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Infrared properties of the gluon mass equation

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The gauge-invariant generation of a dynamical, momentum-dependent gluon mass is intimately connected with the presence of non-perturbative massless poles in the vertices of the theory, which trigger the well-known Schwinger mechanism. In the deep infrared the integral equation that governs this effective gluon mass assumes a particularly simple form, which may be derived following two seemingly different, but ultimately equivalent procedures. In particular, it may be obtained either as a deviation from a special identity that enforces the masslessness of the gluon in the absence of massless poles, or as a direct consequence of the appearance of a non-vanishing bound-state wave function, associated with the details of the actual formation of these massless poles. In this presentation we demonstrate that, due to profound relations between the various ingredients, the two versions of the gluon mass equation are in fact absolutely identical.

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1. Introduction

It is well-established by now that the dynamical generation of an effective gluon mass [1] explains in a natural and self-consistent way the infrared finiteness of the (Landau gauge) gluon propagator and ghost dressing function, observed in large volume lattice simulations for both $SU(2)$ [2] and $SU(3)$ [3, 4, 5] gauge groups (for an alternative approach see [6]). Given the non-perturbative nature of the mass generation mechanism, the usual starting point in the continuum is the Schwinger-Dyson equations (SDEs) governing the Green's functions under scrutiny. In the framework provided by the synthesis of the pinch technique (PT) [1, 7, 8] with the background field method (BFM) [9], these complicated integral equations are endowed with a variety of important properties, which allow a much tighter control on the truncations adopted and the approximation schemes employed.

Probably the most crucial theoretical ingredient in this context is a special type of vertex, denoted by V, which contains massless, longitudinally coupled poles. This vertex complements the all-order three-gluon vertex entering into the SDE governing the gluon self-energy, and captures the underlying mass generating mechanism, which is none other than the Schwinger mechanism. Specifically, the QCD dynamics is assumed to generate massless bound-state excitations, which, in turn, give rise to the aforementioned poles that appear inside the vertex V.

At the level of the SDE for the gluon propagator, the analysis finally boils down to the derivation of an integral equation that governs the evolution of the dynamical gluon mass, $m^2(q^2)$, as a function of the momentum q^2, in a way similar to the more familiar case of the dynamical generation of a constituent quark mass. However, unlike what happens with the quark gap equation, which, due to its Dirac structure, is unambiguously separated into two equations, governing the quark mass and wave-function, the derivation of the corresponding equations for the gluon mass and wave-function requires the introduction of some additional kinematic criteria [10]. For the purposes of this presentation, we will focus on the integral equation for $m^2(q^2)$ in the infrared limit, i.e, as $q^2 \to 0$, where this separation becomes unique and totally unambiguous: one assigns to the mass equation all contributions that do not vanish as $q^2 \to 0$.

It turns out that the equation for $m^2(0)$ may be derived following two rather distinct procedures, which eventually express the answer in terms of seemingly different quantities. Roughly speaking, the first procedure, which is operationally closer to the standard SDE treatment, identifies the contribution to the gluon mass equation as the deviation produced to the so-called "seagull-identity" by the inclusion of the vertex V. Interestingly enough, this can be accomplished without knowledge of the explicit closed form of the vertex V, relying only on its general features, most notably its longitudinal nature and the Slavnov-Taylor identities that it satisfies. The second procedure expresses the answer in terms of quantities such as "the bound-state wave-function", denoted by B_1, appearing in the study of the Schwinger mechanism from the point of view of the actual formation of the required bound state excitations. This particular quantity satisfies a homogeneous Bethe-Salpeter equation, which, as has been shown recently, admits indeed non-trivial solutions [11].

To be sure, these two procedures, despite their apparent differences, must be ultimately equivalent. In this talk we will argue that this is indeed the case, by showing that the two mass equations obtained coincides in the deep infrared limit, thus providing an important self-consistency check for this entire approach.

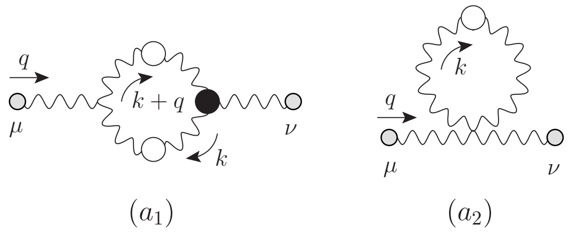

Figure 1: The "one-loop dressed" gluon contribution to the PT-BFM gluon self-energy. White (black) circles denote fully dressed propagators (vertices); a gray circle attached to the external legs indicates that they are background gluons. Within the PT-BFM framework these two diagrams constitute a transverse subset of the full gluon SDE.

2. Dynamical gluon mass: general concepts and ingredients

Let us start by setting up the notation and reviewing some of the most salient features of the dynamical gluon mass generation scenario formulated within the PT-BFM framework. The full gluon propagator $\Delta^{ab}_{\mu\nu}(q) = \delta^{ab}\Delta_{\mu\nu}(q)$ in the Landau gauge is defined as

$$\Delta_{\mu\nu}(q) = -iP_{\mu\nu}(q)\Delta(q^2); \qquad P_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \tag{2.1}$$

where the scalar factor $\Delta(q^2)$ and the gluon self-energy $\Pi_{\mu\nu}(q)$ are related through

$$\Delta^{-1}(q^2) = q^2 + i\Pi(q^2); \qquad \Pi_{\mu\nu}(q) = P_{\mu\nu}(q)\Pi(q^2). \tag{2.2}$$

For later convenience let us also define the *inverse* gluon dressing function, $J(q^2)$, as

$$\Delta^{-1}(q^2) = q^2 J(q^2); \tag{2.3}$$

then, in the presence of a dynamically generated mass, a massive gluon propagator is naturally described in term the natural form of the expression

$$\Delta_m^{-1}(q^2) = q^2 J_m(q^2) - m^2(q^2), \tag{2.4}$$

where the first term corresponds to the "kinetic term", or "wave function" contribution, whereas the second is the (positive-definite) momentum-dependent mass. Notice that the symbol J_m indicates that effectively one has now a mass inside the corresponding expressions: for example, whereas perturbatively $J(q^2) \sim \ln q^2$, after dynamical gluon mass generation has taken place, one has $J_m(q^2) \sim \ln(q^2 + m^2)$.

The usual starting point of our dynamical analysis is the SDE governing the gluon propagator. Within the PT-BFM framework that we employ, one may safely truncate the SDE series down to its "one-loop dressed version" containing gluonic contributions only, given by the diagrams (a_1) and (a_2) shown in Fig. 1 [12, 13, 14]. In fact, the resulting (approximate) gluon self-energy $\Pi_{\mu\nu}(q)$ is transverse, *i.e.*,

$$q^\mu \Pi_{\mu\nu}(q) = 0. \tag{2.5}$$

The PT-BFM equation for the conventional propagator reads, in this case,

$$\Delta^{-1}(q^2)P^{\mu\nu}(q) = \frac{q^2P^{\mu\nu}(q) + ig^2C_A\sum_{i=1}^{5}A_i^{\mu\nu}(q)}{[1+G(q^2)]^2}, \tag{2.6}$$

where, in the Landau gauge,

$$\begin{aligned}
A_1^{\mu\nu}(q) &= \frac{1}{2}\int_k \Gamma^{(0)\mu}_{\alpha\beta}P^{\alpha\rho}(k)P^{\beta\sigma}(k+q)\mathbb{\Gamma}^{\nu}_{\rho\sigma}\Delta(k)\Delta(k+q),\\
A_2^{\mu\nu}(q) &= \int_k P^{\alpha\mu}(k)\frac{(k+q)^\beta\Gamma^{(0)\nu}_{\alpha\beta}}{(k+q)^2}\Delta(k),\\
A_3^{\mu\nu}(q) &= \int_k P^{\alpha\mu}(k)\frac{(k+q)^\beta\mathbb{\Gamma}^{\nu}_{\alpha\beta}}{(k+q)^2}\Delta(k),\\
A_4^{\mu\nu}(q) &= -\frac{(d-1)^2}{d}g^{\mu\nu}\int_k\Delta(k),\\
A_5^{\mu\nu}(q) &= \int_k\frac{k^\mu(k+q)^\nu}{k^2(k+q)^2},
\end{aligned} \tag{2.7}$$

and the d-dimensional integral measure (in dimensional regularization) is defined as $\int_k \equiv \frac{\mu^\varepsilon}{(2\pi)^d}\int \mathrm{d}^dk$. The vertex $\mathbb{\Gamma}$ is the fully-dressed *BQQ* vertex, connecting a background gluon (*B*) to two quantum gluons (*Q*), which naturally appears in the PT-BFM framework and that has been studied in detail in [15] (see also below); in addition,

$$\Gamma^{(0)}_{\mu\alpha\beta}(q,r,p) = g_{\alpha\beta}(r-p)_\mu + g_{\beta\mu}(p-q)_\alpha + g_{\alpha\mu}(q-r)_\beta. \tag{2.8}$$

Finally, the function $G(q^2)$ represents the scalar co-factor of the $g_{\mu\nu}$ component of the special two-point function $\Lambda_{\mu\nu}(q)$, defined as

$$\begin{aligned}
\Lambda_{\mu\nu}(q) &= -ig^2C_A\int_k \Delta^\sigma_\mu(k)D(q-k)H_{\nu\sigma}(-q,q-k,k)\\
&= g_{\mu\nu}G(q^2) + \frac{q_\mu q_\nu}{q^2}L(q^2),
\end{aligned} \tag{2.9}$$

where we have introduced the ghost propagator $D^{ab}(q^2) = \delta^{ab}D(q^2)$, which is related to the ghost dressing function $F(q^2)$ through $D(q^2) = \frac{F(q^2)}{q^2}$ Notice that in the Landau gauge, an important exact (all-order) relation exists, linking $G(q^2)$ and $L(q^2)$ to the ghost dressing function $F(q^2)$, namely [16, 17]

$$F^{-1}(q^2) = 1 + G(q^2) + L(q^2); \tag{2.10}$$

in addition G coincides with the well-known Kugo-Ojima function [16, 17].

The Schwinger mechanism [18] is integrated into the SDE of the gluon propagator through the form of the three-gluon vertex [19, 20, 21, 22]. In particular, a crucial condition for the realization of the gluon mass generation scenario is the existence of a special vertex, to be denoted by $V_{\alpha\mu\nu}(q,r,p)$, which must be completely *longitudinally coupled*, *i.e.*, must satisfy

$$P^{\alpha'\alpha}(q)P^{\mu'\mu}(r)P^{\nu'\nu}(p)V_{\alpha\mu\nu}(q,r,p) = 0. \tag{2.11}$$

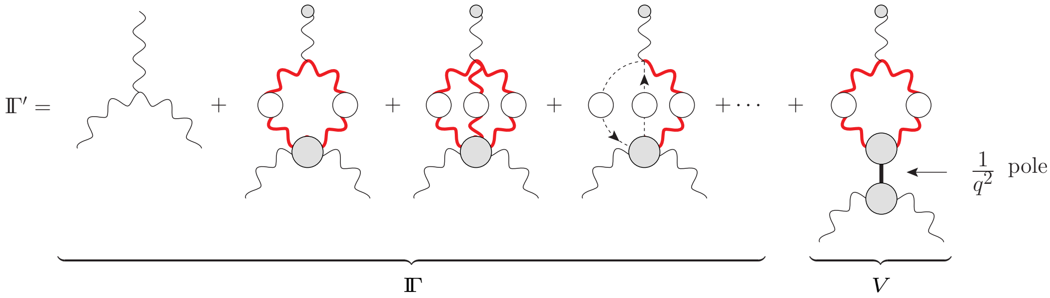

Figure 2: The $\mathbb{I}\Gamma'$ three-gluon vertex. Thick (red online) internal gluon lines indicates gluon propagators with an effective mass.

This vertex is instrumental for maintaining gauge invariance, given that the massless poles that it must contain in order to trigger the Schwinger mechanism, act, at the same time, as composite, longitudinally coupled Nambu-Goldstone bosons. Specifically, in order to preserve the gauge invariance of the theory in the presence of masses, the vertex $V_{\alpha\mu\nu}(q,r,p)$ must be added to the *BQQ* (fully-dressed) three-gluon vertex $\mathbb{I}\Gamma_{\alpha\mu\nu}(q,r,p)$, giving rise to the new full vertex, $\mathbb{I}\Gamma'_{\alpha\mu\nu}(q,r,p)$, defined as [10]

$$\mathbb{I}\Gamma'_{\alpha\mu\nu}(q,r,p) = \mathbb{I}\Gamma_{\alpha\mu\nu}(q,r,p) + V_{\alpha\mu\nu}(q,r,p). \tag{2.12}$$

To see in detail how gauge invariance is preserved, notice that when the Schwinger mechanism is turned off, the *BQQ* vertex alone satisfies the WI

$$q^\alpha \mathbb{I}\Gamma_{\alpha\mu\nu}(q,r,p) = p^2 J(p^2) P_{\mu\nu}(p) - r^2 J(r^2) P_{\mu\nu}(r), \tag{2.13}$$

when contracted with respect to the momentum of the background gluon. By requiring that

$$q^\alpha V_{\alpha\mu\nu}(q,r,p) = m^2(r^2) P_{\mu\nu}(r) - m^2(p^2) P_{\mu\nu}(p), \tag{2.14}$$

we see that, after turning the Schwinger mechanism on, the corresponding WI satisfied by $\mathbb{I}\Gamma'$ would read

$$\begin{aligned} q^\alpha \mathbb{I}\Gamma'_{\alpha\mu\nu}(q,r,p) &= q^\alpha \left[\mathbb{I}\Gamma(q,r,p) + V(q,r,p)\right]_{\alpha\mu\nu} \\ &= [p^2 J(p^2) - m^2(p^2)] P_{\mu\nu}(p) - [r^2 J(r^2) - m^2(r^2)] P_{\mu\nu}(r) \\ &= \Delta_m^{-1}(p^2) P_{\mu\nu}(p) - \Delta_m^{-1}(r^2) P_{\mu\nu}(r)\,, \end{aligned} \tag{2.15}$$

which is indeed the identity (2.13) satisfied by $\mathbb{I}\Gamma$ with the replacement of the conventional gluon propagators appearing on the rhs by massive propagators: $\Delta^{-1} \to \Delta_m^{-1}$. The remaining (more difficult) STIs, triggered when contracting $\mathbb{I}\Gamma'_{\alpha\mu\nu}(q,r,p)$ with respect to the other two legs are realized in exactly the same fashion [10].

The next step is to insert $\mathbb{I}\Gamma'_{\alpha\mu\nu}(q,r,p)$ into the SDE equation satisfied by the gluon propagator, thus obtaining the new SDE of Fig. 3. From the resulting SDE one can obtain two separate equations, the first one governing the behavior of $J_m(q^2)$, and the second one describing the dynamical mass $m^2(q^2)$. The general idea is the following: the terms appearing on the rhs of the SDE may be separated systematically into two contributions, one that vanishes as $q \to 0$ and one that does not;

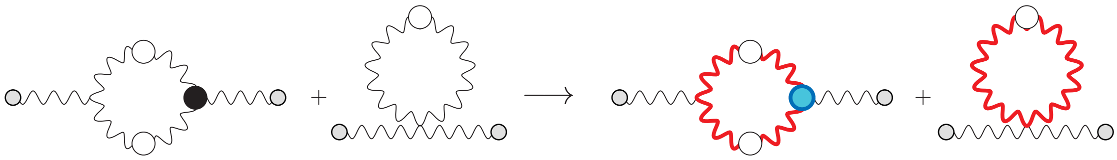

Figure 3: Diagrammatic representation of the gluon one-loop dressed diagrams before and after dynamical gluon mass generation has taken place: the propagators and vertices on the rhs have now become Δ_m and $\mathrm{I\Gamma}'$.

the latter contribution must be set equal to the corresponding non-vanishing term on the lhs, namely $-m^2(q)$, while the former will be set equal to the vanishing term of the lhs, namely $q^2 J_m(q^2)$.

The exact way how to implement this particular separation is rather elaborate, and requires the use of the so-called "seagull identity" [23]

$$\int_k k^2 \Delta'(k) + \frac{d}{2} \int_k \Delta(k) = 0, \tag{2.16}$$

where the "prime" denotes differentiation with respect to k^2. This identity plays a a crucial role, forcing all possible quadratic divergences, associated with integrals of the type $\int_k \Delta(k)$, and variations thereof, to cancel exactly. In the case of the dimensional regularization that is used throughout, the presence of such integrals would give rise to divergences of the type $m_0^2(1/\varepsilon)$, where m_0 is the value of the dynamically generated gluon mass at $q^2 = 0$, *i.e.*, $m_0 = m(0)$; if a hard cutoff Λ were to be employed, these latter terms would diverge quadratically, as Λ^2. The disposal of such divergences would require the introduction in the original Lagrangian of a counter-term of the form $m_0^2 A_\mu^2$, which is, however, forbidden by the local gauge invariance, which must remain intact. Notice that the two types of integrals appearing on the lhs of Eq. (2.16) are individually non-vanishing (in fact, they both diverge); it is only when they come in the particular combination shown above that they sum up to zero.

The seagull cancellation implemented by Eq. (2.16) is a point of paramount importance, because it further re-enforces the fundamental assertion that the gluon mass generation, when implemented correctly, *is absolutely compatible with the underlying BRST symmetry*. Indeed, in the mass generation picture advocated in a series of recent articles [14, 10, 24, 25] *the Lagrangian of the Yang-Mills theory (or that of QCD) is never altered*; therefore, the only other possible ways of violating the gauge (or BRST) symmetry would be (*i*) by not respecting, at some intermediate step, some of the WIs and STIs satisfied by the Green's functions involved; for example, in the conventional SDE formulation, a naive truncation would compromise the transversality of the resulting gluon self-energy, *i.e.*, the fundamental transversality condition (2.5) would be no longer valid; and/or (*ii*) by introducing seagull-type divergences, of the type mentioned above. Evidently, neither (*i*) nor (*ii*) happens, thanks to the powerful field-theoretic properties encoded into the PT-BFM formalism that we employ.

Returning to the gluon SDE, a rather elaborate analysis [10] gives rise to two coupled integral equations, one for $J_m(q)$ and one for the momentum-dependent gluon mass, of the generic type

$$\begin{aligned} J_m(q^2) &= 1 + \int_k \mathscr{K}_1(q^2, m^2, \Delta_m), \\ m^2(q^2) &= \int_k \mathscr{K}_2(q^2, m^2, \Delta_m). \end{aligned} \tag{2.17}$$

such that $q^2\mathscr{K}_1(q^2,m^2,\Delta_m)\to 0$, as $q^2\to 0$, whereas $\mathscr{K}_2(q^2,m^2,\Delta_m)\neq 0$ in the same limit, precisely because it includes the term $1/q^2$ contained inside $V_{\alpha\mu\nu}(q,r,p)$.

There is a relatively straightforward way to derive the closed form of the gluon mass equation in the limit $q^2\to 0$, without knowledge of the explicit form of V; all one needs is to postulate its existence and the properties mentioned above, most notably the identities (2.11) and (2.14). In order to do that, we start by noticing that, since the vertex $\mathrm{I\!\Gamma}'$ satisfies the identity (2.15), the gluon self-energy is transverse *even in the presence of masses*; as a result, restoring the Lorentz structure into the second equation of (2.17), we have that

$$m^2(q^2)P_{\mu\nu}(q) = P_{\mu\nu}(q)\int_k \mathscr{K}_2(q^2,m^2,\Delta_m). \tag{2.18}$$

Now, due to its longitudinal nature, the vertex V can only furnish the part on the rhs proportional to $q_\mu q_\nu/q^2$. The question is, where will the $g_{\mu\nu}$ part come from. A bit of thought reveals that this term can only emerge as a deviation from the seagull identity, which enforces the masslessness of the gluon when $V=0$. In fact, this identity, given its nature and function, can only operate among terms that are proportional to $g_{\mu\nu}$; this is so because the basic seagull contribution proportional to $\int_k\Delta(k)$ stems entirely from graph (a_2), which has no momentum dependence, *i.e.*, it can only be proportional to $g_{\mu\nu}$.

Specifically, the required contribution proportional to $g_{\mu\nu}$ stems from the term

$$\int_k k_\mu k_\nu \Delta(k)\Delta(k+q)\frac{(k+q)^2 J_m(k+q)-k^2 J_m(k)}{(k+q)^2-k^2} = g_{\mu\nu}C_1(q^2)+\frac{q_\mu q_\nu}{q^2}C_2(q^2), \tag{2.19}$$

which originates from $A_1^{\mu\nu}(q)$. Then, since for any function $f(k^2)$ one has the result

$$\int_k \cos^2\theta f(k^2) = \frac{1}{d}\int_k f(k^2), \tag{2.20}$$

it is easy to demonstrate that

$$\begin{aligned} C_1(0) &= \int_k k^2\Delta^2(k)[k^2 J(k)]' \\ &= \int_k k^2\Delta^2(k)[\Delta^{-1}(k)+m^2(k)]' \\ &= \int_k k^2\Delta^2(k)[m^2(k)]' - \int_k k^2\Delta'(k). \end{aligned} \tag{2.21}$$

The first term on the rhs forms part of $g_{\mu\nu}\mathscr{K}_2(q^2,m^2,\Delta_m)$, while the second, after taking all relevant multiplicative factors correctly into account, cancels against the contribution of graph (a_2), by virtue of the seagull identity (2.16). This basic observations can be generalized to include the additional structures coming from the remaining terms in (2.7), and in particular $A_3^{\mu\nu}(q)$, whose net effect is the one of reducing by one power the $1+G$ factor appearing in the denominator of Eq. (2.6) [10].

Then, restoring all relevant factors, and using the fact that in 4 dimension $L(0)=0$ [17] so that by virtue of Eq. (2.10) one can trade the $1+G$ combination for the inverse of the ghost dressing function, one finally obtains, in Euclidean space,

$$\begin{aligned} m^2(0) &= \frac{3}{2}g^2 C_A F(0)\int_k k^2[m^2(k)]'\Delta^2(k) \\ &= -3g^2 C_A F(0)\int_k m^2(k)\Delta(k)\left[k^2\Delta(k)\right]'. \end{aligned} \tag{2.22}$$

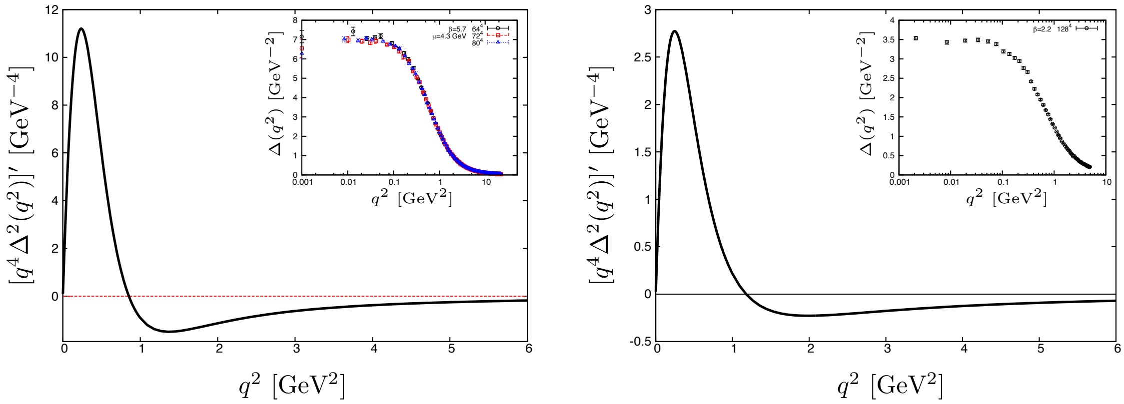

Figure 4: The kernel $[q^4\Delta^2(q^2)]'$ obtained from the $SU(3)$ (left) and $SU(2)$ (right) lattice data. In the inset we show the corresponding lattice results for the gluon propagator, renormalized at $\mu = 4.3$ GeV and $\mu = 2.2$ GeV respectively.

and, after carrying out the angular integration, the final equation

$$m^2(0) = -\frac{3C_A}{8\pi}\alpha_s F(0)\int_0^{\infty} \mathrm{d}y\, m^2(y)[y^2\Delta^2(y)]', \tag{2.23}$$

where $\alpha_s = g^2/4\pi$ and $y = k^2$ (and therefore the "prime" indicates now derivatives with respect to y).

It turns out that the specific form of the mass equation (2.23) introduces a non-trivial constraint on the precise behavior that Δ must display in the region between (1-5) GeV^2. Specifically, in order for the gluon mass to be positive definite, the first derivative of the quantity $q^2\Delta(q^2)$ (the "gluon dressing function") must furnish a sufficiently *negative* contribution in the aforementioned range of momenta. Interestingly enough, the Δ obtained from the lattice, shown in Fig. 4, has indeed this particular property; in the plots shown in Fig. 4 we clearly see that both derivatives change their sign in the intermediate momenta region, which constitutes precisely the required behavior. This is to be contrasted to what happens, for example, in the case of a simple massive propagator $1/(q^2+m^2)$ or with the Gribov-Zwanziger propagator $q^2/(q^4+m^4)$ (with m constant); the derivatives of the corresponding dressing functions, $q^2/(q^2+m^2)$ and $q^4/(q^4+m^4)$, respectively, are positive in the entire range of (Euclidean) momenta, thus excluding, in this context, the possibility of a positive-definite gluon mass.

3. Schwinger mechanism in the bound-state language: a self-consistency check

The key assumption when invoking the Schwinger mechanism in Yang-Mills theories, such as QCD, is that the required poles may be produced due to purely dynamical reasons; specifically, one assumes that, for sufficiently strong binding, the mass of the appropriate bound state may be reduced to zero [19, 20, 21, 22]. It is precisely these poles that constitute the main ingredient of the special vertex V. Specifically, all terms appearing in this vertex are proportional to $1/q^2$, $1/r^2$, $1/p^2$, and products thereof, and can be separated in two distinct parts, according to [11]

$$V_{\alpha\mu\nu}(q,r,p) = U_{\alpha\mu\nu}(q,r,p) + R_{\alpha\mu\nu}(q,r,p). \tag{3.1}$$

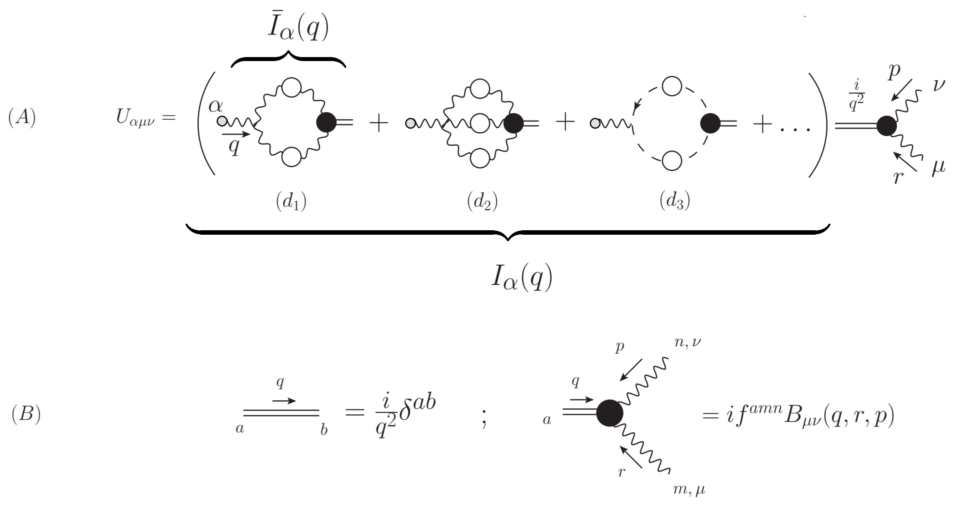

Figure 5: (A) The vertex $U_{\alpha\mu\nu}$ is composed of three main ingredients: the transition amplitude, I_α, which mixes the gluon with a massless excitation, the propagator of the massless excitation, and the massless excitation–gluon–gluon vertex. (B) The Feynman rules (with color factors included) for (i) the propagator of the massless excitation and (ii) the "proper vertex function", or, "bound-state wave function", $B_{\mu\nu}$.

The R term contains structures proportional to r_μ/r^2 and/or p_ν/p^2, which vanish when contracted with the product $P_{\mu'\mu}(r)P_{\nu'\nu}(p)$ coming from the Landau gauge gluon propagators, and therefore are not relevant for us. On the other hand, the U term can be written as [see Fig. 5 (A)]

$$U_{\alpha\mu\nu}(q,r,p) = iI_\alpha(q)\left(\frac{i}{q^2}\right)B_{\mu\nu}(q,r,p), \tag{3.2}$$

where the factor i/q^2 represents the propagator of the scalar massless excitation, while the nonperturbative quantity

$$B_{\mu\nu}(q,r,p) = B_1 g_{\mu\nu} + B_2 q_\mu q_\nu + B_3 p_\mu p_\nu + B_4 r_\mu q_\nu + B_5 r_\mu p_\nu, \tag{3.3}$$

is the effective vertex describing the interaction between the massless excitation and two gluons [Fig. 5 (B)]. Finally, $I_\alpha(q) = q_\alpha I(q)$ is the (nonperturbative) transition amplitude introduced in Fig. 5, allowing the mixing between a gluon and the massless excitation[1]. Ii is important to notice that, due to Bose symmetry with respect to the quantum legs, *i.e.*, $\mu \leftrightarrow \nu$ and $p \leftrightarrow r$ one has $B_{1,2}(q,r,p) = -B_{1,2}(q,p,r)$, and therefore these two form factors vanish in the limit. $q \to 0$.

The WI (2.14) furnishes an exact relation between the dynamical gluon mass, the transition amplitude at zero momentum transfer, and the form factor B_1. Specifically, contracting both sides of the WI with two transverse projectors, one obtains,

$$P^{\mu'\mu}(r)P^{\nu'\nu}(p)q^\alpha V_{\alpha\mu\nu}(q,r,p) = [m^2(r) - m^2(p)]P_\sigma^{\mu'}(r)P^{\sigma\nu'}(p). \tag{3.4}$$

[1] Notice that we do not absorb the extra factor of "i" coming from the Feynman rule in Fig.5 into the definition of the transition function I_α, as was done in [11]; this definition has the advantage that the function $I(q)$ is real in the Euclidean space.

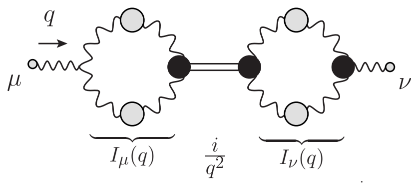

Figure 6: The "squared" diagram.

On the other hand, contracting the full expansion of the vertex (3.2) by the same transverse projectors and next contracting the resulting expression with the momentum of the background leg, we get

$$q^\alpha P^{\mu'\mu}(r)P^{\nu'\nu}(p)V_{\alpha\mu\nu}(q,r,p) = -I(q)[B_1 g_{\mu\nu} + B_2 q_\mu q_\nu]P^{\mu'\mu}(r)P^{\nu'\nu}(p), \tag{3.5}$$

Thus, equating both results, one arrives at the exact relations

$$I(q)B_1(q,r,p) = m^2(p) - m^2(r); \qquad B_2(q,r,p) = 0. \tag{3.6}$$

Finally, carrying out the Taylor expansion around $q = 0$ of both sides of the first relation in (3.6), and using the fact that the form factors $B_{1,2}$ vanish in this limit, we arrive at (Minkowski space)

$$[m^2(p)]' = -I(0)B_1'(p), \tag{3.7}$$

where we have defined

$$B_1'(p) = B_1'(0,-p,p) \equiv \lim_{q\to 0}\left\{\frac{\partial B_1(q,-p-q,p)}{\partial\,(p+q)^2}\right\}, \tag{3.8}$$

and have implicitly assumed that $I(0)$ is finite.

We will next approximate the transition amplitude $I_\alpha(q)$, connecting the gluon with the massless excitation, by considering only diagram (d_1) in Fig. 5, corresponding to the gluonic "one-loop dressed" approximation; we will denote the resulting expression by $\bar{I}_\alpha(q)$. In the Landau gauge, the amplitude $\bar{I}_\alpha(q)$ reads

$$\bar{I}_\alpha(q) = \frac{1}{2}iC_A\int_k \Delta(k)\Delta(k+q)\Gamma^{(0)}_{\alpha\beta\lambda}P^{\lambda\mu}(k)P^{\beta\nu}(k+q)B_{\mu\nu}(-q,-k,k+q), \tag{3.9}$$

where the origin of the $1/2$ factor is combinatoric, and $\Gamma^{(0)}_{\alpha\beta\lambda}$ is the tree-level vertex of Eq. (2.8).

At this point we are in the position to determine how the transition function is related to the dynamically generated gluon mass which originates from the inclusion of the vertex V in the corresponding gluon SDE. Indeed, at the level of approximation employed here we find that the contribution to the gluon self energy is

$$\begin{aligned}
A_1^{\mu\nu}(q)|_V &= \frac{1}{2}g^2C_A\int_k \Delta(q+k)\Delta(k)\Gamma^{(0)\mu}_{\alpha\beta}P^{\alpha\rho}(k)P^{\beta\sigma}(k+q)U^\nu_{\rho\sigma} \\
&= -g^2\left[\frac{1}{2}C_A\int_k \Delta(q+k)\Delta(k)\Gamma^{(0)\mu}_{\alpha\beta}P^{\alpha\rho}(k)P^{\beta\sigma}(k+q)B_{\rho\sigma}\right]\left(\frac{i}{q^2}\right)\bar{I}_\nu(-q) \\
&= -i\frac{q^\mu q^\nu}{q^2}\,g^2\bar{I}^2(q),
\end{aligned} \tag{3.10}$$

where we have used Eq. (3.2) (with $I_V \to \bar{I}_V$), together with the property $\bar{I}_V(-q) = -\bar{I}_V(q)$ as well as Eq. (3.9). Notice that as anticipated this contribution is purely longitudinal.

Since the inclusion of V in A_3 has the same effect previously found, namely to effectively remove one power of $(1+G)$ in the denominator[2], we obtain the positive-definite result

$$m^2(0) = g^2 F(0)\bar{I}^2(0). \tag{3.11}$$

Of course, for the PT-BFM framework to be self-consistent, the two infrared equations for the gluon mass ought to coincide. To see that this is indeed the case, consider again the original mass equation (2.22) , and let's insert the relation (3.7) – at the same level of approximation used in the derivation of (3.11), *i.e.*, with the replacement $I(0) \to \bar{I}(0)$ – into the integral in the first line, to get

$$\begin{aligned} m^2(0) &= \frac{3}{2} g^2 C_A F(0) \int_k k^2 [m^2(k)]' \Delta^2(k) \\ &= g^2 F(0) I(0) \left[-\frac{3}{2} g^2 C_A \int_k k^2 \Delta^2(k) B_1'(k) \right]. \end{aligned} \tag{3.12}$$

On the other hand, a a bit of algebra reveals that, in Euclidean space, on has [11]

$$\bar{I}(0) = \frac{3}{2} i C_A \int_k k^2 \Delta^2(k) B_1'(k), \tag{3.13}$$

so that we indeed (3.12) gives rise to the relation

$$m^2(0) = g^2 F(0)\bar{I}^2(0), \tag{3.14}$$

which is none other than Eq. (3.11).

4. Conclusions

In this work we have presented a highly non-trivial self-consistency check related to the integral equation describing the dynamics of the gluon mass in the deep infrared.

We started by showing how relation (2.22) can be derived only by assuming the existence of the special vertex V, without providing any information either on its own diagrammatic structure, nor on its composition in terms of other dynamical quantities, such as the bound-state wave function B. Next, we presented an alternative formulation, where all the intricate dynamical ingredients that trigger the Schwinger mechanism, subject to the constraints imposed by gauge invariance, were furnished. Both methods give rise to the same equation, proving that they represent nothing but different facets of the same underlying dynamics.

In the future it would be particularly important to generalize the consistency check described to the full momentum range, *i.e.*, extend it to the full dynamical equation governing the evolution of the mass $m^2(q^2)$ and not only to the deep infrared limit $m^2(0)$. In addition, one should go beyond the one-loop approximation by including the missing two-loop dressed gluon diagrams in both the SDE as well as the bound-state analysis. Work along these directions is already in progress.

[2] It is precisely the inclusion of this term that account for the difference of Eq. (3.11) with respect to the analogous equation (3.38) of [11].

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