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# Aspects of Gribov-Zwanziger theory and QCD

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We review recent loop calculations using the Gribov-Zwanziger Lagrangian where the most general BRST invariant dimension two operator using the localizing ghost fields is included. We argue that the natural colour channel for this operator to condense into is the $\mathscr{R}$ channel. The exact structure of the 3-point QCD vertices at the symmetric subtraction point at two loops is also discussed in relation to the MOM subtraction schemes.

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## 1. Introduction

In recent years there has been an intense activity into understanding the infrared behaviour of the gluon and Faddeev-Popov ghosts of QCD. The reason for this is that it gives an insight into colour confinement and thus the absence of free gluons in nature. From a theoretical point of view Gribov indicated that the gauge fixed Lagrangian in non-abelian gauge theories led to an overcounting of gauge field configurations in the path integral, [1]. In order to handle this Gribov introduced a nonlocal Lagrangian which restricted the gauge fields to lie in a region of configuration space. Consequently the propagators of Yang-Mills theory were modified in such a way that the gluon was suppressed in the infrared whereas the Faddeev-Popov ghost was enhanced, [1]. Whilst this was in a semiclassical approach a Lagrangian analysis in the full quantum theory was established over a period of years by Zwanziger, [2 – 4]. This Gribov-Zwanziger Lagrangian was local and renormalizable and meant that loop calculations could be performed. The predictions of a suppressed gluon and an enhanced Faddeev-Popov ghost propagator were established as fundamental properties of the gauge field restriction to a region of configuration space, [2 – 4]. However, with the improvement of lattice algorithms, analyses and computing power it has been possible to probe the propagators to smaller values of the momentum. The upshot is that it is generally accepted that the gluon propagator is not suppressed but freezes and the ghost propagator behaves essentially as a free particle. See, for instance, [5 – 7] and contributions to this meeting. To accommodate this behaviour in the Gribov-Zwanziger context an additional dimension two BRST invariant operator was included in the Lagrangian, [8]. This can mimic the infrared propagator behaviour observed on the lattice. However, that analysis was not complete in that the most general colour structure was not considered. This has been carried out now, [9], and we review that result here as well as discussing latest work. This will include an alternative way of writing the original Gribov gap equation defining the Gribov mass as the vacuum expectation value of purely localizing bosonic ghost fields, [9].

Whilst the infrared properties of the propagators are interesting the overall structure of the Green's functions of QCD are also fundamental to understanding the strong interactions. For instance, another analytic technique which allows one to probe the zero momentum limit is the Schwinger-Dyson equations. This is a method to solve the tower of $n$-point Green's functions. Though in practice one has to make a truncation in order to have a manageable set of equations. However, whether one uses the lattice or Dyson-Schwinger equations both approaches have to be consistent with the ultraviolet structure of the Green's functions. The second part of this article therefore reviews recent activity into computing the 3-point vertex functions of QCD at the symmetric subtraction point analytically at two loops, [10]. It extends the original one loop momentum subtraction scheme analysis of Celmaster and Gonsalves, [11]. So the full exact three loop MOM $\beta$-functions and renormalization group functions are known exactly now as well as the full two loop structure of the vertex amplitudes, [10, 12]. This extends the earlier numerical approximation carried out in [13] based on the use of MINCER, [14], in approximating the symmetric point Feynman diagrams in an expansion in 2-point diagrams.

The article is organized as follows. We recall aspects of the Gribov-Zwanziger Lagrangian in section 2 including the recent work on the infrared behaviour of the bosonic localizing ghost. Properties of the theory when the most general dimension two BRST invariant localizing ghost operator is included are discussed in section 3 including the gap equation for the $\mathscr{R}$ channel and the infrared structure of the bosonic localizing ghost propagator in that case. A summary of the construction of the effective potential for the general ghost operator is provided in section 4 as well as recent results on the gap equation. Section 5 deals with the two loop structure of the QCD 3-point vertices at the symmetric point in conventional perturbation theory based on the non-Gribov QCD Lagrangian. Concluding remarks are given in section 6.

## 2. Gribov-Zwanziger theory

The restriction of the gauge field, $A^a_\mu$, in the Landau gauge to lie within the first Gribov region means that the QCD Lagrangian in the path integral construction gets modified by additional non-local terms, [1]. In [2, 3] Zwanziger managed to localize this Lagrangian by introducing additional localizing ghost fields in such a way as to retain the horizon condition and not upset the accepted ultraviolet structure of QCD. Specifically the renormalizable localized Lagrangian is

$$\begin{aligned} L^{GZ} &= L^{QCD} + \frac{1}{2}\rho^{ab\,\mu}\partial^\nu\left(D_\nu\rho_\mu\right)^{ab} + \frac{i}{2}\rho^{ab\,\mu}\partial^\nu\left(D_\nu\xi_\mu\right)^{ab} - \frac{i}{2}\xi^{ab\,\mu}\partial^\nu\left(D_\nu\rho_\mu\right)^{ab} \\ &+ \frac{1}{2}\xi^{ab\,\mu}\partial^\nu\left(D_\nu\xi_\mu\right)^{ab} - \bar{\omega}^{ab\,\mu}\partial^\nu\left(D_\nu\omega_\mu\right)^{ab} - \frac{1}{\sqrt{2}}gf^{abc}\partial^\nu\bar{\omega}^{ae}_\mu\left(D_\nu c\right)^b\rho^{ec\,\mu} \\ &- \frac{i}{\sqrt{2}}gf^{abc}\partial^\nu\bar{\omega}^{ae}_\mu\left(D_\nu c\right)^b\xi^{ec\,\mu} - i\gamma^2 f^{abc}A^{a\,\mu}\xi^{bc}_\mu - \frac{dN_A\gamma^4}{2g^2} \end{aligned} \tag{2.1}$$

where $L^{QCD}$ is the usual Landau gauge fixed Lagrangian with $N_f$ massless quarks and Faddeev-Popov ghosts. We have recorded the version where the real localizing ghosts, $\xi^{ab}_\mu$ and $\rho^{ab}_\mu$, are used together with their Grassmann counterparts, $\omega^{ab}_\mu$ and $\bar{\omega}^{ab}_\mu$. The parameter $\gamma$ is the Gribov mass, $g$ is the coupling constant, $d$ is the spacetime dimension and $N_A$ is the dimension of the adjoint representation. The mixed quadratic term means a more involved set of propagators which are

$$\begin{aligned} \langle A^a_\mu(p)A^b_\nu(-p)\rangle &= -\frac{\delta^{ab}p^2}{[(p^2)^2+C_A\gamma^4]}P_{\mu\nu}(p) \;,\; \langle A^a_\mu(p)\xi^{bc}_\nu(-p)\rangle = \frac{if^{abc}\gamma^2}{[(p^2)^2+C_A\gamma^4]}P_{\mu\nu}(p) \\ \langle A^a_\mu(p)\rho^{bc}_\nu(-p)\rangle &= 0 \;,\; \langle\xi^{ab}_\mu(p)\xi^{cd}_\nu(-p)\rangle = -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}\gamma^4}{p^2[(p^2)^2+C_A\gamma^4]}P_{\mu\nu}(p) \\ \langle\xi^{ab}_\mu(p)\rho^{cd}_\nu(-p)\rangle &= 0 \;,\; \langle\rho^{ab}_\mu(p)\rho^{cd}_\nu(-p)\rangle = \langle\omega^{ab}_\mu(p)\bar{\omega}^{cd}_\nu(-p)\rangle = -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} \end{aligned} \tag{2.2}$$

where $P_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu/p^2$ and $f^{abc}$ are the colour group structure constants. The gluon propagator is clearly suppressed in the infrared in keeping with Gribov's analysis, [1]. Over a period of years certain general properties of the propagators in the quantum theory have been established. Aside from the Faddeev-Popov and localizing Grassmann ghost enhancement, it has recently been shown that the bosonic localizing ghost enhances too, [15]. This has been explicitly verified by one loop computations which relies on the gap equation being satisfied. Indeed unless $\gamma$ satisfies this gap equation one is not in the gauge theory as $\gamma$ is not an independent parameter of the theory. For verifying the enhancement one computes the one loop corrections to the 2-point function matrix of the theory, imposes the gap equation and then inverts this matrix to deduce the zero momentum behaviour of the propagators. At one loop one has, in four dimensions, [16],

$$\langle \xi^{ab}_\mu(p) \xi^{cd}_\nu(-p) \rangle \sim \frac{4\gamma^2}{\pi\sqrt{C_A}(p^2)^2 a} \left[ \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd} \right] \eta_{\mu\nu} + \frac{8\gamma^2}{\pi C_A^{3/2}(p^2)^2 a} f^{abe} f^{cde} P_{\mu\nu}(p)$$
$$\langle \rho^{ab}_\mu(p) \rho^{cd}_\nu(-p) \rangle \sim - \frac{8\gamma^2}{\pi\sqrt{C_A}(p^2)^2 a} \delta^{ac}\delta^{bd} \eta_{\mu\nu} \quad (2.3)$$

as $p^2 \to 0$. One feature of Zwanziger's general analysis, [15], was that the adjoint projection of $\xi^{ab}_\mu$ was not enhanced which can be seen from (2.3),

$$f^{apq} f^{brs} \langle \xi^{pq}_\mu(p) \xi^{rs}_\nu(-p) \rangle \sim - \delta^{ab} \left[ \frac{69\pi C_A^2 a}{128\sqrt{C_A}\gamma^2} + \frac{p^2}{\gamma^4} \right] P_{\mu\nu}(p) - \delta^{ab} \left[ \frac{8 C_A \gamma^2}{\pi\sqrt{C_A}(p^2)^2 a} + \frac{4}{\pi^2 p^2 a} \right] L_{\mu\nu}(p) \quad (2.4)$$

where $L_{\mu\nu}(p) = p_\mu p_\nu / p^2$. However, this adjoint colour projection does not vanish at zero momentum in the transverse part because of the constant term. The enhancement in the longitudinal sector accords with the expectation of [15]. The infrared behaviour of the transverse part of this spin-1 adjoint index object appears to be consistent with the non-suppressed gluon observed over many years now on the lattice, [5–7]. Indeed if one takes the same projection in (2.2) then the *original* propagator is

$$f^{apq} f^{brs} \langle \xi^{pq}_\mu(p) \xi^{rs}_\nu(-p) \rangle = - \frac{C_A p^2}{[(p^2)^2 + C_A \gamma^4]} \delta^{ab} P_{\mu\nu}(p) - \frac{C_A}{p^2} \delta^{ab} L_{\mu\nu}(p) . \quad (2.5)$$

So the transverse term is suppressed similar to the original $A^a_\mu$ field of Gribov's analysis. This similarity is perhaps not surprising as $\xi^{ab}_\mu$ is used to localize a non-local operator which depends purely on $A^a_\mu$. It may be that the behaviour of the $\xi^{ab}_\mu$, which is the remnant of the original $A^a_\mu$ field, is being observed in lattice work. The three dimensional case is similar, [17]. However, this is not a full explanation of lattice data as the Faddeev-Popov ghost enhancement is not seen. Though that may be a case of not comparing the same quantities as the continuum analysis. This comment is driven by the fact that Faddeev-Popov ghost fields are not constructed on the lattice and that the object which is regarded as being the ghost propagator on the lattice is merely the inverse of the Faddeev-Popov operator. This is driven by the Lagrangian construction and there appears to be an assumption that this operator remains relevant in the infrared despite claims that BRST symmetry is broken. Aside from this there have been Lagrangian approaches, [8], to try and understand what is known as the decoupling solution. Though it transpires that that analysis was not complete, [9].

## 3. BRST invariant operator

To model the non-enhanced Faddeev-Popov ghost and frozen gluon propagators of the decoupling solution in the Gribov-Zwanziger context, a BRST invariant dimension two operator built from the localizing ghost fields was included in the Lagrangian, [8]. The justification for the inclusion of such an additional mass term is that it is dynamically generated through the condensation of the operator itself. This has been analysed using the local composite operator formalism, [8]. However, the construction of [8] was not comprehensive and did not take into account all possible colour channels, [9]. Moreover it was shown in [9] that a non-enhanced Faddeev-Popov ghost and a frozen gluon could be modelled in various colour channels which were different from that analysed in [8]. Hence the construction of [8] was not unique. Specifically the most general BRST invariant operator is

$$\mathscr{O} \;=\; \left[ \mu^2_{\mathscr{Q}} \delta^{ac} \delta^{bd} + \mu^2_{\mathscr{W}} f^{ace} f^{bde} + \frac{\mu^2_{\mathscr{R}}}{C_A} f^{abe} f^{cde} + \mu^2_{\mathscr{S}} d_A^{abcd} + \frac{\mu^2_{\mathscr{P}}}{N_A} \delta^{ab} \delta^{cd} + \mu^2_{\mathscr{T}} \delta^{ad} \delta^{bc} \right] \mathscr{O}^{abcd} \tag{3.1}$$

where

$$\mathscr{O}^{abcd} \;=\; \frac{1}{2} \left[ \rho^{ab} \rho^{cd} \,+\, i \xi^{ab} \rho^{cd} \,-\, i \rho^{ab} \xi^{cd} \,+\, \xi^{ab} \xi^{cd} \right] \,-\, \bar{\omega}^{ab} \omega^{cd} \tag{3.2}$$

and the parameters $\mu_i^2$ label the different colour channels in the notation of [8]. With the inclusion of (3.1) in (2.1) then one can study the structure of the propagators. It transpires that decoupling behaviour emerges when one has non-zero $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{T}$ or $\mathscr{W}$ masses in any combinations, [9]. Whilst the $\mathscr{Q}$ sector was studied at length in [8] we focus on the $\mathscr{R}$ case for reasons which will be clear later. With only $\mu^2_{\mathscr{R}}$ non-zero the propagators are

$$\begin{aligned}
\langle A^a_\mu(p) A^b_\nu(-p) \rangle_{\mathscr{R}} &= -\, \frac{\delta^{ab}[p^2 + \mu^2_{\mathscr{R}}]}{[(p^2)^2 + \mu^2_{\mathscr{R}} p^2 + C_A \gamma^4]} P_{\mu\nu}(p) \;\;,\;\; \langle A^a_\mu(p) \rho^{bc}_\nu(-p) \rangle_{\mathscr{R}} \;=\; 0 \\
\langle A^a_\mu(p) \xi^{bc}_\nu(-p) \rangle_{\mathscr{R}} &= \frac{i f^{abc} \gamma^2}{[(p^2)^2 + \mu^2_{\mathscr{R}} p^2 + C_A \gamma^4]} p_{\mu\nu}(p) \;\;,\;\; \langle \xi^{ab}_\mu(p) \rho^{cd}_\nu(-p) \rangle_{\mathscr{R}} \;=\; 0 \\
\langle \xi^{ab}_\mu(p) \xi^{cd}_\nu(-p) \rangle_{\mathscr{R}} &= -\, \frac{\delta^{ac}\delta^{bd}}{p^2} \eta_{\mu\nu} + \frac{f^{abe} f^{cde} [\mu^2_{\mathscr{R}} p^2 + C_A \gamma^4] P_{\mu\nu}(p)}{C_A p^2 [(p^2)^2 + \mu^2_{\mathscr{R}} p^2 + C_A \gamma^4]} + \frac{f^{abe} f^{cde} \mu^2_{\mathscr{R}} L_{\mu\nu}(p)}{C_A p^2 [p^2 + \mu^2_{\mathscr{R}}]} \\
\langle \rho^{ab}_\mu(p) \rho^{cd}_\nu(-p) \rangle_{\mathscr{R}} &= \langle \omega^{ab}_\mu(p) \bar{\omega}^{cd}_\nu(-p) \rangle_{\mathscr{R}} \;=\; -\, \frac{\delta^{ac}\delta^{bd}}{p^2} \eta_{\mu\nu} \,+\, \frac{f^{abe} f^{cde} \mu^2_{\mathscr{R}}}{C_A p^2 [p^2 + \mu^2_{\mathscr{R}}]} \eta_{\mu\nu} \,.
\end{aligned} \tag{3.3}$$

These are very similar to the $\mathscr{Q}$ case but there are massless poles in localizing ghost propagators.

With these one can analyse the zero momentum behaviour akin to (2.3). At one loop the Faddeev-Popov ghost form factor is

$$\begin{aligned}
d_c(p^2) \;=\; -\, \Bigg[ 1 \,-\, C_A \Bigg[ \frac{5}{8} \,-\, \frac{3}{8} \ln \left[ \frac{C_A \gamma^4}{\mu^4} \right] \\
+\, \frac{3\mu^2_{\mathscr{R}}}{8\sqrt{\mu^4_{\mathscr{R}} - 4 C_A \gamma^4}} \ln \left[ \frac{\mu^2_+}{\mu^2_-} \right] \,+\, O\left(p^2\right) \Bigg] a \,+\, o(a^2) \Bigg]^{-1}
\end{aligned} \tag{3.4}$$

where we have set $\mu^2_\pm \,=\, \frac{1}{2} \left[ \mu^2_{\mathscr{R}} \pm \sqrt{\mu^4_{\mathscr{R}} - 4 C_A \gamma^4} \right]$, $\mu$ is the mass scale introduced in dimensional regularization, which we use throughout, to ensure the coupling constant is massless and $a = g^2/(16\pi^2)$. The gap equation satisfied by $\gamma$ is deduced from the horizon condition and to one loop is, [9],

$$1 \;=\; C_A \left[ \frac{5}{8} \,-\, \frac{3}{8} \ln \left[ \frac{C_A \gamma^4}{\mu^4} \right] \,-\, \frac{3\mu^2_{\mathscr{R}}}{8\sqrt{\mu^4_{\mathscr{R}} - 4 C_A \gamma^4}} \ln \left[ \frac{\mu^2_+}{\mu^2_-} \right] \right] a \,+\, O(a^2) \,. \tag{3.5}$$



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The sign difference in the last terms of the previous two equations means that there is no Faddeev-Popov ghost enhancement for non-zero $\mu^2_{\mathscr{R}}$. This can be traced through the actual computation and if, for instance, the gluon propagator was suppressed and not frozen then there would be ghost enhancement. This appears to be a general feature when one analyses other colour channels with a frozen gluon propagator, [9]. Whilst this part of the $\mathscr{R}$ case is the same as that for the $\mathscr{Q}$ channel the bosonic ghost sector is different due to the massless poles in (3.3). Repeating the calculation which led to (2.3) for the $\mathscr{R}$ case we find the zero momentum limit for the localizing ghosts is

$$\langle \xi^{ab}_\mu(p) \xi^{cd}_\nu(-p) \rangle_{\mathscr{R}} \sim \frac{1}{2\mathscr{Q}_0 p^2 a} \left[ \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} - \frac{2}{C_A} f^{abe} f^{cde} \right] \eta_{\mu\nu}$$
$$\langle \rho^{ab}_\mu(p) \rho^{cd}_\nu(-p) \rangle_{\mathscr{R}} \sim \frac{1}{2\mathscr{Q}_0 p^2 a} \left[ \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} - \frac{2}{C_A} f^{abe} f^{cde} \right] \eta_{\mu\nu} \qquad (3.6)$$

where

$$\mathscr{Q}_0 = \left[ \frac{1}{8} \sqrt{\mu^4_{\mathscr{R}} - 4C_A\gamma^4} \ln \left[ \frac{\mu^2_+}{\mu^2_-} \right] - \frac{1}{8} \ln \left[ \frac{C_A\gamma^4}{(p^2)^2} \right] - \frac{11}{24} \right] \frac{\mu^2_{\mathscr{R}}}{C_A\gamma^4} + \frac{1}{4\sqrt{\mu^4_{\mathscr{R}} - 4C_A\gamma^4}} \ln \left[ \frac{\mu^2_+}{\mu^2_-} \right] \qquad (3.7)$$

and is equivalent to that of the Faddeev-Popov ghost factor. So whilst there is no enhancement, in contrast to the $\mathscr{Q}$ case, the propagator does not freeze. Moreover, it is similar to the Faddeev-Popov ghost behaviour. Interestingly the same colour structure as the pure Gribov-Zwanziger case emerges. Hence the adjoint projection freezes to zero or a finite value. This differing behaviour in the localizing ghost sector therefore represents a potential test for lattice simulations if one wished to resolve which of the two cases if either was correct. Though in either situation one would have to construct fields on the lattice which corresponded to $\xi^{ab}_\mu$ and $\rho^{ab}_\mu$ as these are inherently part of the Gribov-Zwanziger construction. In the interim one can study which colour channel is more energetically favourable by applying the local composite operator formalism to (3.1).

## 4. Effective potential

To determine which of the colour channels or combinations of colour channels is energetically favourable, we have computed the one loop effective potential for the operator $\mathscr{O}^{abcd}$, [18]. As we are interested in the colour tensor which leads to a non-zero expectation value the potential is constructed by summing all one loop multi-leg Feynman diagrams with the operator $\mathscr{O}^{abcd}$ as the tag on the external leg. However this is not a simple procedure purely because of the set of free colour indices attached to each leg. Instead we use the colour contracted operator $\mathscr{O}$ which is a colour singlet and then extract the condensation of $\mathscr{O}^{abcd}$ at the end. Due to the structure of the propagators and vertices of (2.1) we have summed the 3 one leg, 4 two leg, 10 three leg, 54 four leg and 408 five leg graphs into logarithms which is the expected structure of the effective potential at one loop. This was then expanded out to the six leg term and shown to agree with the explicit calculation of the 3960 six leg graphs. Performing the Feynman integral and applying the local composite operator formalism the minimum of the potential is located at the combination of parameters

$$N_A C_A \left[ \mu^2_{\mathscr{Q}} + \frac{C_A}{2} \mu^2_{\mathscr{W}} + \mu^2_{\mathscr{R}} - \mu^2_{\mathscr{T}} \right] . \qquad (4.1)$$

It is straightforward to check that this combination is proportional to the contraction of $f^{abg} f^{cdg}$ with the colour tensor term inside the square brackets of (3.1). Hence,

$$\langle \mathscr{O}^{abcd} \rangle \; \propto \; f^{abe} f^{cde} \tag{4.2}$$

which corresponds to the $\mathscr{R}$ channel rather than the $\mathscr{Q}$. This condensation direction appears to be consistent with the original propagators, (2.2). For instance, integrating over the momentum of the mixed propagator produces the gap equation whilst that for the $A^a_\mu$ field indicates that there is a non-zero vacuum expectation value for $\frac{1}{2} A^a_\mu A^{a\,\mu}$. Repeating this for the $\xi^{ab}_\mu$ propagator which has $\mathscr{Q}$ and $\mathscr{R}$ colour tensors, it is the term with the $\mathscr{R}$ tensor which survives due to the presence of a Gribov mass in the denominator factor.

If the $\mathscr{R}$ condensation direction is correct then the propagator analysis we reviewed earlier indicates that there is a natural test. However, as it rests on lattice data for the $\xi^{ab}_\mu$ propagator, we need to address how one can mimic that field. A clue resides in the gap equation and its definition in terms of the horizon condition. The original definition in Gribov's construction, [1], is

$$\left\langle A^a_\mu(x) \frac{1}{\partial^\nu D_\nu} A^{a\,\mu}(x) \right\rangle \; = \; \frac{dN_A}{C_A g^2} \tag{4.3}$$

which translates to

$$f^{abc} \left\langle A^{a\,\mu}(x) \xi^{bc}_\mu(x) \right\rangle \; = \; \frac{idN_A \gamma^2}{g^2} \tag{4.4}$$

for the Gribov-Zwanziger Lagrangian, (2.1). The former is expressed solely in terms of $A^a_\mu$ fields and is non-local whilst the latter is local and depends on the localizing ghost as well as $A^a_\mu$. These two definitions can be replaced by a third, [17], which has no explicit $A^a_\mu$ dependence. Using the relation

$$A^a_\mu \; = \; - \, \frac{i}{C_A \gamma^2} f^{abc} \left( \partial^\nu D_\nu \xi_\mu \right)^{bc} \tag{4.5}$$

and solving it recursively in perturbation theory for $A^a_\mu$, we can rewrite (4.4) as

$$f^{abcd}_4 \left\langle \partial^\nu \xi^{ab\,\mu} \left[ \partial_\nu \xi^{cd}_\mu \; - \; \frac{ig}{C_A \gamma^2} f^{cfrs}_4 \left( \partial^\sigma \partial_\sigma \xi^{rs}_\nu \right) \xi^{fd}_\mu - \frac{g^2}{C_A^2 \gamma^4} f^{cfrs}_4 f^{rqmn}_4 \partial^\sigma \left[ \left( \partial^\rho \partial_\rho \xi^{mn}_\sigma \right) \xi^{qs}_\nu \right] \xi^{fd}_\mu \right.\right.$$
$$\left.\left. + \; O(g^3) \right] \right\rangle \; = \; \frac{dC_A N_A \gamma^4}{g^2} \tag{4.6}$$

with $f^{abcd}_4 \; = \; f^{abp} f^{cdp}$. This is also a local expression but is an infinite series. Indeed in some sense it is dual to Gribov's original definition and is similar to the expansion of the geometric series. As a check on whether this is in fact equivalent to the gap equation for $\gamma$ we have evaluated (4.6) by computing all the two loop diagrams for the vacuum expectation value, [17]. It agrees exactly with that derived from (4.4), [19]. As a further check on the relation of $A^a_\mu$ and $\xi^{ab}_\mu$, (4.5), we have explicitly verified that replacing $A^a_\mu$ in $\frac{1}{2} \langle A^a_\mu A^{a\,\mu} \rangle$ with the expression in terms of $\xi^{ab}_\mu$ defined implicitly in (4.5) that one obtains precisely the same algebraic expression for both vacuum expectation values to *two* loops. This suggests that (4.5) is key to deriving a non-local projection of $A^a_\mu$ which could be used on the lattice to obtain data for the $\xi^{ab}_\mu$ field and hence test what its infrared behaviour is in relation to the analysis here and in [9].

## 5. QCD vertices

We now turn to a completely separate area of (massless) QCD and that is the structure of the 3-point vertices at the symmetric subtraction point which corresponds to a non-exceptional momentum configuration. The original work by Celmaster and Gonsalves at one loop led to the set of MOM subtraction schemes, [11]. However, with the need for more precision it is of interest to extend this to the next order analytically which has recently been performed in [10]. A numerical estimate of the two loop vertex structure was given in [13]. Whilst knowledge of the exact three loop MOM $\beta$-functions will be of use for precision, having information on the full vertex amplitudes will aid lattice matching of the same Green's function. Moreover, in the symmetric subtraction point approach one avoids potential infrared problems which can occur in the extraction of the amplitudes at the asymmetric point, for instance. Here we will briefly summarize highlights of [10] to provide a flavour of the issues and the results. As with the work reviewed earlier we use the symbolic manipulation language FORM, [20], with all diagrams for each 3-point vertex generated by QGRAF, [21]. Unlike the approach used by [13] to approximate the 3-point vertices at the symmetric point and for the asymmetric subtraction, the MINCER algorithm, [14], cannot be applied for the exact symmetric point calculation. Instead we used the Laporta algorithm, [22], to build a database of relevant integration by parts relations between all the integrals contributing to the Feynman diagrams. Specifically we used the REDUZE package, [23], encoding of the Laporta algorithm where REDUZE is written in C++ using the underlying GINAC computer algebra system, [24].

For reference here we will focus on the triple gluon vertex with the Green's function defined by, [10],

$$\left\langle A^a_\mu(p) A^b_\nu(q) A^c_\sigma(-p-q) \right\rangle \Big|_{p^2=q^2=-\mu^2} ~=~ f^{abc} \, \Sigma^{ggg}_{\mu\nu\sigma}(p,q) \Big|_{p^2=q^2=-\mu^2} \tag{5.1}$$

where the structure constants have been factored off and the Lorentz scalar amplitudes, $\Sigma^{ggg}_{(k)}(p,q)$, are given by

$$\Sigma^{ggg}_{\mu\nu\sigma}(p,q) \Big|_{p^2=q^2=-\mu^2} ~=~ \sum_{k=1}^{14} \mathcal{P}^{ggg}_{(k)\,\mu\nu\sigma}(p,q) \, \Sigma^{ggg}_{(k)}(p,q) \tag{5.2}$$

and $\mathcal{P}^{ggg}_{(k)\,\mu\nu\sigma}(p,q)$ are the tensors of the basis at the symmetric point. The explicit forms are given in [10]. Applying the Laporta algorithm and computing the 8 one loop and 106 two loop graphs the numerical version of the exact amplitudes are

$$\begin{aligned}
& \Sigma^{ggg}_{(1)}(p,q) \Big|_{\overline{\mathrm{MS}}} ~=~ \Sigma^{ggg}_{(2)}(p,q) \Big|_{\overline{\mathrm{MS}}} ~=~ -~ \frac{1}{2} \, \Sigma^{ggg}_{(3)}(p,q) \Big|_{\overline{\mathrm{MS}}} \\
& = -~ \Sigma^{ggg}_{(4)}(p,q) \Big|_{\overline{\mathrm{MS}}} ~=~ \frac{1}{2} \, \Sigma^{ggg}_{(5)}(p,q) \Big|_{\overline{\mathrm{MS}}} ~=~ -~ \Sigma^{ggg}_{(6)}(p,q) \Big|_{\overline{\mathrm{MS}}} \\
& = -\,1 ~-~ \left[ 1.1212444 - 3.7618956\alpha - 1.2890232\alpha^2 + 0.1250000\alpha^3 - 0.0417366 N_f \right] a \\
& + \big[ 29.7530676 + 16.4600770\alpha - 9.7794300\alpha^2 - 3.2060809\alpha^3 - 1.6522848\alpha^4 + 0.2812500\alpha^5 \\
& \quad - [11.5677203 - 0.9686976\alpha - 0.9112399\alpha^2 + 0.4166667\alpha^3] N_f \big] a^2 ~+~ O(a^3)
\end{aligned} \tag{5.3}$$

where $\alpha$ is the linear covariant gauge parameter. The explicit analytic forms are too large to record here. Indeed an indication of their analytic structure can be gained from the two loop mapping of the coupling constant in the MOMggg scheme to that in the $\overline{\text{MS}}$ scheme. For instance, [10],

$$
\begin{aligned}
a_{\text{MOMggg}} = {} & a_{\overline{\text{MS}}} + \left[ \left[ 69\psi'(\tfrac{1}{3}) - 46\pi^2 + 1188 \right] C_A + \left[ 128\pi^2 - 192\psi'(\tfrac{1}{3}) - 432 \right] T_F N_f \right] \frac{a^2_{\overline{\text{MS}}}}{162} \\
& + \Big[ \Big[ 19044(\psi'(\tfrac{1}{3}))^2 - 25392\pi^2\psi'(\tfrac{1}{3}) - 6938784\psi'(\tfrac{1}{3}) - 100602\psi'''(\tfrac{1}{3}) \\
& \quad - 72643392 s_2(\tfrac{\pi}{6}) + 145286784 s_2(\tfrac{\pi}{2}) + 121072320 s_3(\tfrac{\pi}{6}) - 96857856 s_3(\tfrac{\pi}{2}) \\
& \quad + 276736\pi^4 + 4625856\pi^2 - 113724\Sigma + 8301852\zeta(3) + 40126833 \\
& \quad - 504468\frac{\ln^2(3)\pi}{\sqrt{3}} + 6053616\frac{\ln(3)\pi}{\sqrt{3}} + 541836\frac{\pi^3}{\sqrt{3}} \Big] C_A^2 \\
& \quad + \Big[ 141312\pi^2\psi'(\tfrac{1}{3}) - 105984(\psi'(\tfrac{1}{3}))^2 - 2960064\psi'(\tfrac{1}{3}) + 33592320 s_2(\tfrac{\pi}{6}) \\
& \qquad - 67184640 s_2(\tfrac{\pi}{2}) - 55987200 s_3(\tfrac{\pi}{6}) + 44789760 s_3(\tfrac{\pi}{2}) - 47104\pi^4 \\
& \qquad + 1973376\pi^2 + 2239488\Sigma - 8957952\zeta(3) - 26695008 \\
& \qquad + 233280\frac{\ln^2(3)\pi}{\sqrt{3}} - 2799360\frac{\ln(3)\pi}{\sqrt{3}} - 250560\frac{\pi^3}{\sqrt{3}} \Big] C_A T_F N_f \\
& \quad + \Big[ 124416\psi'''(\tfrac{1}{3}) - 1492992\psi'(\tfrac{1}{3}) - 331776\pi^4 + 995328\pi^2 \\
& \qquad - 4478976\Sigma + 6718464\zeta(3) - 7138368 \Big] C_F T_F N_f \\
& \quad + \Big[ 147456(\psi'(\tfrac{1}{3}))^2 - 196608\pi^2\psi'(\tfrac{1}{3}) + 2322432\psi'(\tfrac{1}{3}) + 65536\pi^4 \\
& \qquad - 1548288\pi^2 + 2923776 \Big] T_F^2 N_f^2 \Big] \frac{a^3_{\overline{\text{MS}}}}{419904} + O\left( a^4_{\overline{\text{MS}}} \right)
\end{aligned} \tag{5.4}
$$

in the Landau gauge. The arbitrary linear covariant gauge expression is significantly larger than this. Here $s_n(z)$ is related to the polylogarithm and $\Sigma$ is a combination of harmonic polylogarithms. Indeed this mapping means that the three loop MOMggg $\beta$-function can be determined analytically for arbitrary $\alpha$, [10]. Again space does not permit the full analytic form but numerically for non-zero $\alpha$ we have, [10],

$$
\begin{aligned}
\beta^{\text{MOMggg}}(a,\alpha) = {} & - [11.0000000 - 0.6666667 N_f] a^2 \\
& - \Big[ 102.0000000 + 19.6546434\alpha - 0.2710840\alpha^2 - 5.8591391\alpha^3 \\
& \quad + 1.1250000\alpha^4 \\
& \quad - \left[ 12.6666667 + 2.0158609\alpha + 0.4373952\alpha^2 - 0.5000000\alpha^3 \right] N_f \Big] a^3 \\
& - \Big[ 1570.9843804 + 658.0709292\alpha + 269.2238338\alpha^2 + 43.0029610\alpha^3 \\
& \quad - 99.2797189\alpha^4 + 14.8550247\alpha^5 + 5.3345924\alpha^6 - 0.7031250\alpha^7 \\
& \quad + \big[ 0.5659290 - 43.2393672\alpha - 22.7471960\alpha^2 - 19.8709555\alpha^3 \\
& \qquad + 14.8347569\alpha^4 + 0.9764184\alpha^5 - 0.2812500\alpha^6 \big] N_f \\
& \quad - \left[ 67.0895364 + 4.6479610\alpha + 0.8898051\alpha^2 - 2.3056953\alpha^3 \right] N_f^2 \\
& \quad + 2.6581155 N_f^3 \Big] a^4 + O(a^5) \, .
\end{aligned} \tag{5.5}
$$

This together with the expressions for the MOMh and MOMq $\beta$-functions at three loops agree very closely with the numerical estimates given in [13].

## 6. Discussion.

We have reviewed recent work, [9], on the structure of the Gribov-Zwanziger Lagrangian with the inclusion of the most general BRST invariant dimension two operator built from the localizing ghost fields. From the analysis it appears that the natural colour channel for the operator to condense in is that corresponding to $\mathscr{R}$ rather than $\mathscr{Q}$. Indeed the former case is in accord with what one would expect from the original propagators of the Gribov-Zwanziger Lagrangian. Moreover the infrared behaviour of the bosonic localizing ghost propagator has been deduced at one loop after the implementation of the gap equation for the Gribov mass. In the $\mathscr{R}$ case there are massless modes which survive in the non-perturbative region unlike the $\mathscr{Q}$ case. In addition the residue of the zero momentum behaviour is in one-to-one correspondence with that of the Faddeev-Popov ghost propagator behaviour in the same limit. For this reason it would be interesting if the lattice could derive data for the localizing ghost propagator at zero momentum as, aside from providing more information on the structure of the theory, it would give insight into the full ghost sector. This is important due to the fact that one cannot include Faddeev-Popov ghosts on the lattice. The second part of the article concentrated on the ultraviolet behaviour of QCD at the symmetric subtraction point of the 3-point vertices of the theory in an arbitrary linear covariant gauge, [10]. This is useful for lattice computations of the Green's functions as one can now match onto the ultraviolet region with more precision due to the provision of the amplitudes to two loops. Moreover, the original one loop construction of the momentum subtraction schemes of Celmaster and Gonsalves, [11], has now been extended to two loops analytically, [10].

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