\{ \left( \gamma_5 e^{-ipx} e^{-\frac{(i\mathcal{D}^{(\theta)}(a))^2}{\Lambda^2}} e^{ipx} \right) \right\}, \\ \mathcal{D}^{(\theta)}(a) &= \mathcal{D} + \mathcal{R}, \quad \mathcal{R} = -[\frac{1}{4}\theta^{\alpha\beta} f_{\alpha\beta}\gamma^\mu D_\mu + \frac{1}{2}\theta^{\alpha\beta} \gamma^\rho f_{\rho\alpha} D_\beta]. \end{aligned}$$

Let us expand next the previous result in powers of  $\theta$  and remove the terms that vanish as  $\Lambda \rightarrow \infty$ . Thus one gets

$$\begin{aligned} \mathcal{A}[\omega, a, \theta]_{cov} &= \lim_{\Lambda \rightarrow \infty} - \int d^4x \text{Tr} \omega \int \frac{d^4p}{(2\pi)^4} \text{tr} \left\{ \left( \gamma_5 e^{-ipx} e^{-\frac{(i\mathcal{D}^{(\theta)}(a))^2}{\Lambda^2}} e^{ipx} \right) \right\} = \\ &= \mathcal{A}^{(ordinary)}[\omega, a] + \mathcal{A}^{(1)}[\omega, a, \theta] + o(\theta^2) \\ \mathcal{A}[\omega, a]^{(ordinary)} &= -\frac{1}{32\pi^2} \int d^4x \text{Tr} \omega \varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}, \\ \mathcal{A}^{(1)}[\omega, a, \theta] &= \int d^4x \text{Tr} \omega(x) [\mathcal{A}_1(x) + \mathcal{A}_2(x) + \mathcal{A}_3(x)] \end{aligned}$$

$$\begin{aligned}
\mathcal{A}_1 &= -\sum_{l=0}^1 \lim_{\Lambda \rightarrow \infty} 2i \int \frac{d^4q}{(2\pi)^4} e^{-q^2} \frac{1}{2} \text{tr} \gamma_5 \mathcal{P}^{2l}(\Lambda q) \{ \mathcal{D}(\Lambda q), \mathcal{R}(\Lambda q) \} \mathcal{P}^{2(1-l)}(\Lambda q) \mathbb{I}, \\
\mathcal{A}_2 &= -\sum_{l=0}^2 \lim_{\Lambda \rightarrow \infty} 2i \int \frac{d^4q}{(2\pi)^4} e^{-q^2} \frac{1}{3! \Lambda^2} \text{tr} \gamma_5 \mathcal{P}^{2l}(\Lambda q) \{ \mathcal{D}(\Lambda q), \mathcal{R}(\Lambda q) \} \mathcal{P}^{2(2-l)}(\Lambda q) \mathbb{I}, \\
\mathcal{A}_3 &= -\sum_{l=0}^3 \lim_{\Lambda \rightarrow \infty} 2i \int \frac{d^4q}{(2\pi)^4} e^{-q^2} \frac{1}{4! \Lambda^4} \text{tr} \gamma_5 \mathcal{P}^{2l}(\Lambda q) \{ \mathcal{D}(\Lambda q), \mathcal{R}(\Lambda q) \} \mathcal{P}^{2(3-l)}(\Lambda q) \mathbb{I}.
\end{aligned}$$

Some lengthy algebra and the fact that the  $a_\mu$ 's commute –U(1) case– lead to

$$\begin{aligned}
\mathcal{A}_1 &= -\frac{i}{8\pi^2} \theta^{\alpha\beta} \varepsilon^{\mu\nu\rho\sigma} \left( -\frac{1}{2} f_{\alpha\beta} f_{\mu\nu} f_{\rho\sigma} - f_{\nu\alpha} f_{\mu\beta} f_{\rho\sigma} \right) \\
&\quad + \frac{1}{16\pi^2} \theta^{\alpha\beta} \varepsilon^{\mu\nu\rho\sigma} \left[ f_{\mu\nu} (\partial_\rho f_{\sigma\alpha} D_\beta \mathbb{I} + \frac{1}{2} \partial_\rho f_{\alpha\beta} D_\sigma \mathbb{I}) + \partial_\mu f_{\nu\alpha} f_{\rho\sigma} D_\beta \mathbb{I} \right. \\
&\quad \quad \left. + \frac{1}{2} \partial_\mu f_{\alpha\beta} f_{\rho\sigma} D_\nu \mathbb{I} \right], \\
\mathcal{A}_2 &= -\frac{i}{2(4\pi)^2} \theta^{\alpha\beta} \varepsilon^{\mu\nu\rho\sigma} f_{\alpha\beta} f_{\mu\nu} f_{\rho\sigma} + \\
&\quad - \frac{1}{16\pi^2} \theta^{\alpha\beta} \varepsilon^{\mu\nu\rho\sigma} \left[ f_{\mu\nu} \partial_\rho f_{\sigma\alpha} D_\beta \mathbb{I} + \partial_\mu f_{\nu\alpha} f_{\rho\sigma} D_\beta \mathbb{I} + \frac{1}{2} (f_{\mu\nu} \partial_\rho f_{\alpha\beta} D_\sigma \mathbb{I} \right. \\
&\quad \quad \left. + \partial_\mu f_{\alpha\beta} f_{\rho\sigma} D_\nu \mathbb{I}) \right], \\
\mathcal{A}_3 &= 0.
\end{aligned}$$

So, finally the first order in  $\theta$  correction to the ordinary anomaly vanishes:

$$\begin{aligned}
\mathcal{A}^{(1)}[\omega, a, \theta] &= \int d^4x \text{Tr} \omega(x) [\mathcal{A}_1(x) + \mathcal{A}_2(x) + \mathcal{A}_3(x)] = \\
&\quad \frac{i}{32\pi^2} \theta^{\alpha\beta} \varepsilon^{\mu\nu\rho\sigma} \int d^4x \text{Tr} \omega(f_{\alpha\beta} f_{\mu\nu} f_{\rho\sigma} + 4 f_{\nu\alpha} f_{\mu\beta} f_{\rho\sigma}) = 0, \\
\mathcal{A}[\omega, a, \theta]_{cov} &= \mathcal{A}^{(ordinary)}[\omega, a] + o(\theta^2).
\end{aligned}$$

This shows complete agreement with the result obtained by using Dimensional Regularization. Higher order corrections in  $\theta$  and the nonabelian case are still to be worked out.

### 3. Renormalisability

The issue of the renormalisability of noncommutative gauge theories formulated within the enveloping-algebra formalism started off splendidly, for it was shown by Bichl, Grimstrup, Grosse, Popp, Schweda and Wulkenhaar [11] that the photon two-point function is renormalisable at any order in  $\theta$ . Unfortunately, Wulkenhaar showed [12] that this  $\theta$ -expanded QED was not renormalisable mainly due to the infamous four-point fermionic divergence:

$$\frac{c}{D-4} \theta^{\alpha\beta} \varepsilon_{\mu\nu\rho\sigma} \int d^4x \bar{\psi} \gamma_5 \gamma^\rho \psi \bar{\psi} \gamma^\sigma \psi.$$

Four years after Wulkenhaar's paper, there came along the encouraging results by Buric, Latas and Radovanovic [13], and, Buric, Radovanovic and Trampetic [14], that the gauge sector of SU(N) and the noncommutative Standard Model were one-loop renormalisable at first order in  $\theta$ . And yet, due to the infamous four-point fermionic divergence above, the construction of theories with a renormalisable one-loop and first-order-in- $\theta$  matter sector remained an open issue. Then it appeared a new paper by Buric, Latas, Radovanovic and Trampetic [15], where they showed that the divergence of the four-point fermionic function vanishes for a noncommutative SU(2) chiral theory with the matter sector being an SU(2)-doublet of noncommutative left-handed fermions. This result was later generalized in ref. [16] to any noncommutative GUT-inspired theory with only fermions as matter fields. Let me recall that by noncommutative GUT-inspired theories I mean

gauge theories whose noncommutative fermions are all –this is capital– left-handed multiplets, which transforms under a Grand Unification group. Thus, one of the obstacles –what about the renormalisability of the other 1PI functions?– to achieve one-loop and first-order-in- $\theta$  renormalisability had been removed by selecting Grand Unification –and, as we shall see, family unification, besides– as a guiding principle.

The absence of the infamous four-point fermionic divergence opened up the possibility of building noncommutative theories with massless fermionic noncommutative chiral matter that are one-loop renormalisable at first order in  $\theta$ . Actually, Wulkenhaar had already pointed out in ref. [12] that, in the massless case, noncommutative QED is (off-shell) one-loop renormalisable at first order in  $\theta$ , if one forgets about the fermionic four-point function. At long last, it was shown in ref. [17] that noncommutative GUT-inspired theories, with a matter sector made out of fermions and no scalars, were, on-shell and at first order in  $\theta$ , one-loop-renormalisable for any anomaly safe compact simple gauge group, if, and only if, all the flavour fermionic multiplets carry irreps with the same quadratic Casimir, ie, renormalisability is very partial to family unification. This selects SO(10), E<sub>6</sub>, and drops SU(5), as noncommutative Grand Unification groups–see [18].

We shall close this section with a quick recap of the results in ref. [17]. The action of the noncommutative GUT-inspired models in ref. [17] reads

$$S = \int d^4x - \frac{1}{2g^2} \text{Tr} F_{\mu\nu} \star F^{\mu\nu} + \bar{\Psi}_L i \not{D} \Psi_L,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star, \quad D_\mu \Psi_L = \partial_\mu \Psi_L - i\rho_\Psi(A_\mu) \star \Psi_L,$$

$\rho_\Psi$  denotes an arbitrary unitary representation, which is a direct sum of irreducible representations,  $\rho_\Psi = \bigoplus_{r=1}^F \rho_\Psi^r$ . Then, lengthy computations led to the following result:

Once  $\psi_L^r$ ,  $g$  and  $\theta$  have been renormalised as follows

$$\psi^r = (Z_\Psi^r)^{1/2} \psi_R^r, \quad g = \mu^{-\varepsilon} Z_g g_R, \quad \theta^{\mu\nu} = Z_\theta \theta_R^{\mu\nu},$$

$$Z_\Psi^r = 1 + \frac{g^2 C_2(r)}{16\pi^2 \varepsilon}, \quad Z_g = 1 + \frac{g^2}{16\pi^2 \varepsilon} \left[ \frac{11}{6} C_2(G) - \frac{2}{3} \sum_r c_2(r) \right],$$

$$Z_\theta = -Z_\Psi^r - \frac{g^2}{48\pi^2 \varepsilon} (13C_2(r) - 4C_2(G)),$$

the UV divergences, at one-loop and first order in  $\theta$ , which remain in the background-field effective action are given by the on-shell vanishing expression

$$S^{\text{ct}} = \int d^4x \frac{\delta S}{\delta a_\mu^a(x)} F_\mu^a[a, \psi] + \left( \sum_r \frac{\delta S}{\delta \psi^r(x)} G_r[a, \psi] + \text{c.c.} \right),$$

where

$$F_\mu = y_1 \theta^{\alpha\beta} \mathcal{D}_\mu f_{\alpha\beta} + y_2 \theta_\mu^\alpha D^\nu f_{\nu\alpha} + \sum_r y_3^r \theta_\mu^\alpha (\bar{\psi}_r \gamma_\alpha P_L T^a \psi^r) T^a$$

$$+ i \sum_r y_4^r \theta^{\alpha\beta} (\bar{\psi}_r \gamma_{\mu\alpha\beta} P_L T^a \psi^r) T^a + y_5 \tilde{\theta}_\mu^\beta D^\nu f_{\nu\beta},$$

$$G_{r,L} = k_1^r \theta^{\alpha\beta} f_{\alpha\beta} P_L \psi^r + k_2^r \theta^{\alpha\beta} \gamma_{\alpha\mu} P_L f_\beta^\mu \psi^r$$

$$+ k_3^r \theta^{\alpha\beta} \gamma_{\alpha\mu} P_L D_\beta D^\mu \psi^r + k_4^r \theta^{\alpha\beta} \gamma_{\alpha\beta} P_L D^2 \psi^r$$

$$+ k_5^r \tilde{\theta}^{\alpha\beta} \gamma_5 P_L f_{\alpha\beta} \psi^r; \quad y_i \in \mathbb{R}, \quad k_i \in \mathbb{C},$$

with

$$\begin{aligned}
y_1 &= \text{Im}k_1^r, y_3^r = 2g^2 y_2, \\
y_4^r &= -y_5 g^2 - \frac{g^4}{384\pi^2} (16C_2(r) - 13C_2(G)), \\
\text{Re}k_1^r &= -\frac{1}{2} \text{Im}k_3^r - \frac{g^2}{384\pi^2 \varepsilon} (13C_2(r) - 8C_2(G)), \\
\text{Im}k_5^r &= -\frac{g^2}{384\pi^2 \varepsilon} (11C_2(r) - 8C_2(G)), \\
\text{Im}k_4^r &= \frac{g^2 C_2(r)}{384\pi^2 \varepsilon}, \text{Re}k_2^r = -\frac{5g^2}{192\pi^2 \varepsilon} (2C_2(r) - C_2(G)), \\
\text{Im}k_2^r &= \text{Re}k_3^r = 2\text{Re}k_5^r = -2\text{Re}k_4^r.
\end{aligned}$$

Let me stress that  $y_1, y_2, y_5$  and  $Z_\theta$  above must be flavour independent, and so must be  $y_3, y_4$ . Hence.  $C_2(r)$ , must be the same for all irreps, which in turn demands family unification.

#### 4. Yukawa Terms in Noncommutative GUTs

Here I shall address the issue of constructing Yukawa terms in noncommutative SO(10) and  $E_6$  GUTs. For details I refer the reader to ref. [19].

Let us recall that Yukawa terms of ordinary SO(10) and  $E_6$  read

$$\mathcal{Y}^{(\text{ord})} = \int d^4x \mathcal{Y}_{ff'} \mathcal{C}_{AiB} \tilde{\Psi}_{Af}^\alpha \Psi_{\alpha Bf'} \phi_i, \quad (4.1)$$

where  $\tilde{\Psi}_f^\alpha \equiv (\Psi_f^\alpha)^t$ , and  $\mathcal{Y}_{ff'}$  denotes the Yukawa coefficients. For SO(10), each fermionic multiplet  $\Psi_{\alpha f'}$  carry the 16 irrep of SO(10), whereas, in the  $E_6$  GUT,  $\Psi_{\alpha f'}$  transforms under the 27 irrep of  $E_6$ . The Higgs multiplets in SO(10) carry any of the following irreps: 10, 120 and  $\overline{126}$ . In the  $E_6$  case the Higgs multiplets furnish any of the irreps of  $E_6$  that I enumerate now: 27, 351' and 351. Indeed, one has the following Clebsch-Gordan decompositions

$$16 \otimes 16 = (10 \oplus 126)_s \oplus 120_{\text{as}}, 27 \otimes 27 = (\overline{27} \oplus \overline{351}')_s \oplus \overline{351}_{\text{as}}$$

In eq. (4.1),  $\mathcal{C}_{AiB}$  is an invariant tensor:

$$\tilde{\Sigma}_{AC}^a \mathcal{C}_{CiB} + \mathcal{C}_{AjB} M_{ji}^a + \mathcal{C}_{AjC} \Sigma_{CB}^a = 0,$$

where  $\tilde{\Sigma}^a$ ,  $M^a$  and  $\Sigma^a$  denote the group generators in the irreps furnished by  $\tilde{\Psi}_{Af}^\alpha$ ,  $\phi_i$  and  $\Psi_{Bf'}^\alpha$ , respectively.

Let  $\tilde{\Psi}_{Af}^\alpha$ ,  $\Psi_{\alpha Bf'}$  and  $\Phi_i$  denote the noncommutative fermionic and Higgs fields defined by standard Seiberg-Witten maps, ie, solutions to

$$s_{NC}(NCField) \equiv i\Lambda \star (NCField) = s(NCField)$$

that match the ordinary fields at  $\theta = 0$ . Then, a naive noncommutative version

$$\mathcal{Y}_{(\text{naive})}^{(NC)} = \int d^4x \mathcal{Y}_{ff'} \mathcal{C}_{AiB} \tilde{\Psi}_{Af}^\alpha \star \Psi_{\alpha Bf'} \star \Phi_i$$

of the ordinary Yukawa term in eq. (4.1) would not do! Indeed,

$$\begin{aligned}
0 \neq s_{NC} \mathcal{Y}_{(\text{naive})}^{(NC)} &= \\
&\int d^4x \mathcal{Y}_{ff'} \mathcal{C}_{AiB} (i\tilde{\Lambda}_{AC} \star \tilde{\Psi}_{Cf}^\alpha \star \Psi_{\alpha Bf'} \star \Phi_i + \tilde{\Psi}_{Af}^\alpha \star i\Lambda_{BC} \star \Psi_{\alpha Cf'} \star \Phi_i \\
&\quad + \tilde{\Psi}_{Af}^\alpha \star \Psi_{\alpha Bf'} \star i\Lambda_{ij} \star \Phi_j),
\end{aligned}$$



for the  $\star$ -product is not commutative and  $\mathcal{C}_{AiB}$  is not invariant for enveloping-algebra valued  $\Lambda$ 's.

I shall now explain my strategy for constructing noncommutative Yukawa terms. To carry over the properties of  $\mathcal{C}_{AiB}$  to the noncommutative theory in a consistent way, one first combines  $\mathcal{C}_{AiB}$  with the ordinary fields  $\tilde{\Psi}_{Af}^\alpha$ ,  $\Psi_{\alpha Bf'}$  and  $\phi_i$ , and, then, defines new ordinary fields that transform under tensor products of ordinary irreps of the gauge group, but have the very same number of freedom as  $\tilde{\Psi}_{Af}^\alpha$ ,  $\Psi_{\alpha Bf'}$  and  $\phi_i$ :

$$\phi_{AB} = \mathcal{C}_{AiB} \phi_i, \quad \tilde{\Psi}_{iBf}^\alpha = \tilde{\Psi}_{Af}^\alpha \mathcal{C}_{AiB}, \quad \Psi_{\alpha Aif'} = \mathcal{C}_{AiB} \Psi_{\alpha Bif'}.$$

The BRS transformations of these new fields run thus:

$$\begin{aligned} s\phi_{AB} &= -i\tilde{\lambda}_{AC}^{(\psi)} \phi_{CB} - i\phi_{AC} \lambda_{CB}^{(\psi)}, \\ s\tilde{\Psi}_{iBf}^\alpha &= -i\tilde{\lambda}_{ij}^{(\phi)} \tilde{\Psi}_{jBf}^\alpha - i\tilde{\Psi}_{iCf}^\alpha \lambda_{CB}^{(\psi)}, \\ s\Psi_{\alpha Aif'} &= -i\tilde{\lambda}_{AC}^{(\psi)} \Psi_{\alpha Cif'} - i\Psi_{\alpha Aif'} \lambda_{ji}^{(\phi)}. \end{aligned}$$

Next, to each ordinary field  $\phi_{AB}$ ,  $\tilde{\Psi}_{iBf}^\alpha$  and  $\Psi_{\alpha Aif'}$ , one associates a noncommutative counterpart

$$\Phi_{AB}[\phi_{AB}, a_\mu^a, \theta], \quad \tilde{\Psi}_{iBf}^\alpha[\tilde{\Psi}_{iBf}^\alpha, a_\mu^a, \theta] \quad \text{and} \quad \Psi_{\alpha Aif'}[\Psi_{\alpha Aif'}, a_\mu^a, \theta],$$

which, respectively, are solutions to the following Hybrid Seiberg-Witten map equations:

$$s_{NC}\Phi_{AB} = s\Phi_{AB}, \quad s_{NC}\tilde{\Psi}_{iBf}^\alpha = s\tilde{\Psi}_{iBf}^\alpha, \quad s_{NC}\Psi_{\alpha Aif'} = s\Psi_{\alpha Aif'}, \quad (4.2)$$

where one defines

$$\begin{aligned} s_{NC}\Phi_{AB} &\equiv -i\tilde{\Lambda}_{AC}^{(\psi)} \star \Phi_{CB} - i\Phi_{AC} \star \Lambda_{CB}^{(\psi)}, \\ s_{NC}\tilde{\Psi}_{iBf}^\alpha &\equiv -i\tilde{\Lambda}_{ij}^{(\phi)} \star \tilde{\Psi}_{jBf}^\alpha - i\tilde{\Psi}_{iCf}^\alpha \star \Lambda_{CB}^{(\psi)}, \\ s_{nc}\Psi_{\alpha Aif'} &\equiv -i\tilde{\Lambda}_{AC}^{(\psi)} \star \Psi_{\alpha Cif'} - i\Psi_{\alpha Aif'} \star \Lambda_{ji}^{(\phi)}. \end{aligned}$$

Let me point out that the action from the left and from the right (as opposed to both actions from the left or both from the right) of the  $\Lambda$ 's is the only choice consistent with  $(s_{NC})^2 = 0!$ . The solutions to the equations in eq. (4.2) are Seiberg-Witten maps of hybrid type, a notion introduced by Schupp [7].

We are now in the position to obtain in a natural (naive) way noncommutative  $\text{SO}(10)$ ,  $E_6$  Yukawa terms from their ordinary counterparts. Indeed, in terms of  $\phi_{AB}$ , the ordinary Yukawa term reads:

$$\mathcal{Y}_1^{(\text{ord})} \equiv \mathcal{Y}^{(\text{ord})} = \int d^4x \mathcal{Y}_{ff'} \tilde{\Psi}_{Af}^\alpha \phi_{AB} \Psi_{\alpha Bf'},$$

so that, its noncommutative counterpart is

$$\mathcal{Y}_1^{(\text{nc})} = \int d^4x \mathcal{Y}_{ff'}^{(1)} \tilde{\Psi}_{Af}^\alpha \star \Phi_{AB} \star \Psi_{\alpha Bf'}.$$

In words: the noncommutative Yukawa term associated to  $\mathcal{Y}_1^{(\text{ord})}$  is obtained by replacing each ordinary field in the latter with its noncommutative counterpart and the ordinary product with the  $\star$ -product.

By construction  $\mathcal{Y}_1^{(\text{nc})}$  is invariant under the following noncommutative BRS transformations:

$$\begin{aligned} s_{\text{NC}} \tilde{\Psi}_{Af}^\alpha &= i \tilde{\Psi}_{Bf}^\alpha \star \tilde{\Lambda}_{BA}^{(\psi)}, & s_{\text{nc}} \Psi_{\alpha Bf'} &= i \Lambda_{BC}^{(\psi)} \star \Psi_{\alpha Cf'}, \\ s_{\text{NC}} \Phi_{AB} &= -i \tilde{\Lambda}_{AC}^{(\psi)} \star \Phi_{CB} - i \Phi_{AC} \star \Lambda_{CB}^{(\psi)}, \\ s_{\text{NC}} \tilde{\Lambda}_{BA}^{(\psi)} &= -i \tilde{\Lambda}_{BC}^{(\psi)} \star \tilde{\Lambda}_{CA}^{(\psi)}, & s_{\text{nc}} \Lambda_{BC}^{(\psi)} &= i \Lambda_{BD}^{(\psi)} \star \Lambda_{DC}^{(\psi)}. \end{aligned}$$

The Seiberg-Witten maps which define the noncommutative fields are

$$\begin{aligned} \tilde{\Psi}_{Af}^\alpha &= \tilde{\Psi}_{Af}^\alpha - \frac{1}{2} \theta^{\mu\nu} \partial_\mu \tilde{\Psi}_{Bf}^\alpha \tilde{a}_{\nu BA}^{(\psi)} + \frac{i}{4} \theta^{\mu\nu} \tilde{\Psi}_{Cf}^\alpha \tilde{a}_{\mu CB}^{(\psi)} \tilde{a}_{\nu BA}^{(\psi)} + O(\theta^2), \\ \Phi_{AB} &= \phi_{AB} + \frac{1}{2} \theta^{\mu\nu} \tilde{a}_{\mu AC}^{(\psi)} \partial_\nu \phi_{CB} + \frac{i}{4} \theta^{\mu\nu} \tilde{a}_{\mu AC}^{(\psi)} \tilde{a}_{\nu CD}^{(\psi)} \phi_{DB} + \\ &\quad + \frac{1}{2} \theta^{\mu\nu} \partial_\mu \phi_{AC} \tilde{a}_{\nu CB}^{(\psi)} + \frac{i}{4} \theta^{\mu\nu} \phi_{AC} \tilde{a}_{\mu CD}^{(\psi)} \tilde{a}_{\nu DB}^{(\psi)} \\ &\quad + \frac{i}{2} \theta^{\mu\nu} \tilde{a}_{\mu AC}^{(\psi)} \phi_{CD} \tilde{a}_{\nu DB}^{(\psi)} + O(\theta^2), \\ \Psi_{\alpha Bf'} &= \psi_{\alpha Bf'} - \frac{1}{2} \theta^{\mu\nu} a_{\mu BC}^{(\psi)} \partial_\mu \psi_{\alpha Cf'} + \frac{i}{4} \theta^{\mu\nu} a_{\mu BC}^{(\psi)} \tilde{a}_{\nu CD}^{(\psi)} \psi_{Df'}^\alpha + O(\theta^2). \end{aligned}$$

Let me now point out that if we use  $\tilde{\Psi}_{iBf}^\alpha$  and  $\Psi_{\alpha Aif'}$  to formulate an ordinary Yukawa term, we obtain the same ordinary Yukawa term:

$$\begin{aligned} \mathcal{Y}_2^{(\text{ord})} &= \int d^4x \mathcal{Y}_{ff'} \tilde{\phi}_i \tilde{\Psi}_{iBf}^\alpha \Psi_{\alpha Bf'}, \\ \mathcal{Y}_3^{(\text{ord})} &= \int d^4x \mathcal{Y}_{ff'} \tilde{\Psi}_{Af}^\alpha \Psi_{\alpha Aif'} \phi_i, \\ \mathcal{Y}_1^{(\text{ord})} &= \mathcal{Y}_2^{(\text{ord})} = \mathcal{Y}_3^{(\text{ord})}. \end{aligned}$$

And yet, the noncommutative counterparts of  $\mathcal{Y}_2^{(\text{ord})}$  and  $\mathcal{Y}_3^{(\text{ord})}$  are not equal:

$$\begin{aligned} \mathcal{Y}_2^{(\text{nc})} &= \int d^4x \mathcal{Y}_{ff'}^{(2)} \tilde{\Phi}_i \star \tilde{\Psi}_{iBf}^\alpha \star \Psi_{\alpha Bf'} \\ \mathcal{Y}_3^{(\text{nc})} &= \int d^4x \mathcal{Y}_{ff'}^{(3)} \mathcal{C}_{AiB} \tilde{\Psi}_{Af}^\alpha \phi_i \Psi_{\alpha Bf'} \\ \mathcal{Y}_1^{(\text{nc})} &\neq \mathcal{Y}_2^{(\text{nc})} \neq \mathcal{Y}_3^{(\text{nc})} \neq \mathcal{Y}_1^{(\text{nc})} \end{aligned}$$

Hence, I propose the following noncommutative Yukawa term

$$\mathcal{Y}^{(\text{nc})} = \mathcal{Y}_1^{(\text{nc})} + \mathcal{Y}_2^{(\text{nc})} + \mathcal{Y}_3^{(\text{nc})}.$$

It can be shown –see ref [19]– that at first order in  $\theta$  this is the most general BRS invariant Yukawa-type term

$$\theta^{\mu\nu} \int d^4x \mathcal{Y}_{ff'} \psi_{Af}^\alpha \mathcal{Y}_{\mu\nu}^{AiB} [\theta^{\rho\sigma}, \partial_\mu, a_\nu^a] \phi_i \Psi_{\alpha Bf'}$$

that one can write. This Yukawa term is therefore renormalisable at first order in  $\theta$ .

## 5. What about SUSY?

For  $U(N)$  in the fundamental rep.,  $\mathcal{N} = 1$  SYM exists in the enveloping-algebra formalism as a classical theory:

$$S_{\text{NCSYM}} = \frac{1}{2g^2} \text{Tr} \int d^4x \left[ -\frac{1}{2} F^{\mu\nu} \star F_{\mu\nu} - 2i \Lambda^\alpha \star \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \bar{\Lambda}^{\dot{\alpha}} + D \star D \right]$$

where

$$A_\mu = A_\mu[a, \lambda_\alpha, d, \theta], \Lambda_\alpha[a, \lambda_\alpha, d, \theta] \text{ and } D = D[a, \lambda_\alpha, d, \theta]$$

are SW maps.  $S_{NCSYM}$  is invariant under  $\mathcal{N} = 1$  SUSY:

- linearly realized in terms of the noncommutative fields ( there is a local superfield formulation) and
- nonlinearly realized in terms of the ordinary fields (no local superfield formulation exists, but a nonlocal one does, at least for U(1) –see [20]).

The  $\mathcal{N} = 1$  SUSY transformations of the noncommutative fields read

$$\begin{aligned} A_\mu[\varphi, \theta] &\rightarrow A_\mu^{(\varepsilon)}[\varphi, \theta] = A_\mu[\varphi, \theta] + \delta_\varepsilon A_\mu[\varphi, \theta] \\ \Lambda_\alpha[\varphi, \theta] &\rightarrow \Lambda_\alpha^{(\varepsilon)}[\varphi, \theta] = \Lambda_\alpha[\varphi, \theta] + \delta_\varepsilon \Lambda_\alpha[\varphi, \theta] \\ D[\varphi, \theta] &\rightarrow D^{(\varepsilon)}[\varphi, \theta] = D[\varphi, \theta] + \delta_\varepsilon D[\varphi, \theta] \end{aligned}$$

where  $\varphi$  denotes generically the ordinary fields and

$$\begin{aligned} \delta_\varepsilon A^\mu &= i\varepsilon^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\Lambda}^{\dot{\alpha}} + i\bar{\varepsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu \Lambda^\alpha, \\ \delta_\varepsilon \Lambda_\alpha &= (\sigma^{\mu\nu})_\alpha{}^\beta \varepsilon_\beta F_{\mu\nu} + i\varepsilon_\alpha D, \\ \delta_\varepsilon D &= -\varepsilon^\alpha \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \bar{\Lambda}^{\dot{\alpha}} + \bar{\varepsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu D_\mu \Lambda^\alpha. \end{aligned}$$

Now, the SUSY transformations have just introduced –do not forget that we are in the  $U(N)$  case in the fundamental representation– can be induced by performing a nonlinear variation of the ordinary fields, which up to first order in  $\theta$ , reads

$$\begin{aligned} \delta_\varepsilon a_\mu &= \frac{1}{4}\varepsilon\sigma_\mu\bar{\lambda} - \frac{1}{4}\bar{\varepsilon}\bar{\sigma}_\mu\lambda + \frac{1}{16}\theta^{\nu\rho} \left[ \{a_\nu, 2D_\rho(\varepsilon\sigma_\mu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\mu\lambda) - i[a_\rho, \varepsilon\sigma_\mu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\mu\lambda]\} \right. \\ &\quad \left. - \{\varepsilon\sigma_\nu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\nu\lambda, \partial_\rho a_\mu + f_{\rho\mu}\} - \{a_\nu, \partial_\rho(\varepsilon\sigma_\mu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\mu\lambda) + D_\rho(\varepsilon\sigma_\mu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\mu\lambda) \right. \\ &\quad \left. - D_\mu(\varepsilon\sigma_\rho\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\rho\lambda)\} \right] + \theta^2, \\ \delta_\varepsilon \lambda_\alpha &= -\varepsilon_\alpha d + 2i\varepsilon_\gamma (\sigma^{\mu\nu})^\gamma{}_\alpha f_{\mu\nu} + \frac{1}{4}\theta^{\nu\rho} \left[ -\frac{1}{4}\{\varepsilon\sigma_\nu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\nu\lambda, 2D_\rho\lambda_\alpha - i[a_\rho, \lambda_\alpha]\} \right. \\ &\quad \left. - \{a_\nu, 4iD_\rho(\varepsilon_\gamma(\sigma^{\mu\rho})^\gamma{}_\alpha f_{\mu\lambda}) + 2[a_\rho, \varepsilon_\gamma(\sigma^{\mu\lambda})^\gamma{}_\alpha f_{\mu\lambda}] + \frac{i}{4}[\varepsilon\sigma_\rho\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\rho\lambda, \lambda_\alpha]\} \right] \\ &\quad + \theta^2 \\ \delta_\varepsilon d &= i\bar{\varepsilon}\bar{\sigma}^\mu D_\mu\lambda + i\varepsilon\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{4}\theta^{\nu\rho} \left[ 2i\{f_{\mu\nu}, \bar{\varepsilon}\bar{\sigma}^\mu D_\rho\lambda + \varepsilon\sigma^\mu D_\rho\bar{\lambda}\} \right. \\ &\quad \left. + i\{a_\nu, (\partial_\rho + D_\rho)(\bar{\varepsilon}\bar{\sigma}^\mu D_\mu\lambda + \varepsilon\sigma^\mu D_\mu\bar{\lambda})\} - \frac{1}{4}\{\varepsilon\sigma_\nu\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\nu\lambda, 2D_\rho d - i[a_\rho, d]\} \right. \\ &\quad \left. - \{a_\nu, 2D_\rho(i\bar{\varepsilon}\bar{\sigma}^\mu D_\mu\lambda + i\varepsilon\sigma^\mu D_\mu\bar{\lambda}) - i[a_\rho, i\bar{\varepsilon}\bar{\sigma}^\mu D_\mu\lambda + i\varepsilon\sigma^\mu D_\mu\bar{\lambda}] \right. \\ &\quad \left. + \frac{i}{4}[\varepsilon\sigma_\rho\bar{\lambda} - \bar{\varepsilon}\bar{\sigma}_\rho\lambda, d]\} \right] + \theta^2. \end{aligned} \tag{5.1}$$

The following comments concerning the nonlinear variations of the ordinary fields in eq. (5.1) are now in order:

- They are truly  $\mathcal{N} = 1$  SUSY transformations,

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}](\text{fields}) = i(\varepsilon_2\sigma^\mu\bar{\varepsilon}_1 - \varepsilon_1\sigma^\mu\bar{\varepsilon}_2)\partial_\mu(\text{fields}) + \text{gauge transformations},$$

due to the fact that the noncommutative fields carry a linear realisation of  $\mathcal{N} = 1$  SUSY. This holds at any order in  $\theta$  –see ref. [20].

- $\delta_\varepsilon a_\mu$ ,  $\delta_\varepsilon \lambda_\alpha$  and  $\delta_\varepsilon d$  belong to the Lie algebra of the ordinary gauge group only for  $U(N)$  in the fundamental rep. and its siblings, i.e.,
- for an arbitrary Lie algebra they take values on the enveloping-algebra: they are not ordinary field variations which are also ordinary fields.

The question that one should ask next is whether we can have SUSY noncommutative GUTs. It is apparent that for simple gauge groups in any representation, it still makes sense to consider the theory defined by the action

$$S = \frac{1}{2g^2} \text{Tr} \int d^4x \left[ -\frac{1}{2} F^{\mu\nu} \star F_{\mu\nu} - 2i \Lambda^\alpha \star \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \bar{\Lambda}^{\dot{\alpha}} + D \star D \right]$$

where

$$A_\mu = A_\mu[a, \lambda_\alpha, d, \theta], \quad \Lambda_\alpha = \Lambda_\alpha[a, \lambda_\alpha, d, \theta] \quad \text{and} \quad D = D[a, \lambda_\alpha, d, \theta]$$

are Seiberg-Witten maps. This action looks like a SUSY invariant noncommutative action, for it is invariant under the following transformations

$$\begin{aligned} A_\mu[\varphi, \theta] &\rightarrow A_\mu^{(\varepsilon)}[\varphi, \theta] = A_\mu[\varphi, \theta] + \delta_\varepsilon A_\mu[\varphi, \theta], \\ \Lambda_\alpha[\varphi, \theta] &\rightarrow \Lambda_\alpha^{(\varepsilon)}[\varphi, \theta] = \Lambda_\alpha[\varphi, \theta] + \delta_\varepsilon \Lambda_\alpha[\varphi, \theta], \\ D[\varphi, \theta] &\rightarrow D^{(\varepsilon)}[\varphi, \theta] = D[\varphi, \theta] + \delta_\varepsilon D[\varphi, \theta], \\ \delta_\varepsilon A^\mu &= i\varepsilon^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\Lambda}^{\dot{\alpha}} + i\bar{\varepsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu \Lambda^\alpha, \\ \delta_\varepsilon \Lambda_\alpha &= (\sigma^{\mu\nu})_{\alpha\beta} \varepsilon_\beta F_{\mu\nu} + i\varepsilon_\alpha D, \\ \delta_\varepsilon D &= -\varepsilon^\alpha \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \bar{\Lambda}^{\dot{\alpha}} + \bar{\varepsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu D_\mu \Lambda^\alpha, \end{aligned} \tag{5.2}$$

and these transformations satisfy the  $\mathcal{N} = 1$  SUSY algebra commutation relationship

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}](\text{NCfields}) = i(\varepsilon_2^\mu \bar{\varepsilon}_1 - \varepsilon_1^\mu \bar{\varepsilon}_2) \partial_\mu (\text{NCfields}) + \text{NC gauge transformations}.$$

Notice that  $\varphi$  denotes the ordinary fields  $a_\mu$ ,  $\lambda_\alpha$  and  $d$ , and NC stands for noncommutative. This all goes in the right direction, but there is a catch:  $A_\mu^{(\varepsilon)}[\varphi, \theta]$ ,  $\Lambda_\alpha^{(\varepsilon)}[\varphi, \theta]$  and  $D^{(\varepsilon)}[\varphi, \theta]$  are not Seiberg-Witten maps in the sense that there are no ordinary fields  $\varphi^{(\varepsilon)}[\varphi, \partial, \theta]$ ,

$$\varphi^{(\varepsilon)}[\varphi, \partial, \theta] = \varphi + \varepsilon^\alpha \phi_\alpha[\varphi, \partial, \theta] + \bar{\varepsilon}_{\dot{\alpha}} \bar{\varphi}^{\dot{\alpha}}[\varphi, \partial, \theta],$$

such that

$$A_\mu^{(\varepsilon)}[\varphi, \theta] = A_\mu[\varphi^{(\varepsilon)}[\varphi, \partial, \theta]; \theta], \quad \Lambda_\alpha^{(\varepsilon)}[\varphi, \theta] = \Lambda_\alpha[\varphi^{(\varepsilon)}[\varphi, \partial, \theta]; \theta], \quad D^{(\varepsilon)}[\varphi, \theta] = D[\varphi^{(\varepsilon)}[\varphi, \partial, \theta]; \theta],$$

where  $A_\mu[\varphi; \theta]$ ,  $\Lambda_\alpha[\varphi; \theta]$  and  $D[\varphi; \theta]$  are solutions to the Seiberg-Witten equations satisfying  $A_\mu[\varphi; \theta = 0] = a_\mu$ ,  $\Lambda_\alpha[\varphi; \theta = 0] = \lambda_\alpha$  and  $D[\varphi; \theta = 0] = d$ . The transformations in eq. (5.2) are therefore defined from the space of noncommutative "physical" fields –those defined by the Seiberg-Witten map as explained above– into the space of general fields taking values on the enveloping algebra. The so remaining question is whether this invariance has any physical consequences. In this regard, it is worth noticing that –unlike in the  $U(N)$  case– the SUSY noncommutative  $SU(N)$  theory thus obtained is one-loop and first-order-in- $\theta$  (off-shell) renormalisable. This would be just a lucky chance unless there is a symmetry at work, at first order in  $\theta$ , that relates the gluon and gluino dynamics –see [21].

Some additional information regarding noncommutative SUSY theories defined by means of the Seiberg-Witten map can be found in refs. [22–24] and [25].

## 6. Open problems

We shall conclude with a list of pressing problems:

- For  $SO(10)$  and  $E_6$ , inclusion of a phenomenologically relevant noncommutative Higgs potential: a non trivial issue as implied by the construction of Yukawa terms.
- Study of the one-loop renormalisability of those noncommutative GUTS at first order in  $\theta$ .
- Construction and analysis of the properties of noncommutative SUSY  $SO(10)$ ,  $E_6$ .
- Study of the phenomenological implications of noncommutative  $SO(10)$ ,  $E_6$  GUTs.
- Gauge anomalies, Fujikawa's method and index theorems. Recall that the index theorem in  $2n+2$  dimensions gives the gauge anomaly in  $2n$  dimensions, that the index of the Dirac operator does not change under small deformations of it, and that in our formalism we are considering small deformations of the ordinary Dirac operator. Putting it all together: no  $\theta$ -dependent anomalous terms.
- A challenging question: Will these noncommutative GUTs eventually find accommodation within F-theory [26]?
- A final question: can one formulate noncommutative GUTs without using the enveloping-algebra formalism? In answering this question in the affirmative, the ideas presented in refs. [27, 28] look most promising; see also [29].

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