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Determination of α_s from lattice QCD

Eigo Shintani*
RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA
E-mail: shintani@riken.jp

The accuaracy of strong coupling constant α_s is further improved by the high-precision caculation of lattice QCD with Monte-Carlo simulation including the dynamical light (up and down) and strange quarks. I explain recent lattice determination of α_s using vacuum polarization function computed in dynamical overlap fermion configurations. I also review some other recent results and discuss the future perspective.

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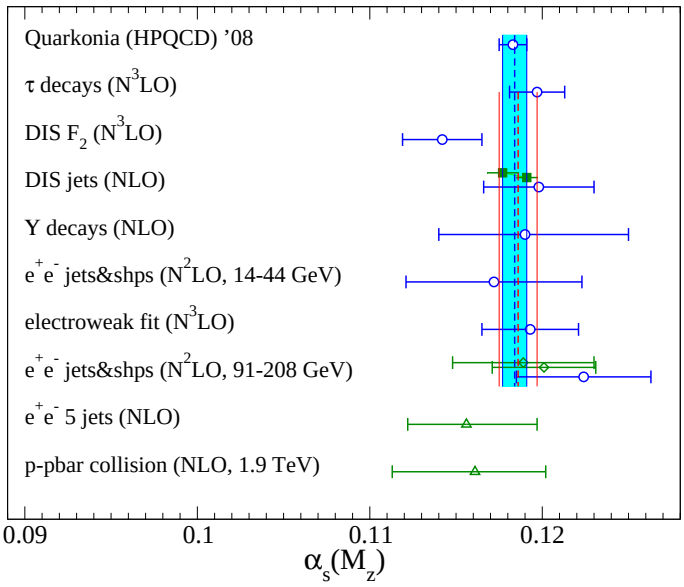

Figure 1: Summary of $\alpha_s(M_Z)$ in $\overline{MS}$ obtained from several methods. Those values refer to [2] and recent publications. The published world average presented in [2] is illustrated as blue dashed line (central value) and blue band (uncertainty). The red dashed line and band denote the result except for lattice calculation. The green squares denote the results in an analysis using data-set of deep-inelastic scattering data and HERA data quoted in [3]. Those results do not contain theoretical uncertainties. The recent results from OPAL [4] (the green diamonds) using NNLO and NNLO+NNLA calculation, and from LEP [5] and Tevatron [6] (the green upper-triangles) are also shown.

1. Introduction

Strong coupling constant α_s is one of the most important fundamental parameter of quantum chromodynamics (QCD). QCD is successful to describe the strong dynamics related to quark-gluon interaction with just a few input parameters, α_s and quark mass. Indeed this theory provides highly consistent results with high-energy experimental measurements using perturbative and non-perturbative calculation such as lattice QCD or operator product expansion (OPE). High precision determination of α_s plays an important role in theoretical prediction of the Standard Model (SM) and the new physics search at extremely high-energy experiments. Recently Higgs production experiment at LHC [1] requires less than 0.5% precision of α_s to keep the accuracy similar to the electroweak contribution since 7%–8% uncertaities of gluon fusion diagram is dominated from the uncertainties of α_s and parton distribution function for 125 GeV Higgs mass.

The current world average of α_s has been published in 2009 [2] as $\alpha_s(M_Z) = 0.1135 \pm 0.0007$ which is evaluated by combination of the measurement of e^+e^- annihilation, τ decay and lattice QCD calculation (see Figure 1). Note that the central value of α_s certainly leads to the preciseness of lattice calculation in which its accuracy has already reached to the five-digit precision, and indeed if lattice result is made excluded from it, the central value shifts about 30% and such error increases by 50% [2].

To obtain α_s, one needs an ingredient of dimensionless quantity which is known as the per-

turbative expansion is converged at sufficiently high energy scale Q where is enough controlled by the power raw $\mathcal{O}(\Lambda_{\rm QCD}/Q)$ contributions. Conventionally R_i denotes such quantity corresponding to the physical observable i, for example in the case of τ decay R_τ is chosen to be hadronic branching ratio and also in the case of e^+e^- annihilation R_i is chosen to be the i-jet production cross section divided by total cross section. In these cases the appropriate energy scale for convergence of perturbative expansion depends on the observable. For R_τ which provides the precise determination of α_s from experimental measurement, the energy scale $Q = M_\tau = 1.78$ GeV corresponds to $\alpha_s(M_\tau) \simeq 0.33$ where the unknown non-perturbative effect from the estimate based on OPE at $Q = M_\tau$ may be about 7%, and when taking to $Q = M_Z$ about three factor of such uncertainty decreases because relative error of running coupling constant can be described as $\Delta\alpha_s/\alpha_s \sim \alpha_s$. In this case theoretical uncertainties can be rather efficiently suppressed by a few % level than other high energy experiments (in which about 5% theoretical uncertainties remain).

In lattice QCD there are many choices of R as short distance observable. The energy scale we can choose is allowed for broad range below $Q \simeq \mathcal{O}(\pi/a)$ by the lattice cut-off a^{-1} which is provided by the simulation parameter β regarding the bare gauge coupling. Presently thanks to the great development of computational algorithm and equipment (supercomputer) lattice QCD simulation makes progress into one of the most reliable theoretical calculation of QCD based on the first principles. Full QCD simulation including sea quarks of degenerate light (up-down) flavor plus strange has been established, and now the new simulation with sea charm quark becomes routine. As a consequence lattice calculations of α_s has reached toward five-digit precision area, which is a few % uncertainty compared with estimate from τ decay measurement.

In this proceedings I first show the recent work in JLQCD collaboration adopting Adler function to dimensionless quantity R in lattice QCD, and next I briefly explain the other lattice works and compare these results.

2. Adler function and vacuum polarization function

Conventionally Adler function D is defined by the derivative of hadronic vacuum polarization function (VPFs) $\Pi_J(Q^2)$ with respect to the Euclidean momentum square Q^2:

$$D(Q^2) = -Q^2 \frac{d\Pi_J(Q^2)}{dQ^2} \tag{2.1}$$

which has been known as scheme-independent quantity calculated by the perturbation at NNLO ($\mathcal{O}(\alpha_s^3)$) [7, 8] and N^{3}LO ($\mathcal{O}(\alpha_s^4)$) [9]. The ingredient Π_J is given by the spin-expansion of non-singlet vector or axial-vector current correlator as

$$\langle J^a_\mu J^b_\nu \rangle(Q) = \delta^{ab}\left[(\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi^{(1)}_{J=V,A} - Q_\mu Q_\nu \Pi^{(0)}_{J=A}\right] \tag{2.2}$$

where the superposition indicates the corresponding spin of state, for example $\Pi^{(1)}_J$ is associated with $\rho(J=V)$ and $a_1(J=A)$ meson pole, $\Pi^{(0)}_A$ is associated with π, a_0 poles, at low energy region. At high energy region, where is equivalent to deep Euclidean region, we can reproduce the perturbative formula for Π_J from Adler function D when the renormalization scheme is chosen. In

general operator production expansion (OPE) describes Π_J as

$$\Pi_{V+A}|_{\rm OPE}(Q^2,\alpha_s) = c + C_0(Q^2,\mu^2,\alpha_s) + C_m^{V+A}(Q^2,\mu^2,\alpha_s)\frac{\overline{m}^2(Q)}{Q^2} + \sum_{q=u,d,s} C_{\bar{q}q}^{V+A}(Q^2,\alpha_s)\frac{\langle m_q\bar{q}q\rangle}{Q^4} + C_{GG}(Q^2,\alpha_s)\frac{\langle(\alpha_s/\pi)GG\rangle}{Q^4} + \mathscr{O}(Q^{-6}) \tag{2.3}$$

with analytic formula C_0 from $\overline{MS}$ scheme at $\mathscr{O}(\alpha_s^2)$ [10] and $\mathscr{O}(\alpha_s^3)$ [11, 12], C_m from $\overline{MS}$ scheme at $\mathscr{O}(\alpha_s^2)$ [13], $C_{X=\bar{q}q,GG}$ is Wilson coefficient of non-perturbative quantity for operator condensate $\langle X\rangle$ [14]. c denotes the scheme-dependent constant which vanishes in Adler function.

3. Strong coupling constant from VPFs on the lattice

Here I would like to show the recent results using the Adler function computed by VPFs in lattice QCD with overlap fermion.

3.1 Lattice formula in the exact chiral fermion

VPFs can be defined as the same way as continuum theory (2.2) on the lattice if vector or axial-vector current is satisfied with Ward-Takahashi (WT) identity. If not, we should take into account the lattice artifact due to violation of current conservation. In overlap fermion formulation, which is defined as lattice fermion with the exact chiral symmetry on the lattice, there is an exact definition of conserved current:

$$V_\mu^{a,\rm cv}(x) = \sum_{w,z}\bar{q}(w)t^aK_\mu(w,z|x)q(z), \quad A_\mu^{a,\rm cv}(x) = \sum_{w,z}\bar{q}(w)t^aK_\mu(w,z|x)[\hat{\gamma}_5 q](z), \tag{3.1}$$

where $K_\mu(w,z|x)$ is defined as the kernel of conserved current whose explicit formula is described in [15, 16]. The current-current correlator used in both conserved ones are computationally hard to construct because of non-locality of the inverse of h_W among the two operator, unless all-to-all quark propagator is employed. However the mixing correlator with local-conserved current, $\langle J_\mu^{a,\rm cv}(x)J_\nu^{b,\rm loc}(y)\rangle$ with $J_\mu^{a,\rm loc}(x) = \bar{q}t^a\gamma_\mu q$, does not matter to computational construction on the lattice. This current-current correlator is satisfied with WT identity for one side,

$$\sum_\mu \hat{Q}_\mu\langle V_\mu^{a,\rm cv}V_\nu^{b,\rm loc}\rangle(Q) = 0, \quad \sum_\mu \hat{Q}_\mu\langle A_\mu^{a,\rm cv}A_\nu^{b,\rm loc}\rangle(Q) - 2m_q\langle P^aA_\nu^{b,\rm loc}\rangle(Q) = 0, \tag{3.2}$$

where $a\hat{Q}_\mu = \sin(aQ_\mu)$ is momentum definition corresponding to the backward derivative operator ∂_μ^{x*}. The second term of AWT in (3.2) represents the correlation function of the pseudo-scalar density operator $P^a(x) = \bar{q}(x)t^a\gamma_5(1-D_{\rm ov}/m_0)q(x)$ and the local axial-vector current $A_\nu^{b,\rm loc}(y)$. A possible term arising from the axial transformation of $J_\nu^{b,\rm loc}(y)$ ($J = V$ or A) vanishes when we take the vacuum expectation value, since the vacuum has axis-interchange symmetry while the index ν remains in $J_\nu^{b,\rm loc}(y)$.

The vector and axial-vector VPFs are now given by [16]

$$\langle J_\mu^{a,\rm cv}J_\nu^{b,\rm loc}\rangle(Q) = \delta^{ab}\left[(\delta_{\mu\nu}\hat{Q}^2 - \hat{Q}_\mu\hat{Q}_\nu)\Pi_J^{(1)}(Q) - \hat{Q}_\mu\hat{Q}_\nu\Pi_J^{(0)}(Q) + \Delta_{\mu\nu}^J(Q)\right]. \tag{3.3}$$

Here, $\Pi_V^{(0)}(Q)$ vanishes because of the conservation of $V_\mu^{a,\mathrm{cv}}$, while $\Pi_A^{(0)}(Q)$ represents a remnant due to PCAC:

$$\Pi_A^{(0)}(Q) = -2m_q\langle P^a A_\nu^{a,\mathrm{loc}}\rangle(Q)/(\hat{Q}^2\hat{Q}_\nu). \tag{3.4}$$

(Repeated indices a's are not summed.) The transverse part $\Pi_J^{(1)}(Q)$ can be extracted as

$$\Pi_J^{(1)}(Q) = \langle J_\mu^{a,\mathrm{cv}} J_\mu^{a,\mathrm{loc}}\rangle(Q)/(\hat{Q}^2 - \hat{Q}_\mu\hat{Q}_\mu), \tag{3.5}$$

(repeated indices μ's are not summed) if one ignores the additional term $\Delta_{\mu\nu}^J(Q)$, which reflects the violation of the current conservation of the local current $J_\nu^{a,\mathrm{loc}}$. Since the current conservation is recovered in the continuum limit, this term can be expanded in terms of small aQ_μ as

$$\Delta_{\mu\nu}^J(Q) = \sum_{m,n=1} \left(\delta_{\mu\nu}\sum_\rho \hat{Q}_\rho^{2m} - \hat{Q}_\mu^{2(m-1)}\hat{Q}_\mu\hat{Q}_\nu\right) Q_\nu^{2n} F_{mn}(\hat{Q}), \tag{3.6}$$

where F_{mn} denotes the scalar function depends on the index m,n and momentum Q. It satisfies the condition $\sum_\mu \hat{Q}_\mu\Delta_{\mu\nu}^J(Q) = 0$ coming from the WT identity for $J_\mu^{a,\mathrm{cv}}$. Fortunately the numerical investigation suggests the corresponding lattice artifact is tiny contribution to VPFs [16, 18] rather than the case when local-local current is used [19].

3.2 α_s from VPFs

OPE formula in Eq.(2.2) is applicable to use as a fitting function with data of VPFs extracted from current-current correlator in Eq.(3.4) and Eq.(3.5) ignoring the $\Delta_{\mu\nu}^J$. Since the value of the operator condensate of gluon can not be independently determined from direct lattice calculation due to renormalon ambiguity [20], we treat it as a free parameter as well as c, otherwise the quark condensate is deterministic in lattice QCD unless the chiral symmetry does not broken on the lattice. The coupling constant is given by $\Lambda_{\overline{MS}}$ in the perturbative expansion up to $\mathscr{O}(\alpha_s^4)$ [21, 22]. Thus three free parameters $(c, \langle GG\rangle, \Lambda_{\overline{MS}})$ are required in this fitting when overlap fermion is adopted.

JLQCD collaboration has curried out the calculation with overlap fermion in dynamical light quark (degenerate up-down) and strange quark in the $16^3\times 64$ lattice at $a^{-1} = 1.83(1)$ GeV [16]. Figure 2 shows that χ^2 fitting has well quality of describing lattice VPFs with OPE formula as a function of $\hat{Q}^2$ in a whole quark mass. Actually after dealing with correlated fit between different momenta the value of χ^2/dof is 1.7 with carefully constrained momentum range.

Mainly there are two sources possibly including as systematic error into this calculation. The details refer to description in [16].

- At small $(a\hat{Q})^2$, truncation effect in higher order than $\mathscr{O}(Q^{-6})$ in OPE may be significant. Figure 3 indicates that in low $(aQ)^2$ region from 0.4–0.5 the behavior of higher order operator can be observed.

- At large $(aQ)^2$, the contribution of $\Delta_{\mu\nu}$ in Eq.(3.6) does not ignore. In [16] that is taken into account by investigation with lattice perturbation (see Fig.3) and comparison between data of $\mu = \nu$ and $\mu \neq \nu$ (see Fig.4). The systematic error is conservatively evaluated from these estimate.

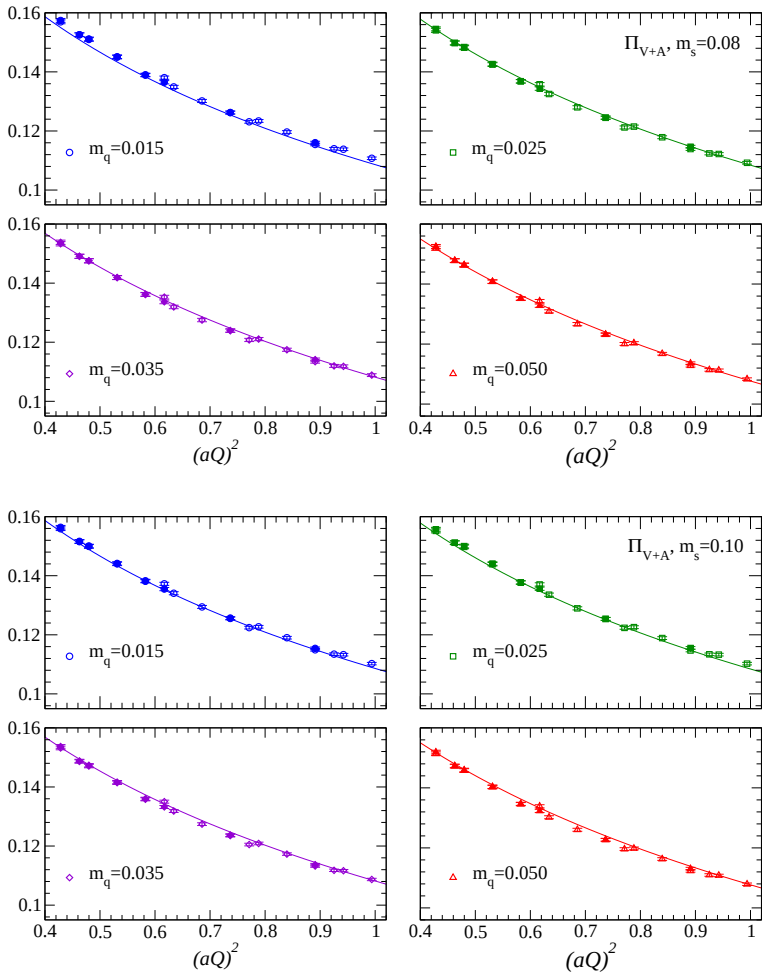

Figure 2: $(aQ)^2$ dependence of VPF, $\Pi_{V+A}(Q)$, at all valence quark masses: $m_q = 0.015$ (circle), 0.025 (square), 0.035 (diamond), and 0.050 (triangle), evaluated in overlap fermion using 2+1 flavor in $16^3 \times 64$ [16]. Top panel is a result at $m_s = 0.08$ while the bottom is at $m_s = 0.10$. Solid curves show a fit function at each quark masses. Filled symbols are the points for which each momentum component is equal to or smaller than $2\pi/16$ in the lattice unit.

After taking account of other systematic uncertainties as determination of lattice spacing, renormalization factor (for local current, quark mass) and physical mass of $m_{c,b}$, we obtain

$$\alpha_s^{(5)}(M_Z) = 0.1181(3)(^{+14}_{-12}). \tag{3.7}$$

The main uncertainty is due to large difference of lattice spacing from determination using f_π and Ω mass [23, 16], however this may be reduced as soon as possible when the precise simulation using more accurate parameter (larger lattice volume etc.) is available in near future [24].

4. Other lattice calculations of strong coupling constant

Here I will introduce the results of other lattice method. We find the brief overview of current lattice status in the literature [25].

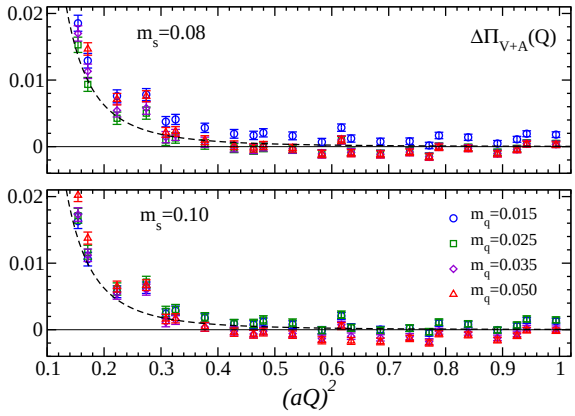

Figure 3: Difference between the lattice data and the fit function. Dashed line shows a guiding line representing the $1/Q^6$ behavior.

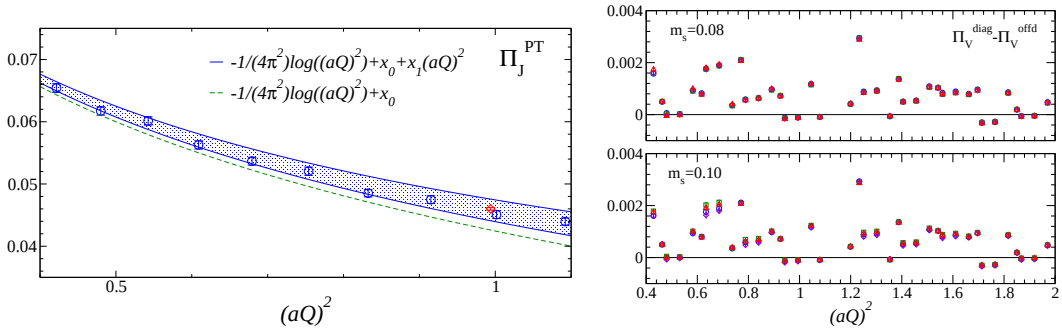

Figure 4: (Left) $(aQ)^2$ dependence of one-loop VPF $\Pi_{J=V,A}(Q^2)$ in lattice perturbation theory. Dashed line shows the leading logarithm term plus a constant, which corresponds to the continuum perturbation theory. Solid lines show the function including lattice artifact of $O((aQ^2))$. The shaded band represents an uncertainty due to the higher order effects. The red diamond denotes the value at the upper limit of our fit of VPF. (Right) Difference between $\Pi_V^{\rm diag}(Q)$ and $\Pi_V^{\rm offd}(Q)$ at all valence quark masses $m_q = 0.015$ (circles), 0.025 (squares), 0.035 (diamonds), and 0.050 (triangles). Top panel of right figure is the result at $m_s = 0.08$ and the bottom one is in $m_s = 0.10$.

4.1 From Wilson loop with lattice perturbation

The perturbative expansion of Wilson loops with some kinds of loop-path shape provides the value of strong coupling constant α_V [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36] as

$$-\ln W_{m,n} = \sum_{i=1} c_i^{(N_f)}(m,n)\left[\alpha_V^{(N_f)}(q^*_{m,n})\right]^i \tag{4.1}$$

by the comparison between the light-hand-side which is provided from Wilson loop for $n \times m$ or non-planer paths in lattice Monte-Carlo simulation, and the right-hand-side in which $c_i^{(N_f)}(n,m)$ is given by computation of Feynman diagram in lattice perturbation. The scale $q^*_{m,n}$ is the average momentum of gluon, and this value has been usually used in evaluation from the one-loop gluon contribution to Wilson loops [28], however more robust procedure of scale setting incorporating

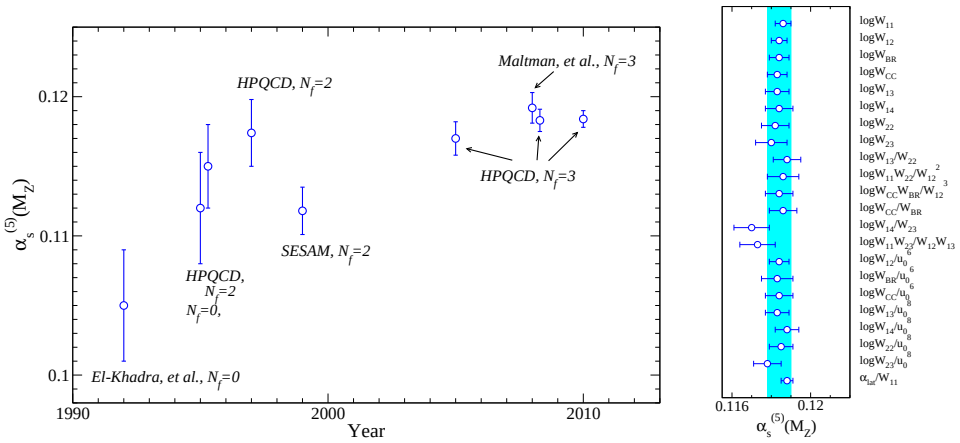

Figure 5: (Left) History of the $\alpha_s^{(5)}(M_Z)$ vale from short-distance quantity of Wilson loop on the lattice. I represent the number of dynamical quark flavor used in the Monte Carlo calculation of Wilson loops as N_f. (Right) $\alpha_s^{(5)}(M_Z)$ from 22 different Wilson loops determined by HPQCD collaboration [36].

higher order contribution is available in [37], and thus conventionally in Ref.[33, 35, 36] the notation of this scale is distinguished as $q^*_{m,n} \to d/a$. To convert the conventional $\overline{MS}$ scheme, the perturbation formula using static-quark potential at third order [38] is available

$$\alpha_{\overline{MS}}^{(N_f)}(Q) = \alpha_P^{(N_f)}(e^{5/6}Q)\Big[1+\frac{2}{\pi}\alpha_P^{(N_f)}+X_{\overline{MS}}(\alpha_P^{(N_f)})^2+\mathscr{O}((\alpha_P^{(N_f)})^3)\Big], \tag{4.2}$$

$$\alpha_V^{(N_f)}(Q) = \alpha_P^{(N_f)}(Q)\big[1+\alpha_P^2(1.86-0.45N_f+X_{\overline{MS}})+\mathscr{O}(\alpha_P^3)\big], \tag{4.3}$$

where the value of $X_{\overline{MS}}$ has been known as $X_{\overline{MS}} \simeq 0.95$ [31, 32] for quenched QCD.

Since starting in Ref.[26] the procedure of lattice calculation of α_s with Wilson loop, the precision has become higher and higher. We can find the results of $\alpha_s^{(5)}(M_Z)$ in $N_f = 0$ [26, 29], $N_f = 2$ [29, 31, 32] and $N_f = 3$ [33, 34, 35, 36] (see Fig. 5). The recent result in [36] is given by

$$\alpha_s^{(5)}(M_Z) = 0.1184(6), \tag{4.4}$$

from combining the 22 determinations of coupling constants evaluated from Wilson loops and lattice perturbation at NNLO ($c^{(3)}_{n\le 3}$) plus simultaneous fitting result from different lattice spacing $c^{(3)}_{4,5}$ (see Fig.5).

4.2 From moment of heavy quark current-current correlators

Determination of α_s from moment of heavy quark current-current correlator is similar idea to determination of α_s from VPFs as explained in section 3.2. Here dimensionless ingredient is defined as

$$G_n \equiv \sum_t \left(\frac{t}{a}\right)^n G(t), \quad G(t) = a^6 \sum_{\vec{x}} (am_{0h})^2 \langle 0|j_5(\vec{x},t)\, j_5(0,0)|0\rangle, \tag{4.5}$$

where G_n corresponds to coefficient at $(n-2)/2$th order of Taylor series for vacuum polarization $\Pi^p(q^2)$, which is defined as $i\int dx e^{iqx}\langle 0|T\, j_5(x)\, j_5(0)|0\rangle = q^2\Pi^p(q^2)$, around zero external momentum. Conventionally in Ref.[39, 36] they define dimensionless "reduced moment" as the ratio of

tree level moment:

$$R_n = \begin{cases} G_4/G_4^{(0)} & \text{for } n=4 \\ \frac{am_{\eta_h}}{2am_{\rm pole,h}^{(0)}}(G_n/G_n^{(0)})^{1/(n-4)} & \text{for } n \geq 6 \end{cases}, \tag{4.6}$$

in which the denominator is used to be the lowest-order moment of lattice perturbation, and $m_{\rm pole,h}^{(0)}$ is the lowest-order of c-mass, which is related to bare heavy quark mass am_{0h} [39]. Reduced moment is equivalent to continuum quantities up to $\mathcal{O}((am_h)^m\alpha_s)$:

$$R_n = \begin{cases} g_4(\alpha_s(\mu),\mu/m_h)/g_4^{(0)} & \text{for } n=4 \\ \frac{m_{\eta_h}}{2m_c(\mu)}\left(g_n(\alpha_s(\mu),\mu/m_h)/g_n^{(0)}\right)^{1/(n-4)} & \text{for } n \geq 6 \end{cases} + \mathcal{O}((am_h)^m\alpha_s), \tag{4.7}$$

and therefore α_s and $m_h(\mu)$ can be evaluated by using the equivalence between lattice calculated R_n and g_n in continuum perturbation theory at NNLO [40, 10] up to $n=62$ and N^{3}LO [11, 41, 42] up to $n=10$ (where n represented in here is equivalent to $(n-2)/2$th moment). The exponent m will take $m=2,4\cdots$, which depends on the order of moments.

HPQCD collaboration have performed the above procedure to calculate $m_{c,b}$ and α_s using up to $n=30$ [36] in simultaneous fit combined with exact coefficients in N^{3}LO perturbation theory and fitting with free parameters, and thus they obtained

$$\alpha_s^{(5)}(M_Z) = 0.1183(7), \tag{4.8}$$

and also $m_c(3\,\text{GeV}) = 0.986(6)$ GeV for $N_f=4$ and $m_b(10\,\text{GeV}) = 3.617(25)$ for $N_f=5$.

4.3 From step scaling function in Schrödinger functional scheme

ALPHA collaboration [43] have developed the new scheme for the calculation of α_s from non-perturbative running coupling constant through step-scaling procedure. So called "Schrödinger functional (SF) scheme" is fabricated by the step scaling function $\Sigma(u)$ of renormalized coupling constant $u=\bar{g}^2$, which is given by the derivative of effective action with respect to the imposed Dirichlet boundary field parametrized by real value η (and ν) in temporal direction [44, 45]. In this procedure scaling the coupling constant follows the step scaling function at finite size L after taking the continuum limit:

$$\sigma(u) = \lim_{a/L\to 0}\Sigma(u,L/a), \quad \Sigma(u,L/a) = \bar{g}^2(2L)\big|_{\bar{g}^2(L)=u}, \tag{4.9}$$

where the second equation indicates the new renormalized coupling constant $\bar{g}^2(2L)$ obtained at 2 times scale-up from the old renormalized coupling constant $\bar{g}^2(L)$ at fixed lattice spacing a. $\sigma(u)$ is evaluated by the continuum extrapolation for $\Sigma(u,L/a)$ using several lattice spacing (bare coupling constants) (see Fig.6). When obtaining the $\sigma(u)$ from starting maximum scale $\mu_{\rm max}=1/L_{\rm max}$ into perturbative region, the Λ-parameter in SF scheme is given by the exact solution of Callan-Symanzik equation

$$\Lambda^{\rm SF} = \mu(b_0\bar{g}^2(\mu))^{-b_1/(2b_0^2)}e^{-1/(2b_0\bar{g}^2(\mu))}\exp\left\{-\int_0^{\bar{g}^2(\mu)}dx\left[\frac{1}{\beta^{\rm SF}(x)}+\frac{1}{b_0x^3}-\frac{b_1}{b_0^2x}\right]\right\} \tag{4.10}$$

with $\mu = 2^i/L_{\rm max}$. In fact, the lattice result of a non-perturbative step scaling function has been well consistent with three-loop perturbation theory at weak coupling region (see Fig. 6). $\beta^{\rm SF}(x)$

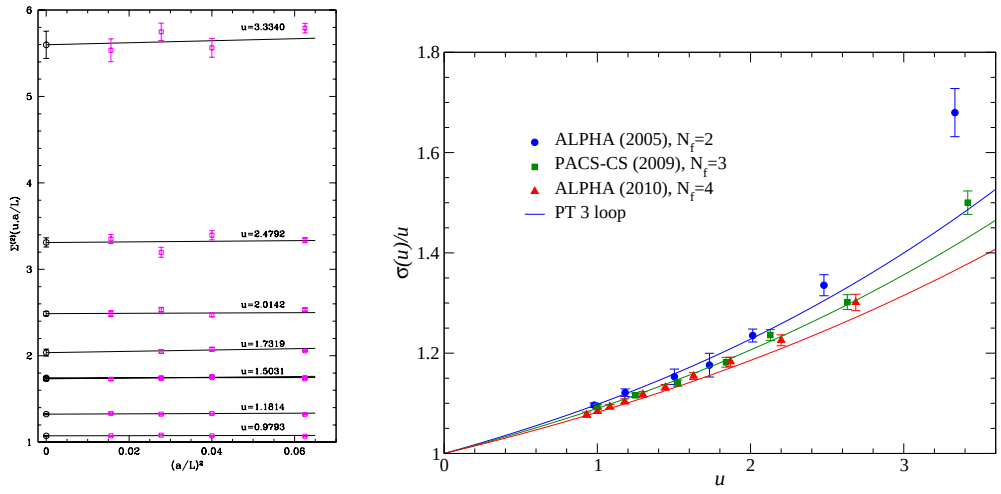

Figure 6: (Left) Continuum extrapolation of step scaling function $\Sigma(u,L/a)$ defined in context starting from $u = 5.5$ [45]. (Right) $\sigma(u)/u$ as a function of u obtained from Schrödinger functional scheme in $N_f = 2$ [45], $N_f = 3$ [46] and $N_f = 4$ [47]. The solid lines denote the comparison with the perturbation at 3 loop order calculated in [44].

denotes the beta function, whose perturbative form has been known as series of up to 3-loop coefficients in SF scheme, $b_0 = (11 - 2N_f/3)/(4\pi)^2$, $b_1 = (102 - 38N_f/3)/(4\pi)^4$, $b_2^{\rm SF} = (0.483(7) - 0.275(5)N_f + 0.0361(5)N_f^2 - 0.00175(1)N_f^3)/(4\pi)^3$ [44]. Note that in this scheme it is necessary to determine the maximum size $L_{\rm max}$ through the renormalized coupling constant $u_{\rm max} = \bar{g}^2(L_{\rm max})$ where is above the inverse of hadronic scale ($\mu_{\rm hadron}^{-1} \ll L_{\rm max}$), for instance, associated with, $L_{\rm max} f_\pi$, $L_{\rm max}/r_0$ [45] or determined lattice spacing a from hadron mass [46] which has been separately obtained in large scale simulation and therefore it will be sensitive to determination of lattice spacing. After conversion into $\Lambda_{\overline{MS}}^{(N_f)}$ with perturbative formula, $\alpha_s^{(5)}(M_Z)$ quoted in [46] is given as

$$\alpha_s^{(5)}(M_Z) = 0.1205(9)(^{+0}_{-17}), \tag{4.11}$$

where the first error is estimated by the quadrature of uncertainties of statistical and systematic coming from perturbative matching of different flavor, and otherwise the second one is due to lattice spacing uncertainty.

4.4 From vertex function of gluon and ghost interaction

Basically from non-perturbative computation of the two-, three-gluon or gluon-ghost vertex function α_s can be straightforwardly derived as a function of external gluon momenta. In Ref.[55, 56, 57, 58, 59, 60, 61] we find several attempts of computing vertex function under gauge fixed configurations with some kinds of flavor of dynamical fermion.

5. Summary

Figure 7 shows the summary of recent published result of $\alpha_s^{(5)}(M_Z)$ with several schemes in lattice QCD as explained in the previous sections. It is obviously consistent with each value ob-

tained in different lattice action and scheme. Assuming that these results are independent between each other, the combined value of $\alpha_s^{(5)}(M_Z)$ is evaluated as

$$\alpha_s^{(5)}(M_Z) = 0.1186(^{+7}_{-4}), \tag{5.1}$$

without correlation among them. Its central value is excellently agreement with world average within 1 σ error, and its accuracy is also compatible with that. To pursue the further high precision α_s below 0.5% accuracy, the control of the following uncertainties has already become routine:

- Perturbative matching where $Q = m_c$ and m_b: To take the coupling constant obtained in $N_f = 2+1$ (or $N_f = 3$) dynamical simulation into $Q = M_Z$, it is necessary to incorporate contamination of vacuum polarization of heavy quark into running coupling constant corresponding threshold near its renormalized mass (m_c or m_b). Ordinarily it is useful to match running coupling constant between different flavor as $N_f = 3 \to 4(4 \to 5)$ at threshold $Q = m_c(m_c)$ ($m_b(m_b)$) using decoupling relation in perturbation formula up to N^{3}LO $\mathcal{O}(\alpha_s^3)$ [48, 49, 50], however there remains additional effect coming from higher-order truncation. Basically implementation of lattice Monte-Carlo simulation including dynamical charm quark (moreover bottom quark if we can control large lattice artifact $\mathcal{O}(am_b)$) will be the most rigorous treatment of heavy quark effect without depending on perturbative expansion. α_s including charm see quark effect will be available from the several on-going projects [51, 52, 53].

- Uncertainty of lattice spacing: Since the ingredient of R is dimensionless quantity, calculation of α_s at proper scale on the lattice is nothing to do with the determination of lattice spacing, however in order to compare other results at different scale or perform the perturbative matching at particular scale it becomes significant. The lattice spacing specified in the bare coupling constant is determined from a physical observable like pion decay constant, omega baryon mass or heavy quark potential. Although in this manner the accuracy of such physical observable obtained in the same parameter space is important, there are slightly large systematic division among different physical inputs. This issue will be overcome by large scale simulation close to realistic pion mass ($\sim$ 140 MeV) and physical volume (~ 4 fm^3) in the near future.

Presently the determination of α_s in lattice QCD has been established, and the prospect of future lattice calculation may lead to further precision compatible with electroweak coupling constant.

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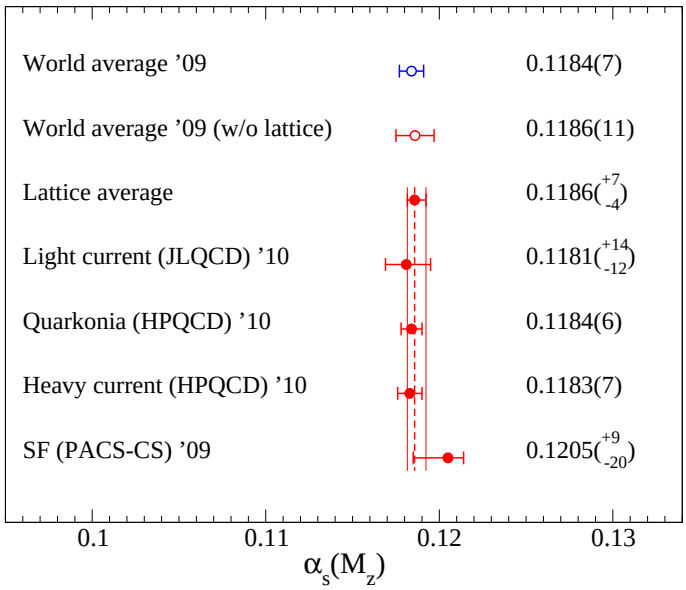

Figure 7: Summary of recent results of $\alpha_s^{(5)}(M_Z)$ calculated in lattice QCD and comparison with world average published in 2009 [2]. The solid (dashed) line shows the central (error) value estimated by using different scheme and fermion action on the lattice.

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