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Determination of the light quark masses from $\eta \to 3\pi$

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Recently, several experimental collaborations have invested considerable effort into new and more precise measurements of the $\eta \to 3\pi$ decays. These experimental advances require revisiting the corresponding theoretical analyses. In this work, we present a new calculation of the $\eta \to 3\pi$ decay amplitude relying on dispersive methods. We show how the study of this decay allows one to extract a fundamental parameter of the Standard Model, namely the quark mass ratio $Q^2 \equiv (m_s^2 - \hat{m}^2)/(m_d^2 - m_u^2)$, with good precision. We find $Q = 21.3 \pm 0.6$. We then discuss the possibility of extracting the individual light quark masses.

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1. Introduction

Studying the $\eta \to 3\pi$ decay is particularly interesting because this decay allows one to have access to the light quark mass difference $m_d - m_u$. Bose statistics does not allow three pions to form a configuration where both the total angular momentum and the total isospin vanish. Since the η meson has isospin $I = 0$, this decay proceeds exclusively through isospin violating operators. In the Standard Model, there are two sources of isospin violation: strong and electromagnetic (EM) interactions. It has been shown that the EM corrections to this decay are very small [1]. To a good approximation the decay rate is therefore proportional to the square of the light quark mass difference. If one were able to accurately calculate the proportionality factor, a measurement of the decay rate would thus provide a determination of this quark mass difference. This is the aim of this work.

It has been shown that due to the Kaplan-Manohar ambiguity [2] appearing at next-to-leading order (NLO) of Chiral Perturbation Theory (χPT) a better quantity to extract from this decay is the quark mass double ratio

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}\ , \ \text{ with } \ \hat{m} = \frac{m_u + m_d}{2}\ . \tag{1.1}$$

Determining Q gives an elliptic constraint on the quark mass ratios m_u/m_d and m_s/m_d [3]. The ratio $S = m_s/\hat{m}$ is known very accurately from lattice QCD, but m_u/m_d is still poorly known. In view of the relation $m_u/m_d = (4Q^2 - S^2 + 1)/(4Q^2 + S^2 - 1)$, a sharp determination of Q would lead to an accurate value for m_u/m_d. The amplitude of this decay can be expressed as

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2(M_K^2 - M_\pi^2)}{3\sqrt{3}M_\pi^2 F_\pi^2}M(s,t,u)\ , \tag{1.2}$$

with s, t and u the three Mandelstam variables satisfying $s + t + u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^\pm}^2 \equiv 3s_0$.

Both the theoretical prediction and the measurement of $\eta \to 3\pi$ are extremely involved. On the experimental side, the value of the decay width has increased by more than 3σ since the eighties, from $\Gamma_{\eta\to\pi^+\pi^-\pi^0} = 197 \pm 29$ eV in 1982 to 295 ± 20 eV today. This large shift is almost entirely due to the increase of the value of the total decay width, which is fixed via the process $\eta \to 2\gamma$. On the theory side, the main difficulty is the evaluation of rescattering effects among the pions in the final state [5]. This can be done perturbatively within $SU(3)$ χPT but the chiral expansion converges rather slowly in this case. At tree level, the decay width evaluation [4] gives $\Gamma_{\text{tree}} = 66$ eV in clear disagreement with the experimental result. The one loop calculation, taking one final state rescattering into account, leads to a sizable correction: $\Gamma_{\text{one loop}} = 160 \pm 50$ eV [6]. This result agrees with the experimental result of the time (PDG 1982) but not with today's value. Higher order corrections should be included, as they are expected to be large. A two-loop calculation in χPT was performed in Ref. [7] but for $\eta \to 3\pi$ a large number of unknown low-energy constants enter, reducing the predictive power of this evaluation. Another approach relies on the use of dispersion relations. As in χPT, some unknowns, called the subtraction constants, enter the calculation and must be determined with other methods. Dispersion relations were applied successfully to $\eta \to 3\pi$ in Refs. [8, 9].

Here we will present a new dispersive analysis following the approach of Ref. [9]. The subtraction constants are fixed from a fit to the precise Dalitz plot measurement from the KLOE collaboration [13], which is a new feature compared to the original work. Moreover, we can rely on new precise inputs for the dispersion relations, the $\pi\pi$ phase shifts that have been extracted recently [10 – 12]. Besides, an intense experimental activity in the sector [13, 14] as well as recent important theoretical works [15 – 18] motivate this study.

2. Dispersive analysis of $\eta \to 3\pi$ decays

The dispersive method applied to the $\eta \to 3\pi$ decays relies on the decomposition of the amplitude $M(s,t,u)$ into *S*- and *P*-partial waves [9]

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s) \ . \tag{2.1}$$

The $M_I(s)$, where I stands for isospin, are functions of one variable only, with only a right-hand cut. This decomposition is exact up to chiral corrections of order p^8. The unitary relations for the $M_I(s)$ are given by

$$\text{disc}\, M_I(s) \doteq \frac{M_I(s+i\varepsilon) - M_I(s-i\varepsilon)}{2i} = \left\{M_I(s) + \hat{M}_I(s)\right\} e^{-i\delta_I(s)} \sin\delta_I(s) \ , \tag{2.2}$$

where $\delta_I(s)$ are the *S*- and *P*-wave $\pi\pi$ scattering phase shifts and $I = 0,1,2$. The inhomogeneities $\hat{M}_I(s)$ contain the left-hand cuts of the partial waves. They are obtained from angular averages over the $M_I(s)$, leading to a set of coupled equations. These angular averages are non-trivial and generate complex analytic structures. Knowing the discontinuities of the $M_I(s)$, one can write a set of dispersion integrals:

$$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin\delta_0(s)\hat{M}_0(s)}{|\Omega_0(s')|(s'-s-i\varepsilon)}\right) \ , \tag{2.3}$$

and similarly for M_1 and M_2. The Omnès functions $\Omega_I(s)$ are given by

$$\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\delta_I(s')}{s'(s'-s)} ds'\right) \ . \tag{2.4}$$

The equations for the $M_I(s)$ are solved by an iterative procedure starting with some initial configuration for the $M_I(s)$. The inputs of the dispersive integrals are the $\pi\pi$ phase shift $\delta_I(s)$ taken from Ref. [10]. The dispersion relations fix the amplitude $M(s,t,u)$ in Eq. (2.1) up to four subtraction constants: α_0, β_0, γ_0 (see Eq. (2.3) for $M_0(s)$) and β_1 (in the analogous relation for $M_1(s)$). In Refs. [9, 19], these constants were determined from a matching to the one-loop χPT result. The present analysis relies on a different determination of the subtraction constants, which invokes the precise Dalitz measurement by the KLOE collaboration [13]. We perform a fit where the parameters α_0, β_0, γ_0 and β_1 are determined by simultaneously minimizing the difference between the dispersive representation of the amplitude and both, the one-loop representation from χPT (in the vicinity of the Adler zero) as well as the observed Dalitz plot distribution (in the physical region of the decay). The position of the Adler zero, but not the shape of the amplitude there, is protected by $SU(2) \times SU(2)$ chiral symmetry. Thus we expect the neglected NNLO corrections to be smaller there than, say, at the center of the Dalitz plot.

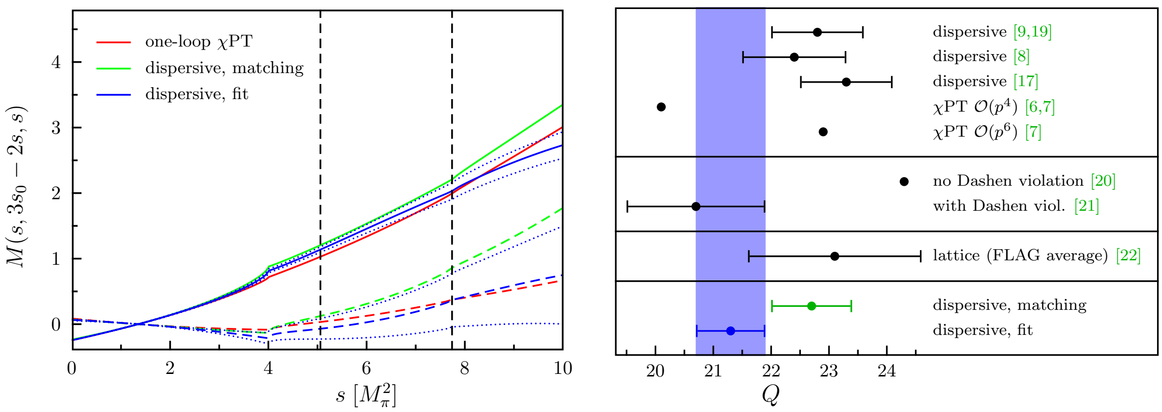

Figure 1: Left: The figure shows several results for the decay amplitude $M(s,t,u)$ along the line $s = u$. The solid and dashed lines stand for the central values of the real and imaginary part, respectively and the dotted lines for the error bands. The vertical dashed lines mark the boundaries of the physical region. Right: Comparison of several results for the quark mass ratio Q.

3. Results

In Fig. 1 (left), we present the results for the amplitude along the line $s = u$. We have repeated the analysis of Refs. [9, 19] using updated inputs. This leads to the green curve labeled as "dispersive, matching". One observes sizable corrections compared to one-loop χPT (in red in Fig. 1 (left)) in the physical region. Using the PDG average for the decay width [23] and the dispersive result for the amplitude, we get $Q = 22.7 \pm 0.7$. One can then express the squared amplitude in terms of the Dalitz plot variables $X = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$ and $Y = \frac{3}{2M_\eta Q_c}\left((M_\eta - M_{\pi^0})^2 - s\right) - 1$ with $Q_c = M_\eta - 2M_{\pi^+} - M_{\pi^0}$, and compare the theoretical distribution to the measurement [13], see Fig. 2. Note that the experimental result is given in terms of the Dalitz plot parameters which are the coefficients of the expansion $|A(X,Y)|^2 \propto 1 + aY + bY^2 + dX^2 + fY^3 + \ldots$ While the Dalitz distribution along the line $Y = 0$ is in very good agreement with the experimental result, it grossly overestimates the data along the line $X = 0.125$ for large negative Y (large positive s).

Our main result, which comes from the fit to the experimental Dalitz distribution as described in Sec. 2, is represented by the blue curves labeled "dispersive, fit" in Figs. 1 and 2. Naturally, the results now agree with experiment. One observes in Fig. 1 (left) that the corrections to the one-loop χPT result are now smaller in the physical region. However, the imaginary part has large uncertainties because the fit only constrains the absolute value squared of the amplitude. This analysis yields $Q = 21.3 \pm 0.6$, which is compared to other results in Fig. 1 (right). Our result stands between the one and two-loop χPT results and is lower than the outcome of the other dispersive analyses. However, it agrees well with the estimate coming from kaon mass splitting including large Dashen violation [21]. The discrepancy with the results of Ref. [17], where a similar dispersive analysis was performed and the same data were used to determine the subtraction constants, is not yet understood. As Q is very sensitive to the normalization of the amplitude, the most likely reason lies in the difference of the procedure to fix it. While we have chosen to perform a fit along the line $s = u$, the authors of Ref. [17] fit along the line $t = u$, which would lead to a very strong violation of the Adler zero position in our analysis.

Only one experimental result is available in the charged mode. However, in the neutral channel

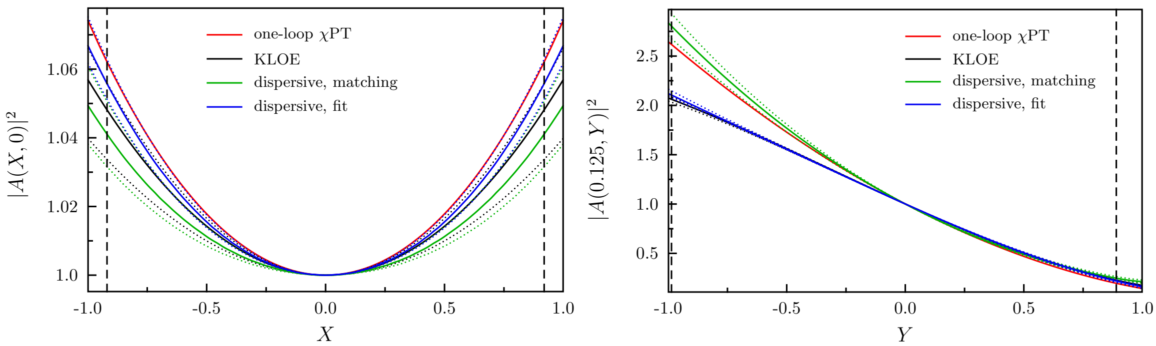

Figure 2: The Dalitz distribution of the amplitude $\eta \to \pi^+\pi^-\pi^0$ along the lines $Y = 0$ (left) and $X = 0.125$ (right). The dashed lines represent the limits of the physical region. The uncertainty band is given by the dotted lines. In the right panel, we choose $X \neq 0$ to probe the term X^2Y of the Dalitz distribution.

there exist a large number of experimental results that predict the slope α of the Dalitz distribution $|\bar{A}(X,Y)|^2 \propto (1+2\alpha Z)$ with $Z = X^2+Y^2$. All the measured values of α are in agreement which each other and with the PDG average $\alpha = -0.0317 \pm 0.0016$ [23]. However, most of the calculations predict the wrong sign for α. One can obtain α from the dispersive analysis of the charged mode by using isospin symmetry to relate the charged and neutral amplitudes. The result from the dispersive analysis with matching to one-loop χPT is $\alpha = 0.030 \pm 0.011$ confirming the previous theoretical results. From the fit, we obtain $\alpha = -0.045 \pm 0.010$, which has the correct sign but is only in marginal agreement with the PDG value. Note that the latter has a very small uncertainty. In order to obtain the best value for Q from the dispersive analysis, we can use all our experimental knowledge and further constrain the subtraction constants by requiring the dispersive analysis to reproduce the experimental value for α. Such a study is currently in progress.

Using the value of Q from the dispersive analysis and the most precise values of $\hat{m}$ and m_s from lattice QCD [24], we obtain an estimate of the reachable precision for the extraction of the light quark masses, $m_u = (2.02 \pm 0.14)$ MeV and $m_d = (4.91 \pm 0.11)$ MeV.

We would like to stress that all the results presented here are still preliminary and the last refinements on the numerical analysis are underway. In particular, the $\mathscr{O}(p^6)$ effects in the determination of the subtraction constants are being investigated.

4. Conclusion

In this talk, we have presented a new dispersive analysis of $\eta \to 3\pi$. This decay represents a very interesting source of information on the light quark masses through the determination of the quark mass ratio Q. To this end, one needs to have the strong rescattering effects in the final state under control. This is possible thanks to dispersion relations, which allow one to know the amplitude up to subtraction constants. Fixing these constants represents the main difficulty of the analysis. Here, we have presented a new analysis where the subtraction constants have been determined using experimental data from the charged channel. This yields $Q = 21.3 \pm 0.6$. The estimate used for the size of the NNLO effects in the vicinity of the Adler zero is a delicate point in the error analysis. This issue still needs to be studied in more detail. Moreover, the present work relies on data from a single experiment, but hopefully new measurements will appear soon and help to improve the analysis.

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