

Large-N reduction with two adjoint Dirac fermions

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We study the single site SU(N) lattice gauge theory with $N_f=2$ adjoint Wilson fermions for values of N up to 53. We determine the phase diagram of the theory as a function of the hopping parameter κ and the inverse 't Hooft coupling b, searching for the region in which the \mathbb{Z}_N^4 center symmetry is unbroken. In this region the theory is equivalent to the infinite volume theory when $N\to\infty$. We find a region of values of κ on both sides of κ_c for which the symmetry is unbroken, including both light physical quarks and masses $\sim \mathcal{O}(1/a)$. This is surrounded by a region with a complicated sequence of partially broken phases. We calculate Wilson loop expectation values and find that using $N \leq 53$ it is possible to extract the heavy-quark potential at small distances (1-3 links) but not at longer distances. For this, larger values of N, or lattices with more sites, are needed.

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1. Introduction

A key property of large-*N* lattice gauge theories is volume reduction, first introduced by Eguchi and Kawai (EK) [1]. They showed that, under some assumptions, Yang-Mills theory on an infinite lattice satisfies the same Dyson-Schwinger equations as the theory reduced to a single space-time point. The crucial assumption was that the center symmetry of the action cannot be spontaneously broken. This assumption was soon shown to be invalid [2]. Over the years, many alternatives have been proposed to restore EK equivalence, but until recently, none has been fully satisfactory.¹

The recent revival of this topic was triggered by the work of Refs. [4, 5] which showed that volume reduction was an example of broad class of large-N orbifold equivalences, holding both in the continuum and on the lattice. If one can construct orbifold projections between the parent and daughter theories (in our case the large-volume and single-site theories, respectively) then the two become equivalent for $N \to \infty$, assuming certain conditions hold. The most non-trivial of these conditions is, as above, the preservation of the center symmetry. The authors of Refs. [5, 6] proposed two possible fixes that might stabilize the ground state to preserve the center symmetry. They calculated the perturbative potential for the eigenvalues of the Polyakov loop and showed that it becomes repulsive if one adds $N_f > 1/2$ massless Dirac fermions in the adjoint representation (with periodic boundary conditions).² In perturbation theory this leads to a center-symmetric ground state, with eigenvalues spread uniformly around the unit circle. Apart from being interesting in their own right, theories with N_f adjoint fermions are connected by a chain of large-N equivalences to QCD with $2N_f$ fundamental flavours. This opens up the possibility of using single-site simulations to give non-perturbative insight into the strong interactions (up to 1/N corrections).

In the following we focus on a single-site SU(N) lattice model with adjoint Wilson fermions—the Adjoint Eguchi-Kawai or AEK model. The partition function is

$$\mathscr{Z} = \int D[U, \psi, \bar{\psi}] e^{(S_{\text{gauge}} + \sum_{j=1}^{N_f} \bar{\psi}_j D_W \psi_j)}, \tag{1.1}$$

where $S_{\rm gauge}$ is the single-site equivalent of the Wilson action:

$$S_{\text{gauge}} = 2Nb \sum_{\mu < \nu} \text{ReTr} U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}$$
(1.2)

 $(b = \frac{1}{g^2N})$ being the inverse 't Hooft coupling' and D_W is the single-site Wilson Dirac operator:

$$D_W = 1 - \kappa \sum_{\mu=1}^{4} \left[\left(1 - \gamma_{\mu} \right) U_{\mu}^{\text{adj}} + \left(1 + \gamma_{\mu} \right) U_{\mu}^{\dagger \text{adj}} \right]. \tag{1.3}$$

Using massless fermions is crucial in the perturbative analysis of Ref. [6] but a lattice simulation by two of us of the $N_f = 1$ AEK model found that, for $N \le 15$, center symmetry is stabilized not only by light fermions but also by fermions with mass 1/a [8]. This is much larger than the expected upper limit of 1/(aN) [9, 10]. If this continues to hold for $N \to \infty$, then the

¹For an overview see, e.g., Ref. [3]. One highly successful alternative approach due to Narayanan and Neuberger is the reduction to a small volume of size $\sim (1 \text{fm})^4$ rather than to a single site.

²The other possibility is the trace-deformed EK model, which, however, becomes very complex when one wants to compactify more than one space-time dimension (see e.g. Ref. [7]).

 $N_f = 1$ AEK model provides a realization of the original Eguchi-Kawai idea. This result was confirmed, and given a semi-analytic understanding (using arguments going beyond perturbation theory) in Ref. [11]. Simulations using massless overlap fermions have also found evidence for reduction [12]. There has also been work on the $N_f = 2$ theory also suggesting that reduction holds [13].

2. Phase diagram of $N_f = 2$ Adjoint Eguchi-Kawai model

From now on we focus on the AEK model with $N_f = 2$. In addition to the above-mentioned connection to QCD with 4 fundamental flavors, the large volume theory to which it might be equivalent is of phenomenological interest as a theory which might exhibit conformal, or near-conformal, behavior in the IR. The N = 2 version of this theory has been extensively studied (see Ref. [14] and references therein).

We use the Hybrid Monte Carlo algorithm adapted to work on a single-site lattice (for details see Ref. [15]). The CPU time scales approximately as N^4 for heavy fermions and $N^{4.5}$ for light fermions. ³ This is a significant improvement over the $\approx N^8$ growth for the Metropolis algorithm used in Ref. [8], and allows us both to reach higher values of N and reduce statistical errors.

To analyze the phase diagram we have performed scans in the $\kappa - b$ plane using values of N up to 30. In addition, we have done several high-statistics runs at selected points in parameter space using $N \le 53$. We have performed simulations with values of b up to b = 200 but we have mostly focused on the region $b \in [0, 1]$. In particular, the two values we have looked at with most care are:

- b = 0.35, which is close to the value used in typical lattice simulations (for N = 3 we have $\beta = 6/g^2 = 6.3$),
- b = 1.0, which is a fairly weak coupling ($\beta = 18$ for N = 3) at which one should be able to compare the results to perturbation theory.

Apart from analyzing the plaquette, we have used two types of observables to study center symmetry breaking:

1. General "open loops":

$$K_n \equiv \frac{1}{N} \text{Tr} \, U_1^{n_1} \, U_2^{n_2} \, U_3^{n_3} \, U_4^{n_4}, \quad \text{with } n_\mu = 0, \pm 1, \pm 2, \dots$$
 (2.1)

where $U^{-n} \equiv U^{\dagger n}$. These are the general order parameters for the breaking of the center symmetry. The simplest example of such loops are the Polyakov loops $P_{\mu} = \text{Tr} U_{\mu}$.

2. Eigenvalues of link matrices: the single-site gauge transformation $U_{\mu} \to \Omega U_{\mu} \Omega^{\dagger}$ obviously leaves the set of eigenvalues unchanged. In the center symmetric-phase one expects the distribution of phases of the eigenvalues to be invariant under translations by $2\pi n/N$.

³The contributions to the scaling are N^3 for fundamental link matrix multiplication (adjoint link matrices are never explicitly constructed in the updating procedure), $\sim N$ for the growth in the number of molecular dynamics steps for constant acceptance, and, for light fermions, $\sim N^{0.5}$ for the increase in the number of CG iterations.

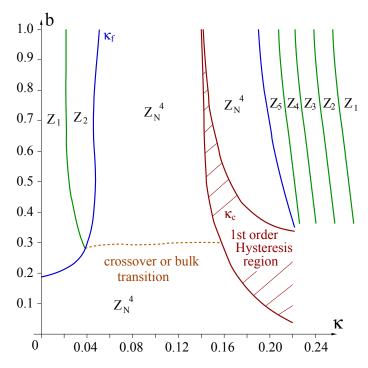


Figure 1: Sketch of the phase diagram for the $N_f = 2$ AEK model. The positions of phase boundaries are approximate, and depend somewhat on N. The funnel gets narrower very slowly when moving to larger values of b. We have also found evidence for a \mathbb{Z}_3 phase on the left-hand side of the funnel for $b \gtrsim 5$.

At weak coupling one also expects that the links are close to being simultaneously diagonalizable. We thus apply the gauge transformation that diagonalizes one link and look at the elements of the other links. Indeed, we find that for $b \gtrsim 1$ the diagonal elements of the links dominate in this basis while as we move to stronger couplings the off-diagonal elements grow the links become increasingly non-commutative.

The resulting phase diagram is pictured in Figure 1. Note that this diagram is qualitatively similar to the one from Ref. [8] for the $N_f = 1$ AEK model. The $\kappa = 0$ line corresponds to the original Eguchi-Kawai model with broken center symmetry at weak coupling. As the fermions become lighter a "funnel" of center-symmetric phase appears on both sides of the critical region. This is the region in which volume reduction holds. Around it there are regions in which the \mathbb{Z}_N^4 center symmetry is broken—moving away from the funnel the breaking is to increasingly small subgroups until we reach a completely broken phase. We label the phases by the approximate remnant of the center symmetry. In particular, a " \mathbb{Z}_i " phase has i clumps in the histograms of the phases of link eigenvalues. In the broken phases there are strong correlations between links in all directions—this is why we label these phases with \mathbb{Z}_i instead of \mathbb{Z}_i^4 . For a more detailed analysis of the phase structure, see Ref. [15].

An important issue is how the width of the funnel depends on N and b. As shown in Fig. 1, we denote the lower limit of the funnel by κ_f , and the corresponding quark mass by $m_f = 1/(2\kappa_f) - 1/(2\kappa_c)$). We first discuss the b dependence. The analysis of Ref. [11] predicts that at weak coupling the center-symmetric and completely broken phases are separated by many phases with

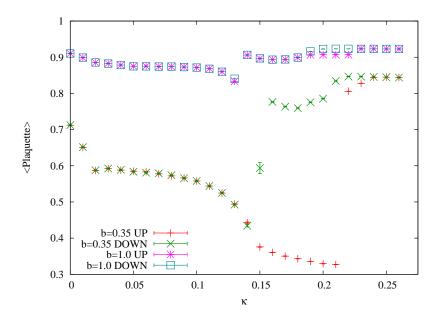


Figure 2: Scans of the average plaquette for N = 23 at b = 0.35 and 1.

partial breakings (which is in agreement with our results) and that the funnel closes as $m_f \sim b^{-1/4}$ for $b \to \infty$. We have performed simulations for N = 10 up to b = 200 and our results are consistent with this form. It is important to note, however, that, although the funnel closes, it does so in such a way that quarks of arbitrarily high physical mass, $m_f = a m_{\rm phys}$, remain inside the funnel for $b \to \infty$.

Using runs up to N=30, we find that κ_f increases with N, so that the width of the funnel shrinks. It is thus very important to go to even higher N in order to perform a reliable extrapolation of κ_f to $N\to\infty$, and thus to establish whether the funnel (at fixed b) has finite width at $N=\infty$ or not. At this stage, the strongest evidence we have for a finite width is obtained by studying the large-N limit of the plaquette within the putative funnel region. We have done this extrapolation at several values of κ for b=0.35 and 1.0, using N up to 53. We find that the extrapolated plaquette for κ away from κ_c is almost independent of κ , and furthermore lies close to the value obtained in the large-N pure gauge theory [15]. This is what we would expect if reduction holds, because the heavy quarks lead only to a small renormalization of the plaquette.

Another interesting aspect of the phase diagram is the order of the critical line. Large volume SU(2) simulations with adjoint fermions find a second order transition line emanating from $b = \infty$, $\kappa = 1/8$ which becomes first order below $b \approx 1/4$. This structure is related to the claimed presence of an IR fixed point. Although N = 2 is far from the $N = \infty$ limit, it is expected for the adjoint theory that its properties should depend only weakly on N.

In fact, we find a very different phase diagram, in which there is a clear first-order transition for b up to at least 1. This is illustrated by the results shown in Fig. 2. The discontinuity in the plaquette drops as N increases, but, for $b \le 1$, we find that it remains non-zero in the $N \to \infty$ limit. For larger values of b our results are less definitive. Although we still see a discontinuity, its magnitude decreases rapidly with increasing b, and it is difficult to determine whether it remains as $N \to \infty$. Nevertheless, it is clear that the first-order transition extends to much larger b than in the large-volume N=2 simulations. The simplest interpretation of our results is that the theory is

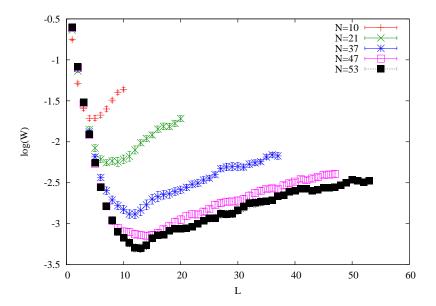


Figure 3: Log-linear plot of $1 \times L$ Wilson loop versus L for $L \le N$.

confining in the IR, with lattice artifacts leading to a first-order transition at κ_c which extends to $b \to \infty$.

3. Towards physical quantities

The ultimate goal of using volume-reduced simulations is to calculate physical quantities that one can apply to large volume theories. One such quantity is the heavy quark potential, which can be obtained using rectangular Wilson loops. In the single-site theory these become

$$W(L_1, L_2) = \frac{1}{12} \sum_{\mu \neq \nu} \langle \frac{1}{N} \text{Tr} U_{\mu}^{L_1} U_{\nu}^{L_2} U_{\mu}^{\dagger L_1} U_{\nu}^{\dagger L_2} \rangle.$$
 (3.1)

For large L_2 we expect $W(L_1, L_2) \sim e^{-V(L_1)L_2}$ but at finite N we must keep $L_j < N$ to avoid finite N effects.

Figure 3 shows a log-linear plot of $1 \times L$ Wilson loops for several values of N. We see convergence to a common envelope for small L followed by a slow rise. The latter is a finite-N effect—as $N \to \infty$ the exponentially falling envelope will extend to $L = \infty$. We stress that this finite-N effect is statistically significant. From the minima of the curves, one can estimate how L_{max} scales with N. The dependence is approximately logarithmic, so that calculation of the potential poses a significant numerical challenge. The problem becomes worse as we move to larger separations and we are only able to see convergence to the linear envelope (and thus extract the potential) for distances of 1-3 links. For larger distances one needs to either use larger values of N or move to larger lattices (e.g. 2^4).

4. Summary & outlook

The single-site lattice gauge theory with 2 flavours of adjoint Wilson-Dirac fermions exhibits a rich phase structure, one that is qualitatively explained by the semi-analytic model of Ref. [11].

Our most important result is that, for N up to 53, we find a broad funnel of center-symmetric phase where large-N reduction holds. It extends from strong to very weak coupling and contains fermions with masses up to $\sim 1/a$. The $N \to \infty$ extrapolations of the plaquette suggest that the funnel does not close in the large-N limit but we cannot completely rule out this possibility at present. A detailed analysis using even higher N is desirable.

We have also analyzed the order of the critical line κ_c , finding a first-order phase transition for all values b. This is different from the large volume SU(2) simulations and favors a confining scenario over the conformal one for the large-N theory.

To use the large-N volume reduction as a practical tool one needs to be able calculate long-distance observables. We have presented a calculation of the heavy-quark potential and showed that 1/N effects limit the precision that one can achieve.

This analysis can be extended by calculating more physical quantities such as particle masses, as well as extending the range over which the heavy-quark potential can be extracted. For the latter it is desirable to reduce the "1/N noise" e.g. by using slightly larger lattices or using twisted model as in Ref. [11]. In addition, making a careful analysis of the vicinity of the critical line, perhaps using improved actions, would be of considerable interest.

Acknowledgments

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