κ-Minkowski differential calculi and star product

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We present the star-product algebra of the κ-deformation of Minkowski space and the formulation of Poincaré covariant differential calculus. We use the tools to construct a twisted *K*-cycle over the algebra and twisted cyclic cocycle.

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1. Introduction

The κ-deformation of Minkowski space was introduced by Lukierski, Nowicki, Ruegg [11] as a homogeneous space of deformed Poincaré symmetries. Treated alone as an algebra it could be described as a nilpotent Lie algebra with d self-adjoint generators x_i, $i = 0, \ldots d-1$, and relations:

$$[x_0, x_i] = \frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0, \quad i, j = 1, \ldots, d-1, \tag{1.1}$$

where $\kappa > 0$ can be viewed as a deformation parameter. Formally, the commutative limit of the coordinate algebra of Minkowski space is obtained in the limit $\kappa \to \infty$. The correct and precise formulation of the κ-Minkowski and its symmetries appeared later, with the discussion of the bicrossproduct structure of the κ-Poincaré by Majid and Ruegg [12].

The differential calculi on κ-Minkowski were thoroughly studied by several authors but almost exclusively in the purely algebraic setup. In this paper we want to present the construction of the fully covariant differential calculus [14] on the star-product version of the κ-Minkowski algebra recently introduced by Durhuus and Sitarz [9].

For simplicity we shall restrict ourselves to $d = 2$ case.

2. Star product

The early attempts to describe the algebra of the κ-Minkowski space through a star-product date to Agostini [1], followed recently by the series of papers of Dabrowski and Piacitelli [5–8]. In a most recent work, Durhuus and Sitarz [9] obtained a much simpler realization of the star product, together with precise description of the algebra, its properties and the symmetries.

We shall briefly sketch here here the main results of that paper, which provides a convenient realization of the κ-Minkowski algebra. This opens several possibilities towards further study of geometric properties of κ-Minkowski within the noncommutative geometry.

Let $\mathscr{B}$ be the linear space of functions over $\mathbb{R}^2$, which are Fourier transforms of functions with a compact support. We define the κ-star product as:

$$(f * g)(\alpha, \beta) = \frac{1}{2\pi} \int dv \int du \, f(\alpha + \frac{u}{\kappa}, \beta) g(\alpha, e^{-v}\beta) e^{-iuv} \tag{2.1}$$

or, equivalently as:

$$(f * g)(\alpha, \beta) = \frac{1}{2\pi} \int dv \int du \, f(\alpha + u, \beta) g(\alpha, e^{-\frac{v}{\kappa}}\beta) e^{-iuv}. \tag{2.2}$$

The algebra $\mathscr{B}$ comes equipped with the involution:

$$f^*(\alpha, \beta) = \frac{1}{2\pi} \int dv \int du \, \overline{f}(\alpha + u, e^{-\frac{v}{\kappa}}\beta) e^{-iuv}. \tag{2.3}$$

For detailed proofs of associativity and other properties, including the derivation of formulas (2.1-2.3) we refer the reader to [9].

The above definition is the Moyal-type formulation of a product over the subspace of smooth functions on $\mathbb{R}^2$. To see that it corresponds exactly to κ-Minkowski it has been demonstrated in [9] that the star product could be extended to a much bigger algebra.

If $\mathscr{C}$ denotes smooth functions of two variables, which are polynomially bounded, and, additionally they are Fourier transforms in the first variable of tempered distributions with compact support (in this variable) then the star product is still well defined. To see it let us denote by $\tilde{f}$ a Fourier transform of a function over $\mathbb{R}^2$ in the first variable. Then, from (2.2) we can rewrite the star product as:

$$f * g(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int dv \, \tilde{f}(v, \beta) g(\alpha, e^{-\frac{v}{\kappa}} \beta) e^{i\alpha v}. \tag{2.4}$$

The integral makes sense, since $\tilde{f}$ is of compact support in v, in fact, it makes sense even if $\tilde{f}$ is a distribution, provided that its support remains a compact subset of $\mathbb{R}$.

Since both $i_\alpha(\alpha, \beta) = \alpha$ and $i_\beta(\alpha, \beta) = \beta$ are of polynomial bound and are Fourier transforms of delta Dirac distribution in the first variable (which is clearly of compact support), then it makes sense to calculate their products. Using (2.4) and the standard calculus of distributions we obtain:

$$(i_\alpha * i_\beta)(\alpha, \beta) = \frac{i}{\kappa} \beta + \alpha \beta, \quad (i_\beta * i_\alpha)(\alpha, \beta) = \beta \alpha,$$

and therefore

$$[i_\alpha, i_\beta] = \frac{i}{\kappa} i_\beta,$$

so we can identify i_α with t and i_β with x.

The main advantage of the star product formulation is that, unlike in almost all considerations so far, we do not need to work with polynomials or apply rather inelegant procedure of normal ordering of coordinates. Moreover, we can make perfect sense of expressions, which are not of polynomial type.

3. The κ-deformed Poincaré algebra

The κ-deformation was treated from its very origins as a homogeneous space of deformed κ-Poincaré algebra [12, 15].

For that reason in addition to the star-product formulation we need to consider the realization of the deformed symmetries, which form a Hopf algebra. We follow the notation of [12].

We recall here the generators of 2-dimensional κ-deformed Poincaré algebra.

$$[P, E] = 0, \qquad\qquad\qquad [N, E] = P,$$
$$\triangle P = P \otimes 1 + e^{i\frac{E}{\kappa}} \otimes P, \qquad \triangle E = E \otimes 1 + 1 \otimes E,$$

and for the relations involving the boost,

$$[N, P] = \frac{i\kappa}{2}(1 - e^{2i\frac{E}{\kappa}}) + \frac{i}{2\kappa} P^2, \qquad \triangle N = N \otimes 1 + e^{i\frac{E}{\kappa}} \otimes N.$$

To make sense of the algebra as a Hopf algebra (as we want to avoid using the exponential function of one of the generators) we formally introduce an operator $\mathscr{E}$ (and its inverse) and rewrite the relations accordingly.

$$[P, \mathscr{E}] = 0, \qquad\qquad\qquad [E, \mathscr{E}] = 0,$$

$$[N, \mathscr{E}] = \frac{i}{\kappa} \mathscr{E} P, \qquad\qquad [N, P] = \frac{i\kappa}{2}(1 - \mathscr{E}^2) + \frac{i}{2\kappa} P^2,$$

$$\triangle P = P \otimes 1 + \mathscr{E} \otimes P, \qquad \triangle N = N \otimes 1 + \mathscr{E} \otimes N, \quad \triangle \mathscr{E} = \mathscr{E} \otimes \mathscr{E}.$$

The action of these generators on the generators of the κ-Minkowski algebra is the same as in the classical case:

$$
\begin{aligned}
P \triangleright t &= 0, & P \triangleright x &= 1, \\
E \triangleright t &= 1, & E \triangleright x &= 0, \\
N \triangleright t &= -x, & N \triangleright x &= -t, \\
\mathscr{E} \triangleright t &= t + \frac{i}{\kappa}, & \mathscr{E} \triangleright x &= x.
\end{aligned}
$$

To proceed further we need to know how to extend the action of the generators of Poincaré algebra to the star-product algebra $\mathscr{B}$.

Definition 3.1. *Let $f \in \mathscr{B}$ so that it is a Fourier transform of $\hat{f} \in C_0(\mathbb{R}^2)$. We define a one parameter group of linear operations on $\mathscr{B}$ in the following way. For any $\gamma \in \mathbb{R}$ let:*

$$T_\gamma(f)(\alpha, \beta) = \frac{1}{2\pi} \int du\, dv\, \hat{f}(u, v) e^{\gamma u} e^{-i(\alpha u + \beta v)}. \tag{3.1}$$

Since $\hat{f}$ is a function on $\mathbb{R}^2$ with compact support, so is its product by $e^{\gamma u}$. Hence the definition is well posed and the obtained function is in $\mathscr{B}$. It is easy to verify that

$$T_\gamma T_\delta = T_{\gamma + \delta}.$$

The explicit formula for the map $T_\gamma : f \to T_\gamma f$ is:

$$T_\gamma(f)(\alpha, \beta) = \frac{1}{2\pi} \int dp\, ds\, f(s, \beta) e^{-\gamma p} e^{-ip(s - \alpha)}. \tag{3.2}$$

Observe that we can formally write

$$(T_\gamma f)(\alpha, \beta) = f(\alpha + i\gamma, \beta). \tag{3.3}$$

If $\tilde{f}$ denotes the Fourier transform of f with respect to the first variable, it is also useful to have the following formula:

$$\widetilde{(T_\gamma f)}(v, \beta) = \tilde{f}(v, \beta) e^{\gamma v}. \tag{3.4}$$

Now we claim:

Proposition 3.2. *The map $T_\gamma : f \to T_g f$ is an algebra automorphism.*

Proof. Using the formula for the star product and the explicit formula for the transformation T_γ we have:

$$T_\gamma(f) * T_\gamma(g)(\alpha,\beta) = \frac{1}{2\pi}\int dudv\, T_\gamma(f)(\alpha+u,\beta)T_\gamma(g)(\alpha,e^{-v}\beta)e^{-iuv}$$

$$= \frac{1}{(2\pi)^3}\int dudv\int dpds\int dp'ds'\, f(s,\beta)g(s',e^{-v}\beta)e^{-\gamma p}e^{-\gamma p'}e^{-ip(s-\alpha-u)}e^{-ip'(s'-\alpha)}e^{-iuv}$$

$$= \frac{1}{(2\pi)^2}\int dvds\int dp'ds'\, f(s,\beta)g(s',e^{-v}\beta)e^{-\gamma(p'+v)}e^{-iv(s-\alpha)}e^{-ip'(s'-\alpha)} = \ldots$$

changing now variables: $s = S+U$, $s' = S$, $P = p'+v$ we obtain:

$$\ldots = \frac{1}{(2\pi)^2}\int dUdv\int dPdS\, f(S+U,\beta)g(S,e^{-v}\beta)e^{-\gamma P}e^{-iP(S-\alpha)}e^{-iUv}$$

$$= \frac{1}{(2\pi)}\int dPdS\,(f*g)(S,\beta)e^{-\gamma P}e^{-iP(S-\alpha)} = T_\gamma(f*g)(\alpha,\beta).$$

$\square$

Now we can state the main theorem of this section.

Theorem 3.3. *The algebra $\mathscr{B}$ is a Hopf module algebra with respect to the following action of the momentum algebra, with the generators $E,P,\mathscr{E}$ represented as linear operators on $\mathscr{B}$.*

$$(E \triangleright f)(\alpha,\beta) = \frac{\partial}{\partial\alpha}, \quad (P \triangleright f)(\alpha,\beta) = \frac{\partial}{\partial\beta}, \quad (\mathscr{E} \triangleright f)(\alpha,\beta) = (T_{\frac{1}{\kappa}}f)(\alpha,\beta).$$

Proof. The only thing to verify is the compatibility of the action of the generators with the product in the algebra $\mathscr{B}$. Since $\mathscr{E}$ is group-like and T_γ is an algebra automorphism this part is already done. As the coproduct for E is classical (Lie-algebra type) and E acts by partial derivation, it is easy to see that this is also satisfied. It remains only to check the action of P:

We calculate:

$$(P \triangleright (f*g))(\alpha,\beta) = \frac{\partial}{\partial\beta}\left(\frac{1}{2\pi}\int dudv f(\alpha+u,\beta)g(\alpha,e^{-\frac{v}{\kappa}}\beta)e^{-iuv}\right)$$

$$= \frac{1}{2\pi}\int dudv\left(\left(\frac{\partial}{\partial\beta}f(\alpha+u,\beta)\right)g(\alpha,e^{-\frac{v}{\kappa}}\beta)+f(\alpha+u,\beta)\left(\frac{\partial}{\partial\beta}g(\alpha,e^{-\frac{v}{\kappa}}\beta)\right)\right)e^{-iuv}$$

$$= ((P \triangleright f)*g)(\alpha,\beta)+\frac{1}{2\pi}\int dudv\, f(\alpha+u,\beta)e^{-\frac{v}{\kappa}}\left(\frac{\partial g}{\partial\beta}\right)(\alpha,e^{-\frac{v}{\kappa}}\beta)e^{-iuv}$$

$$= ((P \triangleright f)*g)(\alpha,\beta)+(\mathscr{E} \triangleright f)*(P \triangleright g)(\alpha,\beta),$$

where in the last line we have used:

$$((\mathscr{E} \triangleright f)*g)(\alpha,\beta) = \frac{1}{\sqrt{2\pi}}\int dv\, \widetilde{(\mathscr{E} \triangleright f)}(v,\beta)g(\alpha,e^{-\frac{v}{\kappa}}\beta)e^{i\alpha v}$$

$$= \frac{1}{\sqrt{2\pi}}\int dv\, \tilde{f}(v,\beta)e^{-\frac{v}{\kappa}}g(\alpha,e^{-\frac{v}{\kappa}}\beta)e^{i\alpha v}$$

$$= \frac{1}{2\pi}\int dudv\, f(\alpha+u,\beta)e^{-\frac{v}{\kappa}}g(\alpha,e^{-\frac{v}{\kappa}}\beta)e^{-iuv}.$$

$\square$

4. Poincaré invariant differential calculus

Lemma 4.1. *We can rewrite the action of the external derivative in the bicovariant and Lorentz invariant differential calculus, described in [14] in the following way:*

$$df = dx(\mathscr{E}^{-1}P \triangleright f) - i\frac{\kappa}{2}\psi_+ \left((\mathscr{E}^{-1} - 1) \triangleright f\right) + \frac{1}{2}\psi_- \left((\frac{i}{\kappa}\mathscr{E}^{-1}P^2 + i\kappa(\mathscr{E} - 1)) \triangleright f\right), \quad (4.1)$$

where the bimodule rules between the functions and the generating one-forms are:

$$f\, dx = dx f + \psi_+ \frac{i}{\kappa}(P \triangleright f),$$
$$f\, \psi_+ = \psi_+ (\mathscr{E}^{-1} \triangleright f) + dx\frac{2i}{\kappa}(\mathscr{E}^{-1}P \triangleright f) - \psi_- \frac{1}{\kappa^2}(\mathscr{E}^{-1}P^2 \triangleright f), \quad (4.2)$$
$$f\, \psi_- = \psi_- (\mathscr{E} \triangleright f).$$

Calculating it explicitly for the generators we have:

$$x\, dx = dx\, x + \frac{i}{\kappa}\psi_+$$
$$t\, dx = dx\, t,$$
$$x\, \psi_+ = \psi_+ x + 2\frac{i}{\kappa}dx,$$
$$t\, \psi_+ = \psi_+ t + \frac{i}{\kappa}\psi_+, \qquad\qquad (4.3)$$
$$x\, \psi_- = \psi_- x,$$
$$t\, \psi_- = \psi_- t - \frac{i}{\kappa}\psi_-,$$

and we can easily see that one recovers the three-dimensional bicovariant and Poincaré covariant calculus.

5. Differential algebra and *K*-cycle

Following the standard procedure how to extend the first order bicovariant differential calculus to higher order forms we obtain (compare with [14]) that the generating one-forms dx, ψ_+, ψ_- are all exact and anticommute with each other.

Our aim here, will be to extend the differential algebra to the *K*-cycle (see [4] for details) - that is to equip $\Omega^*(\mathscr{B})$ with a closed graded trace.

From [9] we know that there exists a unique twisted trace on the algebra, which is invariant under the action of the Poincaré algebra:

Proposition 5.1. *Let $\int f$ denote the standard Lebesgue integral over $\mathbb{R}^2$ for any $f \in \mathscr{B}$. Then it has the following properties:*

- $\int$ *is a twisted trace, that is for all $f,g \in \mathscr{B}$:*

$$\int (f * g) = \int \left(T_{\frac{1}{\kappa}}(g) * f\right),$$

- $\int$ is invariant with respect to the action of κ-Poincaré, that is, for any $h \in \mathscr{P}_\kappa$ and $f \in \mathscr{B}$:

$$\int (h \triangleright f) = \varepsilon(h) \int f.$$

Proof. The proof is by a straightforward computation. for further details we refer to [9]. Let us take $f, g \in \mathscr{B}$. We have:

$$\int f * g = \int d\alpha d\beta \, (f * g)(\alpha, \beta) = \frac{1}{2\pi} \int d\alpha d\beta \int dv \int du \, f(\alpha + u, \beta) g(\alpha, e^{-\frac{v}{\kappa}} \beta) e^{-iuv} \ldots$$

On the other hand, using (3.2):

$$\int (T_{\frac{1}{\kappa}}(f)) * g = \int d\alpha d\beta \, ((T_{\frac{1}{\kappa}}(f)) * g)(\alpha, \beta) =$$

$$= \frac{1}{(2\pi)^2} \int d\alpha d\beta \int dv \int du \int dp ds \, g(s, \beta) e^{-\frac{1}{\kappa} p} e^{-ip(s-\alpha-u)} f(\alpha, e^{-\frac{v}{\kappa}} \beta) e^{-iuv}$$

$$= \frac{1}{2\pi} \int d\alpha d\beta \int dv ds \, g(s, \beta) e^{-\frac{1}{\kappa} v} f(\alpha, e^{-\frac{v}{\kappa}} \beta) e^{-iv(s-\alpha)} = \ldots$$

where by a simple change of variables: $s = a$, $\alpha = a + r$, $v = -q$: $\beta = -\frac{v}{\kappa} b$ we obtain:

$$\ldots = \frac{1}{2\pi} \int da db \int dq dr \, f(a + r, b) g(a, e^{-\frac{1}{\kappa} q}) e^{-iqr}$$

which is the same expression as above, which ends the proof. $\qquad \square$

Using this twisted trace we can construct the twisted K-cycle over the algebra $\mathscr{B}$:

Proposition 5.2. *Let ω be a three-form in the Poincaré covariant differential calculus over $\mathscr{B}$:*

$$\omega = (dx \wedge \psi_+ \wedge \psi_-) f,$$

where $f \in \mathscr{B}$. The following linear functional from $\Omega^3(\mathscr{B}) \to \mathbb{C}$:

$$\int \omega = \int f. \tag{5.1}$$

is a closed graded twisted trace over the differential algebra $\Omega^*(\mathscr{B})$.

Proof. First, let us show that it is closed. Let $\omega = d\rho$, where ρ is a two-form:

$$\rho = (dx \wedge \psi_+) f_+ + (dx \wedge \psi_-) f_- + (\psi_+ \wedge \psi_-) f_{+-}. \tag{5.2}$$

To see that the integral of $d\rho$ vanishes it is sufficient to observe that the integral of each of the component functions of $d\rho$ vanishes. This, however is the consequence of the Poincaré invariance of the trace $\int$. Since the counit of the algebra elements, which define the action of the external derivative d are:

$$\varepsilon \left(\mathscr{E}^{-1} P \right) = 0, \quad \varepsilon \left(\mathscr{E}^{-1} - 1 \right) = 0, \quad \varepsilon \left(\frac{i}{\kappa} \mathscr{E}^{-1} P^2 + i\kappa (\mathscr{E} - 1) \right) = 0. \tag{5.3}$$

Then the trace of their actions on any element of the algebra must vanish. To see the property of the twisted trace, first for the product between a three-form $\omega = (dx \wedge \psi_+ \wedge \psi_-)g$ and a function f, we have:

$$
\begin{aligned}
\int \omega * f &= \int (dx \wedge \psi_+ \wedge \psi_- \, g * f) \\
&= \int (g * f) = \int (T_{\frac{1}{\kappa}}(f) * g) = \int dx \wedge \psi_+ \wedge \psi_- (T_{\frac{1}{\kappa}}(f) * g) \qquad (5.4) \\
&= \int T_{\frac{1}{\kappa}}(f) * (dx \wedge \psi_+ \wedge \psi_- \, g) = \int T_1(f) * \omega.
\end{aligned}
$$

For the product of a one form with a two form observe that the action of T_γ on the generating one-forms is trivial:

$$
T_\gamma(dx) = dx, \quad T_\gamma(\psi_+) = \psi_+, \quad T_\gamma(\psi_-) = \psi_-.
$$

Indeed, using the formula (4.1) and the fact that T_γ is group like and commutes with P and $\mathscr{E}$ we get the above property immediately.

Take now any two-form ρ, as in (5.2). Then, using the relations (4.2) we have:

$$
\begin{aligned}
\rho \wedge dx &= (dx \wedge \psi_+ \wedge \psi_-)f_{+-}, \\
\rho \wedge \psi + &= -(dx \wedge \psi_+ \wedge \psi_-)\left(\mathscr{E}^{-1} \triangleright f_+\right), \\
\rho \wedge \psi - &= (dx \wedge \psi_+ \wedge \psi_-)\left(\mathscr{E} \triangleright f_-\right).
\end{aligned}
$$

Since the twisted trace is invariant under the action of $\mathscr{E}$ we have:

$$
\begin{aligned}
\int \rho \wedge dx &= \int dx \wedge \rho = \int T_{\frac{1}{\kappa}}(dx) \wedge \rho, \\
\int \rho \wedge \psi + &= \int \psi_+ \wedge \rho = \int T_{\frac{1}{\kappa}}(\psi_-) \wedge \rho, \\
\int \rho \wedge \psi - &= \int \psi_- \wedge \rho = \int T_{\frac{1}{\kappa}}(\psi_-) \wedge \rho.
\end{aligned}
$$

which, together with the twisted trace property for the wedge product of a three form and a function completes the proof. $\qquad\square$

As a consequence of the above construction we can present an explicit analytic formula for the three-cyclic cocycle over the κ-Minkowski algebra.

Remark 5.3. *Let $f_0, f_1, f_2, f_3 \in \mathscr{B}$ be the functions over κ-Minkowski space. The multilinear functional:*

$$
\phi(f_0, f_1, f_2, f_3) = \int f_0 \, df_1 \wedge df_2 \wedge df_3,
$$

defines a twisted cyclic cocycle over $\mathscr{B}$.

This property is an immediate consequence of the theorem (2.1) of [10]. The explicit formula for ϕ (which could be derived easily using the definition and bimodule relations for the differential calculus and the closed graded trace) is in no way enlightening, so we skip it.

6. κ-Minkowski and Rieffel's deformation.

Although κ-deformation of the Minkowski space has been obtained by a procedure which is quite different from the Moyal deformation, it is interesting to notice that thanks to the star-product formulation we can find many similarities. In this section we shall show how to write it using integral presentation of deformation by the action of $\mathbb{R}^2$ in Rieffel's [13] approach.

First, let us start with the following action of $\mathbb{R}^2$ on the algebra $\mathscr{B}$:

$$\eta_{(r,s)}(f)(\alpha,\beta) = f(\alpha+r,e^s\beta), \quad (r,s) \in \mathbb{R}^2.$$

Further, define a linear operator $J : \mathbb{R}^2 \to \mathbb{R}^2$:

$$J(r,s) = (s,0).$$

Unlike in the Rieffel's case, the operator is not invertible, in fact, it is nilpotent. However, nothing forbids us from defining the following J-star product on $\mathscr{B}$:

$$(f *_J g)(\alpha,\beta) = \frac{1}{(2\pi)^2} \int du_1 du_2 dv_2 dv_2 \left(\eta_{J(u_1,u_2)}(f)\right)(\alpha,\beta)(\eta_{(v_1,v_2)}(g))(\alpha,\beta)e^{-i(u_1v_1+u_2v_2)}.$$

The involution on the space $\mathscr{B}$ can also be written in the language of Rieffel deformation:

$$f^*(\alpha,\beta) = \int du_1 du_2 \, \eta_{u_1,u_2}(\bar{f})(\alpha,\beta)e^{i(u_1,u_2)J(u_1,u_2)}.$$

We have:

Lemma 6.1. *If $f,g \in \mathscr{B}$ then the $*_J$ product is well-defined and equals the star product (2.1).*

Proof. Indeed, calculating:

$$\begin{aligned}
(f * g)(\alpha,\beta) &= \frac{1}{(2\pi)^2} \int du_1 du_2 dv_2 dv_2 \left(\rho_{J(u_1,u_2)}(f)\right)(\alpha,\beta)(\rho_{(v_1,v_2)}(g))(\alpha,\beta)e^{-i(u_1v_1+u_2v_2)}. \\
&= \frac{1}{(2\pi)^2} \int du_1 du_2 dv_2 dv_2 \, f(\alpha+u_2,\beta)g(\alpha+v_1,e^{v_2}\beta)e^{-i(u_1v_1+u_2v_2)} \\
&= \frac{1}{(2\pi)} \int du_2 dv_2 \, f(\alpha+u_2,\beta)g(\alpha,e^{v_2}\beta)e^{-iu_2v_2}
\end{aligned}$$

$\square$

As an immediate corollary we obtain that the $\mathbb{R}^2$ action on $\mathscr{B}$ is actually an algebra automorphism:

$$\begin{aligned}
\rho_{(r,s)}(f * g)(\alpha,\beta) &= \frac{1}{2\pi} \int f(\alpha+r+u,e^s\beta)g(\alpha+r,e^{v+s}b)e^{iuv}dudv \\
&= \left(\eta_{(r,s)}(f)\right) * \left(\eta_{(r,s)}(g)\right)(\alpha,\beta).
\end{aligned} \tag{6.1}$$

7. Conclusions

We have demonstrated here that the geometry of the κ-deformation can be approached from a different angle, when presented through the star product formulation. The most important result is the construction of the K-cycle, which leads to the first known example of a twisted cyclic cocycle over κ-Minkowski.

We see the construction we provided as the first step towards analysis of Connes-type geometry in the κ case. Of course, this raises a series of interesting questions. First of all, we do not know whether the twisted cyclic cocycle is not trivial. If so, then, surprisingly, κ-deformation will be (in view of cyclic cohomology) of dimension 3. Furthermore, the geometric interpretation of the constructed cocycle will be a challenge: in particular, to find a Dirac operator and a spectral triple (or twisted spectral triple), which provide the geometric realization of the cocycle.

So far, the analysis of κ-deformations was based on the polynomial algebra or formal expressions, like in [2, 3]. Using the star product in this formulation and its symmetries we open new possibilities to go beyond that approach, which might use Connes' noncommutative geometry to study geometric properties of that noncommutative space.

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