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RG flows in 3D scalar field theory

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The possibility of identifying IR fixed points in the RG flow of four-dimensional gauge theories by numerical simulations has attracted a lot of interest in recent years, due to their potential applications for building models of Dynamical Electroweak Symmetry Breaking. Different lattice methodologies have been applied to this task, each of them having their own systematic errors that need to be understood in order to draw robust conclusions. We aim to study some of these systematic errors in the context of three-dimensional scalar field theories. This approach allows us to perform high-precision numerical simulations, whose results can be compared to existing analytical results.

The XXIX International Symposium on Lattice Field Theory - Lattice 2011
July 10-16, 2011
Squaw Valley, Lake Tahoe, California

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PoS(Lattice 2011)061

1. Introduction

To understand the systematic errors in the current lattice studies of RG flows, it would be very useful to have a gauge theory with a non–trivial IRFP which can also be solved analytically, so that the outcome of the numerical simulations can be compared to the analytic results.

In the absence of such a theory, an interesting alternative theory to play with is $(\phi^2)^3_3$, whose RG structure has been studied analytically for large N [1]. This is a scalar gauge theory in three dimensions, where the field ϕ has N real components, with a lagrangian:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda(\phi^2)^2 + \frac{1}{6}\eta(\phi^2)^3 . \tag{1.1}$$

1.1 Gaussian FP

The simplest fixed point is the gaussian one at $\mu^2 = 0$ for the free theory $\lambda = \eta = 0$. The field ϕ has length dimension $-1/2$, so the coupling for a ϕ^{2n} term has dimension $n-3$. Thus the renormalisation of the couplings as the scale is changed by a factor s is:

$$\begin{aligned} \mu^2 &\to s^2\mu^2 \\ \lambda &\to s\lambda \\ \eta &\to \eta \end{aligned} \tag{1.2}$$

so μ and λ are relevant operators, η is marginal and any higher-dimensional operators are irrelevant.

1.2 Wilson-Fisher FP

The theory has another fixed point at $\mu^{*2} < 0$ and λ^*, while $\eta = 0$. The ε–expansion in $\varepsilon = 4-d$ shows that there is only one relevant critical exponent $\nu = 0.6290(25)$, a marginal one; the first irrelevant operator has a critical exponent $\omega = 0.814(18)$. The ε–expansion also predicts a small anomalous dimension for the rescaling of the fields $\eta = 0.036(5)$ (η here refers to the anomalous dimension of the field; this is the conventional notation for this exponent, and it should not to be confused with the coupling of the ϕ^6 term in the action). These values are obtained from a 5-loop computation in the ε–expansion presented in Ref. [2].

2. Implementation

2.1 Lattice model

The model is implemented on a periodic lattice with $\hat{L}^3$ points, lattice spacing a, and physical volume $L^3 = (\hat{L}a)^3$. A discretised langrangian, where $z_x \equiv \sqrt{a}\phi(x)$, is given by [1]

$$\mathscr{L} = \frac{1}{2}\sum_{\mu=1}^{3}[z_{x+\mu} - z_x]^2 + \frac{1}{2}\mu^2 z_x^2 + \frac{1}{4N}\lambda(z_x^2)^2 + \frac{1}{6N^2}\eta(z_x^2)^3, \tag{2.1}$$

which can be rewritten as

$$\mathscr{L} = -\sum_{\mu=1}^{3} z_{x+\mu}.z_x + (3 + \frac{1}{2}\mu^2)z_x^2 + \frac{1}{4N}\lambda(z_x^2)^2 + \frac{1}{6N^2}\eta(z_x^2)^3. \tag{2.2}$$

where the bare couplings μ, λ, η are in units of the lattice spacing.

A lattice set up for $(\phi^4)_4$ and $(\phi^4)_3$ with $N = 1$ is described in Ref. [3], and extended to $(\phi^6)_3$ for $N = 1$ in Ref. [4].

The lattice variables are updated using a mixture of the Metropolis [5] algorithm and a cluster algorithm. The cluster algorithm was originally introduced for updating spin systems [6], and subsequently adapted to scalar field theories [7, 8]. When the system has large correlation lengths (e.g. near Ising limit), adding cluster updates between Metropolis sweeps significantly reduces autocorrelation times.

2.2 Observables

The Fourier transform of the field is given by [4]

$$\widetilde{\phi}(p) = \sum_x e^{ip.x} z_x. \tag{2.3}$$

A renormalised mass can be constructed from the Fourier transfom above, which is used to set the scale,

$$(am_R)^2 = \frac{(ap)^2 \langle |\widetilde{\phi}(p)|^2 \rangle}{\langle \widetilde{\phi}(0)^2 \rangle - \langle |\widetilde{\phi}(p)|^2 \rangle}, \tag{2.4}$$

where $p = (2\pi/\hat{L}a, 0, 0)$, the smallest non-zero momentum. The four and six point renormalised couplings are given by

$$g_4 = -(\hat{L}am_R)^3 \left[\frac{\langle \widetilde{\phi}(0)^4 \rangle}{\langle \widetilde{\phi}(0)^2 \rangle^2} - 3 \right], \tag{2.5}$$

$$g_6 = 10 g_4^2 - (\hat{L}am_R)^6 \left[\frac{\langle \widetilde{\phi}(0)^6 \rangle - 15 \langle \widetilde{\phi}(0)^4 \rangle \langle \widetilde{\phi}(0)^2 \rangle}{\langle \widetilde{\phi}(0)^2 \rangle^3} + 30 \right]. \tag{2.6}$$

Our implementation has been tested against some simple analytical predictions for the free theory, and against existing results [3, 4, 9] in the interacting case. The agreement is always satisfactory, and provides a satisfactory benchmark of our code.

3. MCRG methods

There have been previous MCRG studies of scalar field theories for $N = 1$, $(\phi^4)_3$ [12, 13], $(\phi^4)_4$ [14], and $(\phi^4)_2$ [15]. These all use a single lattice at the critical point with a large basis of operators, rather than the 2–lattice matching method.

3.1 Method

The original MCRG method [16] goes as follows. Consider a hamiltonian that can be written as a sum of couplings K_i and observables S_i,

$$H = \sum_i K_i S_i, \tag{3.1}$$

and an RG transform R_s of scale s such that

$$H^{(n+1)} = R_s H^{(n)} = \sum_i K_i^{(n+1)} S_i^{(n+1)}. \tag{3.2}$$

The fixed point of the RG transform is defined by the condition

$$H^* = R_s H^* = \sum_i K_i^* S_i^* , \tag{3.3}$$

and near this point the flow in the couplings can be expanded linearly to give

$$K_i^{(n+1)} - K_i^* = \sum_j T_{ij}^* (K_j^{(n)} - K_j^*) , \tag{3.4}$$

where

$$T_{ij}^* = \left. \frac{\partial K_i^{(n+1)}}{\partial K_j^{(n)}} \right|_{H^*} . \tag{3.5}$$

The chain rule gives

$$\frac{\partial \langle S_i^{(n)} \rangle}{\partial K_j^{(n-1)}} = \sum_k \frac{\partial K_k^{(n)}}{\partial K_j^{(n-1)}} \frac{\partial \langle S_i^{(n)} \rangle}{\partial K_k^{(n)}} = \sum_k T_{kj} \frac{\partial \langle S_i^{(n)} \rangle}{\partial K_k^{(n)}} . \tag{3.6}$$

From which T_{kj} can be constructed using the identities

$$\frac{\partial \langle S_i^{(n)} \rangle}{\partial K_j^{(n-1)}} = \langle S_i^{(n)} S_j^{(n-1)} \rangle - \langle S_i^{(n)} \rangle \langle S_j^{(n-1)} \rangle \equiv A_{ij}^{(n)} , \tag{3.7}$$

$$\frac{\partial \langle S_i^{(n)} \rangle}{\partial K_j^{(n)}} = \langle S_i^{(n)} S_j^{(n)} \rangle - \langle S_i^{(n)} \rangle \langle S_j^{(n)} \rangle \equiv B_{ij}^{(n)} , \tag{3.8}$$

and finally:

$$\Rightarrow T = B^{-1} A . \tag{3.9}$$

The eigenvalues of T_{ij} give the critical exponents of the system [17], e.g. $\nu = \ln s / \ln \lambda_h$, where λ_h is the largest eigenvalue. Note that, from a single simulation close to the critical point, correlation functions of blocked observables are measured to construct the matrix T_{ij}, and extract the critical exponents.

3.2 Blocking Transforms

The simplest RG blocking transform is a gaussian smeared average given by [18]

$$e^{-S[\vartheta]} = \int d[\phi] e^{-S[\phi]} e^{-\frac{a_W}{2} \sum_{n_B} \left[\vartheta(n_B) - b \sum_{n \in n_B} \phi(n) \right]^2} , \tag{3.10}$$

where a_W is a free parameter, which can be varied to optimise the RG transform. In the limit $a_W \to \infty$ the gaussian becomes a delta–function, and the RG transform becomes a simple average over the points in n_B,

$$\vartheta = b \sum_{n \in n_B} \phi(n) \tag{3.11}$$

The parameter b is not a free parameter here - it needs to be tuned according to the anomalous dimension of the fields at the fixed point,

$$b = s^{-(d+2-\eta)/2} . \tag{3.12}$$

The value of η is tuned by requiring that T^*_{ij} has an exactly marginal eigenvalue $\lambda = 1$.

Blocked fields ϑ that satisfy Eq. 3.10 are given by [15]

$$\vartheta(n_B) = b \sum_{n \in n_B} \phi(n) + \frac{\zeta}{\sqrt{N a_W}}, \tag{3.13}$$

where ζ is a random gaussian variable with zero mean and variance 1.

A basis of local observables $\{S_i\}$ is required for the MCRG method. In Ref. [15] four types of observables were used: $\phi(n)\phi(n+u)$, $\phi(n)^{2k}$, $\phi(n)\phi(n+u)\phi(n)^{2k}$, and $\phi(n+w)\phi(n+u)\phi(n)^{2k}$.

Observables can be added to the analysis until adding more doesn't change the results. (Up to 40 were used in Ref. [15]).

4. MCRG at the Gaussian FP

The most basic test is at the gaussian FP ($N = 1$, $\eta = \lambda = \mu^2 = 0$). Ref. [13] uses $\mu^2 = 0.001$ on 32^3 lattices, with a fourier-space RG blocking transform. We cannot simulate directly at $\mu^2 = 0$ as the correlation length diverges; we performed a first test by using $\mu^2 = 0.01$ on 16^3 lattices. The bare observables $O_k = \langle \phi(n)^{2k} \rangle$ agree reasonably well with Ref. [13]. A more comprehensive study at smaller masses requires larger lattices to ensure $m_R L \gg 1$. Using two observables, and $a_W = 25$, we compute the two largest eigenvalues at the Gaussian FP. The results are reported in Tab. 1 for several values of the bare mass μ^2. The results for the eigenvalues converge nicely towards the analytic prediction as the mass of the field is reduced. Note that the results with 3 steps are more affected by finite-volume effects.

μ^2	$m_R L$	1 step	2 steps	3 steps
0.050	3.52	4.128	3.760	3.368
		1.352	0.736	0.888
0.020	2.24	4.120	3.904	3.488
		1.280	1.600	1.816
0.010	1.55	3.960	3.960	3.864
		1.872	1.960	1.784
0.009	1.48	3.984	3.928	3.976
		1.768	1.888	1.536

μ^2	$m_R L$	1 step	2 steps	3 steps
0.001	0.53	4.000	4.008	3.976
		1.984	2.048	2.112
0.0008	0.44	4.002	4.013	3.997
		1.991	1.979	2.008
0.0005	0.35	4.011	4.031	3.922
		1.976	1.941	2.079
analytic	∞	4		
		2		

Table 1: Results of the stability matrix computations for two eigenvalues at the Gaussian FP.

5. MCRG at the Wilson-Fisher FP

Some critical couplings as calculated from a strong coupling expansion are given in Ref. [12], but in order to do MCRG for a range of fixed points and theories we need to be able to determine the critical couplings on the lattice.

For fixed $\lambda = 10$, μ^2 is varied to look for a critical point. It seems the two quantities best suited for identifying this point are the correlation length $\xi = 1/(am_R)$, where am_R is defined in Eq. 2.4, and the Binder cumulant [20] (also used in e.g. [10])

$$U = \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}, \tag{5.1}$$

where $m = \sum_x z_x$. Figure 1 shows how various observables behave as a function of μ^2 on a 32^3 lattice.

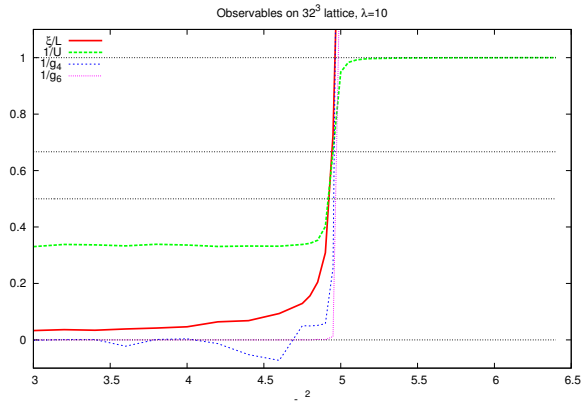

Figure 1: Some observables as a function of μ^2 on a 32^3 lattice at $\lambda = 10$. Strong coupling expansion gives the critical mass as $-\mu^2 = 4.9386$

The Wilson-Fisher FP is found at non–zero λ with $\mu^2 < 0$ tuned to a critical value. This has been investigated in the past using real–space [12] and Fourier–space [13] MCRG methods. Critical pairs of λ, μ^2 from a strong coupling expansion are tabulated in Ref. [12], here we use $\mu^2 = -6.0, \lambda = 12.672$, generating 500 uncorrelated configurations.

From requiring an exactly marginal eigenvalue of 1, in principle one can find $\eta = 0.03$ for the rescaling of the fields as in Eq. 3.12. The critical exponent for this model is known to be $\nu = 0.6290(25)$, which corresponds to a largest eigenvalue of 3.01. The largest eigenvalue that we find after 1/2 blocking steps with $a_w = 25$, as a function of the number of observables and lattice size is shown in the left-hand size of Tab. 2. Fixing the number of observables to six and varying the blocking steps gives the results on the right-hand side of the same table.

L	1 obs	2 obs	3 obs	4 obs	5 obs	6 obs
8	2.60	2.66	3.00	3.08	3.08	3.12
16	2.72	2.72	3.03	3.03	3.03	3.04
32	2.83	2.83	3.04	3.04	3.04	3.04
64	2.80	2.80	3.03	3.02	3.03	3.03
128	2.76	2.75	3.04	3.03	3.03	3.02

L	0/1	1/2	2/3	3/4	4/5	5/6	6/7
8	2.787	3.124					
16	2.701	3.036	3.120				
32	2.740	3.037	3.084	3.164			
64	2.735	3.029	3.109	2.955	2.998		
128	2.904	3.024	3.041	3.170	2.925	2.928	

Table 2: Largest eigenvalue at the WFP, using the stability matrix method.

6. Outlook

In order to understand better the features of MCRG methods, we have started to apply these methods to investigate simple 3D models. In the large-N limit these models can be solved analytically, and the results of the numerical simulations can be compared to the analytical predictions. We expect these models to provide a robust testing ground for the methods that are currently used for the study of IRFP in 4D gauge theories, see e.g. Ref. [21] and references therein.

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