

A geometric approach to sector decomposition

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We present a non-iterative method of sector decomposition, which is used to factorise overlapping singularities from multi-dimensional integrals appearing in perturbative quantum field theory. The problem is converted to a set of ones in convex geometry, and they are solved with algorithms developed in computational geometry. A test implementation shows that this method produces less number of decomposed sectors than usual iterative sector decompositions.

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1. Introduction

Recent high-precision experiments in high energy physics require high-precision theoretical predictions in perturbation theory for a wide variety of physical processes, and the automatization of theoretical calculations are indispensable. In the evaluation of Feynman amplitudes with massless particles, infrared (soft and collinear) divergences can appear in each amplitude separately. However, for suitably defined physical observables, they cancel out among virtual loop corrections and real emission processes. One has to regulate infrared divergences at intermediate calculation steps.

In perturbative QCD, it is common that the divergences are regulated by dimensional regularisation, and divergent parts are expressed as poles in terms of $\varepsilon = (4 - D)/2$, where D is the space-time dimension. An one-dimensional example of an isolation of the divergence is as follows:

$$\begin{aligned} \int_0^1 dx \frac{f(x)}{x^{1-\varepsilon}} &= \int_0^1 dx \frac{f(0)}{x^{1-\varepsilon}} + \int_0^1 dx \frac{f(x) - f(0)}{x^{1-\varepsilon}} \\ &= \frac{f(0)}{\varepsilon} + \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int_0^1 dx \ln^n(x) \frac{f(x) - f(0)}{x}, \end{aligned} \quad (1.1)$$

where we have assumed that $f(x)$ is a finite function in the domain of integration. This integral is divergent for $\varepsilon = 0$ due to the factor $1/x^{1-\varepsilon}$, which produces a logarithmic divergence by integration around $x \sim 0$. In the right-hand side, a *subtraction technique* was used to isolate a pole of ε . The integral in the first term can be analytically performed, and the second term becomes finite for $\varepsilon \rightarrow 0$. This example shows that, if the singular part of the integrand is factored out as powers of an integration variable, one can easily extract divergent part as poles in terms of ε .

The extraction of divergences from more complicated integrals with *overlapping divergences* can be performed by using *sector decomposition* [1] in an algorithmic fashion [2–4]. Let us consider the following two-dimensional example:

$$I = \int_0^1 dx \int_0^1 dy \frac{f(x,y)}{(x+y)^{2-\varepsilon}}, \quad (1.2)$$

with a finite function $f(x,y)$ for the domain of integration. This integral is divergent for $\varepsilon = 0$ due to a singularity around both $x, y \sim 0$. We divide the integration region such that x and y are ordered in each region (sector):

$$I = \int_0^1 dx \int_0^1 dy \frac{f(x,y)}{(x+y)^{2-\varepsilon}} [\Theta(x > y) + \Theta(y > x)]. \quad (1.3)$$

In the sector with $x > y$, the singular behaviour is controlled by the dominant term x in the denominator. We rescale $y \rightarrow xy$ and then we factor out x from the denominator. In the sector with $y > x$, on the other hand, we factor out y with a rescaling $x \rightarrow xy$. We then obtain

$$I = \int_0^1 dx \int_0^1 dy \frac{f(x,xy)}{x^{1-\varepsilon}(1+y)^{2-\varepsilon}} + \int_0^1 dx \int_0^1 dy \frac{f(xy,y)}{y^{1-\varepsilon}(1+x)^{2-\varepsilon}}. \quad (1.4)$$

Now we can extract the poles from each integral by the subtraction technique.

For more complicated integrands, usual sector decomposition method divides the integration region and find appropriate new variables by an iterative way, but it is not trivial whether the

iteration always terminates or not. Actually it was found that a simple iterative method may fail into an infinite loop. To avoid this problem, *strategies* for iterated sector decomposition which are guaranteed to terminate mathematically have been proposed by several authors [5–7]. Since one has to compute the integration for each sector, the sector decomposition method is desired to produce as less number of sectors as possible. Up to now, several practical packages, in which iterated sector decomposition is implemented, are publicly available [5, 6, 8, 9].

We propose another method of sector decomposition, which is based on a classification of asymptotic behaviour of polynomials around the origin. We recast the problem to a set of ones in convex geometry, which can be solved with algorithms developed in computational geometry. These are deterministic algorithms without iterations. In this talk, based on Ref. [10], we present our basic ideas and show results from a test implementation of this method.

2. Factorisation of overlapping singularities

Let us consider the following polynomial:

$$Q(x_1, x_2) = x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2. \quad (2.1)$$

The our goal is to perform sector decomposition such as

$$\int_0^1 d^2x [Q(x_1, x_2)]^\beta = \int_0^1 d^2x (x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2)^\beta = \sum \int_0^1 d^2x x_1^{\alpha_1} x_2^{\alpha_2} [1 + \tilde{Q}(x_1, x_2)]^\beta, \quad (2.2)$$

where \tilde{Q} is a polynomial which vanishes at $x_1 = x_2 = 0$, and then the polynomial $1 + \tilde{Q}$ in the square bracket has a finite value at the origin.

First, we split the integration region into three sectors such that in each sector one of the terms in Q is dominant over other terms:

$$I = \int_0^1 d^2x (x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2)^\beta = \int_0^1 d^2x (x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2)^\beta \times [\Theta(x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2) + \Theta(x_1^2 x_2^2 > x_1 x_2^4, x_1^4 x_2) + \Theta(x_1^4 x_2 > x_1 x_2^4, x_1^2 x_2^2)]. \quad (2.3)$$

In the sector with $x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2$, the singular behaviour at the origin is governed by the dominant term $x_1 x_2^4$. Thus we factor out $x_1 x_2^4$, and the rest of the terms has a finite value by construction:

$$\begin{aligned} I_1 &= \int_0^1 d^2x (x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2)^\beta \Theta(x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2) \\ &= \int_0^1 d^2x (x_1 x_2^4)^\beta \left(1 + \frac{x_1^2 x_2^2}{x_1 x_2^4} + \frac{x_1^4 x_2}{x_1 x_2^4}\right)^\beta \Theta(x_1 x_2^4 > x_1^2 x_2^2, x_1^4 x_2) \\ &= \int_0^1 d^2x x_1^\beta x_2^{4\beta} \left(1 + \frac{x_1}{x_2^2} + \frac{x_1^3}{x_2^3}\right)^\beta \Theta(x_2^2 > x_1). \end{aligned} \quad (2.4)$$

In this form, the singular part of the integrand is factored out as powers of integration variables, but the integration region is not a unit square and the terms in the square bracket are not monomials. These difficulties can be removed with a rescaling $x_1 \rightarrow x_1 x_2^2$. We then obtain

$$I_1 = \int_0^1 d^2x x_1^\beta x_2^{6\beta+2} (1 + x_1 + x_1^3 x_2^3)^\beta. \quad (2.5)$$

One can factor out dominant terms in the other two sectors in a similar way, and rescale $(x_1, x_2) \rightarrow (x_1 x_2^2, x_1^2 x_2)$ and $x_2 \rightarrow x_1^2 x_2$, respectively, to obtain

$$I = \int_0^1 d^2 x x_1^\beta x_2^{6\beta+2} (1 + x_1 + x_1^3 x_2^3)^\beta + 3 \int_0^1 d^2 x x_1^{6\beta+2} x_2^{6\beta+2} (1 + x_1^3 + x_2^3)^\beta + \int_0^1 d^2 x x_1^{6\beta+2} x_2^\beta (1 + x_2 + x_1^3 x_2^3)^\beta. \quad (2.6)$$

Now one can easily isolate poles by the subtraction technique.

The above example shows that sector decomposition can be solved by the following two steps:

1. To determine the integration regions where each term becomes dominant and split sectors by them. Then the dominant term can be factored out.
2. To find a new parametrisation of the integration variables such that the integration region is restored to a unit hypercube and other terms are expressed as monomials. In general, this may require more splitting sectors.

In the next section, we will see that these problems can be recast to ones in convex geometry.

3. Geometric approach

In order to find the dominant terms, we change the integration variables $x_i = e^{-y_i}$. The polynomial Q becomes

$$Q = x_1 x_2^4 + x_1^2 x_2^2 + x_1^4 x_2 = e^{-(y_1+4y_2)} + e^{-(2y_1+2y_2)} + e^{-(4y_1+y_2)} \equiv e^{-\vec{P}_1 \cdot \vec{y}} + e^{-\vec{P}_2 \cdot \vec{y}} + e^{-\vec{P}_3 \cdot \vec{y}}, \quad (3.1)$$

with

$$\vec{P}_1 = (1, 4), \quad \vec{P}_2 = (2, 2), \quad \vec{P}_3 = (4, 1), \quad \vec{y} = (y_1, y_2). \quad (3.2)$$

In this way, each monomial in Q can be mapped one-to-one to a integer vector \vec{P}_i in the two-dimensional Euclidean space (Fig. 1a). For the sector where the first term is dominant, we can write

$$I_1 = \int_0^\infty d^2 y e^{-\vec{1} \cdot \vec{y}} \left(e^{-\vec{P}_1 \cdot \vec{y}} + e^{-\vec{P}_2 \cdot \vec{y}} + e^{-\vec{P}_3 \cdot \vec{y}} \right)^\beta \Theta\left((\vec{P}_2 - \vec{P}_1) \cdot \vec{y} > 0\right) \Theta\left((\vec{P}_3 - \vec{P}_1) \cdot \vec{y} > 0\right), \quad (3.3)$$

where $\vec{1} = (1, 1)$. The integration region is a *convex polyhedral cone* (with the apex at the origin). We denote the region where the first term is dominant as Δ_1 (Fig. 1b):

$$\Delta_1 = \{ \vec{y} \in \mathbb{R}_{\geq 0}^2 \mid (\vec{P}_2 - \vec{P}_1) \cdot \vec{y} > 0 \wedge (\vec{P}_3 - \vec{P}_1) \cdot \vec{y} > 0 \}, \quad (3.4)$$

and factor out the dominant term to obtain

$$I_1 = \int_0^\infty d^2 y e^{-(\beta \vec{P}_1 + \vec{1}) \cdot \vec{y}} \left(1 + e^{-(\vec{P}_2 - \vec{P}_1) \cdot \vec{y}} + e^{-(\vec{P}_3 - \vec{P}_1) \cdot \vec{y}} \right)^\beta \Theta(\vec{y} \in \Delta_1). \quad (3.5)$$

In general, an n -variable polynomial Q corresponds to a set of points in n -dimensional Euclidean space $Z = (\vec{P}_1, \vec{P}_2, \dots, \vec{P}_m)$ with $\vec{P}_i \in \mathbb{Z}_{\geq 0}^n$. Then the region where i -th term is dominant is

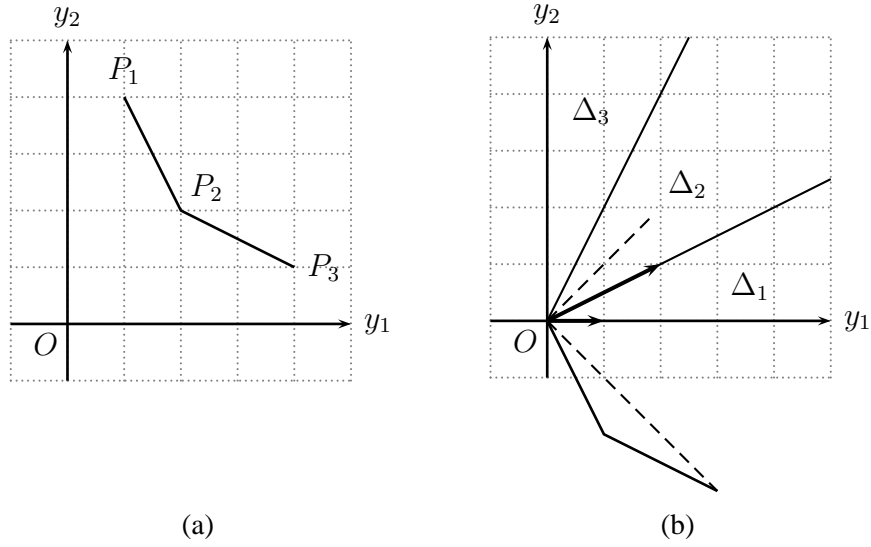


Figure 1: (a) The points in two-dimensional Euclidean space corresponding to the monomials in Q and (b) the corresponding regions where each monomial is dominant.

given as $\Delta_i = \{\vec{y} \in \mathbb{R}_{\geq 0}^n \mid (\vec{P}_j - \vec{P}_i) \cdot \vec{y} > 0, \forall \vec{P}_j \in Z\}$. One can express this region in terms of convex geometry. For a finite set S , a *convex polyhedral cone* C is defined by

$$C(S) = \left\{ \sum_{\vec{v} \in S} r_{\vec{v}} \vec{v} \in \mathbb{R}^n \mid r_{\vec{v}} \geq 0, \forall \vec{v} \in S \right\}. \quad (3.6)$$

A *dual cone* C^\vee of a convex polyhedral cone C is defined by

$$C(S)^\vee = \{\vec{y} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{y} \geq 0, \forall \vec{v} \in C(S)\}. \quad (3.7)$$

Then Δ_i can be expressed as

$$\Delta_i = C(Z_i)^\vee \cap \mathbb{R}_{\geq 0}^n, \quad \text{with } Z_i = (\vec{P}_1 - \vec{P}_i, \dots, \vec{P}_m - \vec{P}_i). \quad (3.8)$$

If there are two or more polynomials one must consider, one should consider an intersection of dual cones, such as $\Delta_{ij} = C(Z_i)^\vee \cap C(Z_j)^\vee \cap \mathbb{R}_{\geq 0}^n$.

The integration region Δ_1 in Eq. (3.5) is a *simplicial cone*, a convex polyhedral cone with n edges in n -dimensional space. This means one can parametrise a point \vec{y} in Δ_1 by *barycentric coordinate*:

$$y_i = \sum_{j=1}^2 (\vec{v}_j)_i u_j, \quad u_j \in \mathbb{R}_{\geq 0}, \quad (3.9)$$

where \vec{v}_i 's are integer vectors corresponding to edges of Δ_1 . From Fig. 1b, one can read $v_1 = (1, 0)$ and $v_2 = (2, 1)$ for Δ_1 . We subsequently change the variable from y_i to u_j then from u_j to $z_i = e^{-u_i}$. We finally obtain

$$\begin{aligned} I_1 &= |\det V| \int_0^1 d^2 z \prod_{i=1}^2 z_i^{(\beta \vec{P}_1 + \vec{1}) \cdot \vec{v}_i - 1} \left[1 + \prod_{i=1}^2 z_i^{(\vec{P}_2 - \vec{P}_1) \cdot \vec{v}_i} + \prod_{i=1}^2 z_i^{(\vec{P}_3 - \vec{P}_1) \cdot \vec{v}_i} \right]^\beta \\ &= \int_0^1 d^2 z z_1^\beta z_2^{6\beta+2} (1 + z_1 + z_1^3 z_2^3)^\beta, \end{aligned} \quad (3.10)$$

Table 1: Comparison of number of sectors among different methods. The numbers in columns “A”, “B”, “C”, “S” and “X” indicate corresponding strategy described in Refs. [5] and [6]. The numbers in column “H” are from Ref. [14]. The numbers marked with “*” are given by [15]. “This method” indicates the number of sectors obtained by our method. “Exponential S.D.” indicates the number of sectors before the triangulation. As described in Ref. [6], “F” means that the sector decomposition fails and “M” means that the memory overflow happened during the sector decomposition on a 8Gb machine.

Diagram	A	B	C	S	X	H	This method	Exponential S.D.
Bubble	2	2	2	2*	2		2	2
Triangle	3	3	3	3*	3		3	3
Box	12	12	12	12	12		12	8
Tbubble	58	48	48	48*	48		48	36
Double box, $p_i^2 = 0$	775	586	586	362	293	282	266	106
Double box, $p_4^2 \neq 0$	543*	245*	245*	230*	192*	197	186	100
Double box, $p_i^2 = 0$ nonplanar	1138	698	698	441*	395		360	120
D420	8898	564	564	180	F		168	100
3 loop vertex (A8)	4617*	1196*	1196*	871*	750*	684	684	240
Triple box	M	114256	114256	22657	10155		6568	856

where V is a matrix constructed as $V = (v_1, v_2)$. Note that $(\vec{P}_i - \vec{P}_1) \cdot \vec{v}_j$ is a non-negative integer since $\vec{v}_j \in \Delta_1$.

In three or higher dimension, a convex polyhedral cone may not be a simplicial cone. In a such case, we need *triangulation*, splitting a convex polyhedral cone into simplicial cones. The triangulation of convex polyhedral cones is not uniquely determined, and the number of sectors will depend on the algorithm.

4. Test implementation

We have made a test implementation of this method. For a convex hull method to construct intersections of dual cones, we adapted incremental algorithm described in Ref. [11], and modified it for convex polyhedral cones. For the triangulation of convex polyhedral cones, we used our own algorithm described in Ref. [10].

The procedure of constructing convex hull was checked by comparison with another package `qhull` [12]. The triangulation of convex polyhedral cones was checked by computing integration volume of resulting simplexes. As non-trivial checks, we compared integrated values of several loop diagrams in the literature. The input to our program was given by the program package partly described in Ref. [13], which builds expressions of loop integrals and performs primary sector decomposition. The output was passed to the same program package, which continues the subsequent isolation of divergences by the subtraction, expansion in terms of ϵ , and numerical integration steps. It was confirmed that integrated values agree well within the statistical errors. We show the numbers of decomposed sectors in Table 1.

5. Conclusion

We have proposed a new method of sector decomposition employing a geometric interpretation of the problem. The original problem is converted to a set of problems in convex geometry. A test implementation shows that the number of decomposed sectors is less than usual iterative sector decompositions.

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