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A scrutiny of hard pion chiral perturbation theory

Gilberto Colangelo*, **Massimiliano Procura, Lorena Rothen, Ramon Stucki**
Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstr. 5, CH-3012 Bern, Switzerland
E-mail: gilberto@itp.unibe.ch

Jaume Tarrús Castellà
Departament d'Estructura i Constituents de la Matèria and Institut de Ciències del Cosmos Universitat de Barcelona, Diagonal, 647, E-08028 Barcelona, Catalonia, Spain

In this talk I have discussed an investigation of factorization of chiral logs as predicted by hard pion chiral perturbation theory from the point of view of standard chiral perturbation theory and dispersion relations. Using as example the pion form factors we were able to explain factorization at the two loop level and even to all orders if one considers only elastic contributions. Inelastic contributions, on the other hand, do not respect factorization, as briefly discussed here.

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*Speaker.

1. Introduction

Some time ago Flynn and Sachrajda [1] made the interesting observation that, after reinterpreting the effective Lagrangian describing the decay of a heavy meson (the K in that case) into a pion and a lepton-neutrino pair, one can unambiguously predict the coefficient of the leading chiral log, in all possible kinematical configurations. This is a nontrivial statement, because it implies that some chiral properties emerge even outside the region of applicability of chiral perturbation theory (χPT). Subsequently, in a series of papers by Bijnens and collaborators [2, 3, 4] it has been claimed that the calculation of such a chiral log is possible in more general processes in which the pion is hard. This approach has been referred to as hard pion chiral perturbation theory (H$\pi\chi$PT).

A particularly clear example of what such a theory is able to predict has been provided in [4] where it has been shown that the scalar and vector form factors of the pion, which have been calculated to two loops in χPT [5], factorize in the limit $M_\pi^2 \ll s$:

$$F_{V,S}(s) = \overline{F}_{V,S}(s)\,(1+\alpha_{V,S}L) + O(M^2) \ , \tag{1.1}$$

where L stands for the chiral log, defined as $L = M^2/(4\pi F)^2 \ln M^2/\mu^2$. M^2 is proportional to the average up and down quark masses $\hat{m}$, $M^2 = 2B\hat{m}$ and F is the decay constant in the chiral limit:

$$M_\pi^2 = M^2 + \mathscr{O}(M^4) \ , \qquad F_\pi = F + \mathscr{O}(M^2) \ . \tag{1.2}$$

$\overline{F}_{V,S}(s)$ are the form factors in the chiral limit.[1] Bijnens and Jemos provide arguments in support of this factorization property [4]. A detailed analytical understanding, however, is still lacking. In this talk we gave a preliminary account of our analysis which investigates in detail how this factorization property emerges. A complete discussion will appear soon [6].

2. Dispersive representation of the pion form factors and leading chiral logs

Consider the vector and scalar pion form factors, respectively $F_V(s)$ and $F_S(s)$ and denote them by the common symbol $F(s)$ (unless it is necessary to distinguish them). We normalize the scalar form factor such that $F_S(0) = 1$ — for the vector form factor this condition need not be imposed as it follows from current conservation. Both these form factors are analytic functions in the cut plane $[4M_\pi^2, \infty)$ and satisfy a dispersion relation of the form

$$F(s) = 1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\,F(s')}{s'(s'-s)} =: 1 + d(s) \ . \tag{2.1}$$

Moreover in the elastic region unitarity relates the imaginary part of the form factor to the form factor itself and the $\pi\pi$ partial wave with the same quantum numbers:

$$\mathrm{Im}\,F(s) = \sigma(s)F(s)t^*(s) \qquad \sigma(s) = \sqrt{1-\frac{4M_\pi^2}{s}} \ . \tag{2.2}$$

We observe that since $t(s)$ starts at $\mathscr{O}(p^2)$, knowing the form factor up to a given chiral order means to know its imaginary part — and if one is able to perform the dispersive integral also its

[1] We denote quantities in the chiral limit by a bar, $\overline{X} = \lim_{M_\pi \to 0} X$.

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real part — at one order higher. We will now address the question, how chiral logs can arise from the dispersive integral.

The first possible mechanism is that these are generated by the integration region around $s = 4M_\pi^2$ — to investigate this we must analyze the behaviour of the integrand around threshold. The form factor takes a constant value, which differs from the value at $s = 0$ by $\mathscr{O}(M^2)$:

$$F(4M_\pi^2) = 1 + \mathscr{O}(M^2) \ , \tag{2.3}$$

whereas the $\pi\pi$ amplitude $t(s)$ either vanishes or goes to a constant, depending on the angular momentum ℓ

$$t_\ell^I(s) \simeq q^{2\ell}\left(a_\ell^I + b_\ell^I q^2 + \mathscr{O}(q^4)\right) \ , \tag{2.4}$$

where $q^2 = s/4 - M_\pi^2$. At $\mathscr{O}(p^2)$ in the chiral counting both a_0^I and b_0^I are nonzero for both isospin channels[2]. In the P wave a_1^1 starts at $\mathscr{O}(p^2)$ whereas b_1^1 vanishes at this order. In summary:

$$a_0^I = \mathscr{O}(M^2/F^2) \ , \quad a_1^1 \sim b_0^I = \mathscr{O}(1/F^2) \ . \tag{2.5}$$

Let us consider the more general case $t(s) \simeq a + bq^2 + \mathscr{O}(q^4)$ and split the dispersive integral into two pieces

$$d(s) = a\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\sigma(s')F(s')}{s'(s'-s)} + \frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\sigma(s')F(s')\,[t^*(s')-a]}{s'(s'-s)} =: d_1(s) + d_2(s) \ . \tag{2.6}$$

We write the first as

$$d_1(s) = 16\pi F(4M_\pi^2) a \bar{J}(s) + a\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\sigma(s')\left[F(s') - F(4M_\pi^2)\right]}{s'(s'-s)} \ . \tag{2.7}$$

In the limit $y := M_\pi^2/s \ll 1$, $\bar{J}$ admits the expansion

$$\bar{J}(s) = \frac{1}{8\pi^2}\left[1 + \frac{\ln(-y)}{2} + y\,(1 - \ln(-y)) + \mathscr{O}(y^2)\right] \tag{2.8}$$

whereas in the second integral, the presence of chiral logs is determined by the behaviour around $s = 4M_\pi^2$. The expansion for $y \ll 1$ of the second term in (2.7), after cutting off the integral gives:

$$I_1(s,\Lambda^2) := \frac{s}{16\pi^2}\int_{4M_\pi^2}^{\Lambda^2} ds' \frac{\sigma(s')\left(s' - 4M_\pi^2\right)}{s'(s'-s)} = \bar{I}_1(s,\Lambda^2) - \frac{6M_\pi^2}{16\pi^2}\ln(-y) + \mathscr{O}(M_\pi^2) \ . \tag{2.9}$$

However, since $a = \mathscr{O}(M^2)$, the chiral log arising from this integral is suppressed by one power of M^2 and is beyond the accuracy of the present analysis. Moreover the integration region from Λ^2 to infinity cannot generate logs. We conclude that

$$d_1(s) = \frac{a}{\pi}\ln(-y) + \mathscr{O}(M^2) \ , \tag{2.10}$$

where we have also used (2.3). The presence of chiral logs in $d_2(s)$ is also determined by the integral $I_1(s,\Lambda^2)$ we just introduced. Indeed by expanding both $F(s)$ and $(t^*(s) - a)$ around threshold we obtain

$$d_2(s) = \overline{d}_2(s) - \frac{3b}{2\pi}M^2\ln(-y) + \mathscr{O}(M^2) \ . \tag{2.11}$$

[2]Higher coefficients do not matter for the calculation of the chiral log, see [6].

Putting $d_1(s)$ and $d_2(s)$ together we obtain

$$d(s) = \bar{d}(s) + 16\pi F^2 \left(\frac{a}{M^2} - \frac{3b}{2} \right) L + \mathscr{O}(M^2) \ . \tag{2.12}$$

For the scalar form factor the relevant parameters are $a = a_0^0$ and $b = b_0^0$ which are given by

$$a_0^0 = \frac{7M^2}{32\pi F^2} + \mathscr{O}(M^4) , \qquad b_0^0 = \frac{1}{4\pi F^2} + \mathscr{O}(M^2) \ , \tag{2.13}$$

leading to

$$\alpha_S = 16\pi F^2 \left(\frac{a_0^0}{M^2} - \frac{3b_0^0}{2} \right) = -\frac{5}{2} \ , \tag{2.14}$$

which reproduces the known result [5, 4]. For the vector form factor we must instead use the parameters: $a = 0$, and $b = a_1^1 = 1/(24\pi F^2) + \mathscr{O}(M^2)$, which leads to

$$\alpha_V = 16\pi F^2 \left(-\frac{3a_1^1}{2} \right) = -1 \ , \tag{2.15}$$

also in agreement with the explicit calculation [5, 4].

2.1 The form factor at $\mathscr{O}(p^2)$ in the chiral limit

As we have seen above the part of the dispersive integral denoted by $d_1(s)$ vanishes in the chiral limit, whereas $d_2(s)$ does not. It is useful for the subsequent discussion to determine the leading order of $d_2(s)$ in the chiral expansion in the chiral limit [7]. To do this we have to find a function which is analytic in the cut plane $[0,\infty)$ and which has as imaginary part along the cut

$$\lim_{M\to 0} \sigma(s) F^{(0)}(s) \left[t^{(2)}(s) - a \right] = \frac{b}{4} s \ . \tag{2.16}$$

Such a function is easily found:

$$\bar{d}^{(2)}(s) = s\frac{b}{4\pi} \ln \frac{\Lambda_2^2}{-s} \ , \tag{2.17}$$

with Λ_2 an unknown scale. The explicit expressions in the case of the scalar and vector form factors read

$$\begin{aligned} \bar{d}_S^{(2)}(s) &= \frac{s}{16\pi^2 F^2} \left[1 + \ln \frac{\mu^2}{-s} + 16\pi^2 \ell_4^r(\mu) \right] \\ \bar{d}_V^{(2)}(s) &= \frac{s}{16\pi^2 F^2} \left[\frac{5}{18} + \frac{1}{6} \ln \frac{\mu^2}{-s} - 16\pi^2 \ell_6^r(\mu) \right] \ . \end{aligned} \tag{2.18}$$

The coefficients of the logs are indeed correctly reproduced by substituting $b = b_0^0 = 1/(4\pi F^2)$ for the scalar form factor and $b = a_1^1 = 1/(24\pi F^2)$ for the vector one. The scale Λ_2 for the scalar (vector) form factor is related to the low energy constant $\ell_4^r(\mu)$ ($\ell_6^r(\mu)$).

2.2 Leading chiral logs beyond $\mathscr{O}(p^2)$

We must now discuss whether and how leading chiral logs can be generated at higher chiral orders — more specifically, we are interested in terms proportional to $s^{n-1}L$ at order p^{2n}. The discussion above made it clear that the behaviour of the integrand around the lower limit of integration may only generate a chiral log at $\mathscr{O}(p^2)$: a term proportional to L but constant in s. There is a second mechanism, however, by which chiral logs may arise from the dispersive integrals at higher orders, namely if the integrand itself contains a chiral log.

Let us consider the dispersion relation at $\mathscr{O}(p^4)$ (*i.e.* contributions to the form factor at the two-loop level). At this order the integrand contains two contributions

$$\mathrm{Im}\, F^{(4)}(s) = \sigma(s)\left[t^{(4)*}(s) + F^{(2)}(s)t^{(2)}(s)\right] \tag{2.19}$$

each of which may contain chiral logs. We consider first the latter term: $t^{(2)}(s)$ is the tree-level contribution to the $\pi\pi$ scattering amplitude and contains no chiral logs, whereas $F^{(2)}(s)$ does. Expanding this term in M^2/s we can write it as

$$F^{(2)}(s)t^{(2)}(s) = \left(\overline{F}^{(2)}(s) + \alpha L\right)\bar{t}^{(2)}(s) + \mathscr{O}(M^2) \ . \tag{2.20}$$

We can similarly expand $t^{(4)}(s)$ and write it as

$$t^{(4)}(s) = \bar{t}^{(4)}(s) + (\beta_0 M^2 + \beta_1 s)L + \mathscr{O}(M^2) \ . \tag{2.21}$$

The term proportional to β_0 generates in $F^{(4)}(s)$ a chiral log suppressed by one order of M^2 which is beyond the accuracy we aim at — β_1 is the term which would be of interest to us, but as we are going to show below this vanishes. We therefore conclude that the form factor at $\mathscr{O}(p^4)$ in the chiral limit, $\overline{d}^{(4)}(s)$ is given by the solution of the dispersion relation with discontinuity

$$\mathrm{Im}\, \overline{F}^{(4)}(s) = \left[\bar{t}^{(4)*}(s) + \overline{F}^{(2)}(s)\bar{t}^{(2)}(s)\right] \ . \tag{2.22}$$

The only term containing the chiral log has as coefficient exactly the absorptive part of the form factor at one chiral order lower, $\bar{t}^{(2)}(s) = b/4s$. As we have discussed above the solution of the corresponding dispersion relation reads

$$s\frac{b}{4\pi}\ln\frac{\Lambda_X^2}{-s} \tag{2.23}$$

with Λ_X an unknown energy scale. We argue, however, that this has to coincide with Λ_2 introduced in Eq. (2.17), for details, see [6]. We conclude that at this order we can write the form factor as

$$F(s) = \left(1 + \overline{d}^{(2)}(s)\right)(1 + \alpha L) + \overline{d}^{(4)}(s) + \mathscr{O}(M^2) + \mathscr{O}(p^6) \ , \tag{2.24}$$

i.e. in factorized form, as predicted by H$\pi\chi$PT.

The same reasoning can be repeated in exactly the same way order by order. At every new step the dangerous terms for factorization are the contributions to $\mathrm{Im}\, F^{(2n)}$ arising from the $\pi\pi$ scattering amplitude at the same order. A chiral log of the form $s^{n-1}L$ in $t^{(2n)}$ would destroy factorization. If these are absent, however, factorization of the leading single chiral log in the limit $M^2 \ll s \ll \Lambda^2$ is respected.

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3. Chiral logs in the $\pi\pi$ scattering amplitude

In order to complete our argument we now turn our attention to the $\pi\pi$ partial waves and their dispersive representation as proposed by Roy [8]:

$$t_\ell^I(s) = k_\ell^I(s) + \sum_{I'=0}^{2}\sum_{\ell'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K_{\ell\ell'}^{II'}(s,s')\mathrm{Im}\, t_{\ell'}^{I'}(s') \ , \tag{3.1}$$

where I and ℓ denote isospin and angular momentum respectively and $k_\ell^I(s)$ is the partial wave projection of the subtraction term given by:

$$k_0^0(s) = a_0^0 + \frac{s-4M_\pi^2}{12M_\pi^2}(2a_0^0 - 5a_0^2) \ , \tag{3.2}$$

and similarly for the other S and P waves. To analyse the possible sources for chiral logs one splits Eq.(3.1) into the S- and P-wave contributions, the higher waves and the integral from a cut-off to infinity:

$$t_\ell^I(s) = k_\ell^I(s) + t_\ell^I(s)_{SP} + d_\ell^I(s) \ , \tag{3.3}$$

where

$$t_\ell^I(s)_{SP} = \sum_{I'=0}^{2}\sum_{\ell'=0}^{2}\int_{4M_\pi^2}^{\Lambda^2} ds' K_{\ell\ell'}^{II'}(s,s')\mathrm{Im}\, t_{\ell'}^{I'}(s') \ , \tag{3.4}$$

and the so called driving terms $d_\ell^I(s)$ contain all the rest [9]. In the elastic region unitarity relates the imaginary part of the $\pi\pi$ partial wave to its modulus squared:

$$\mathrm{Im}\, t_l^I(s) = \sigma(s)|t_l^I(s)|^2 \ . \tag{3.5}$$

These amplitudes can be expanded around threshold, as in Eq. (2.4): in order to see what integrals can generate chiral logs the same analysis we did for the form factors can be repeated here. Only the integral over the S- and P-waves may generate chiral logs:

$$\begin{aligned} t_\ell^I(s)_{SP} &= \int_{4M_\pi^2}^{\Lambda^2} ds' K_{\ell 0}^{I0}(s,s')\left[(a_0^0)^2 + 2a_0^0 b_0^0 q'^2 + (b_0^0)^2 q'^4 + \ldots\right] \\ &+ \int_{4M_\pi^2}^{\Lambda^2} ds' K_{\ell 1}^{I1}(s,s')\left[16(a_1^1)^2 q'^4 + \ldots\right] + \int_{4M_\pi^2}^{\Lambda^2} ds' K_{\ell 0}^{I2}(s,s')\left[(a_0^2)^2 + 2a_0^2 b_0^2 q'^2 + (b_0^2)^2 q'^4 + \ldots\right] , \end{aligned} \tag{3.6}$$

where $q'^2 := s'/4 - M_\pi^2$. The partial wave amplitudes needed for the form factors are $t_0^0(s)$ for $F_S(s)$ and $t_1^1(s)$ for $F_V(s)$. The kernels are known [9]. The result reads

$$t_0^0(s)_{SP} = 6t_1^1(s)_{SP} = \frac{s}{6}\left\{ I_0\left[2a_0^0 b_0^0 - 5a_0^2 b_0^2\right] + \frac{1}{8} I_1\left[2(b_0^0)^2 + 27(a_1^1)^2 - 5(b_0^2)^2\right]\right\} + \ldots \tag{3.7}$$

where

$$I_n(s,\Lambda^2) := \frac{s}{16\pi^2}\int_{4M^2}^{\Lambda^2} ds' \frac{\sigma(s')(s'-4M^2)^n}{s'(s'-s)} \ , \tag{3.8}$$

and the ellipsis stands for all the terms which cannot generate the chiral logs of interest. We find that for both partial waves the coefficient of the chiral log is proportional to

$$\left[2a_0^0 b_0^0 - 5a_0^2 b_0^2\right] - \frac{3M^2}{4}\left[2(b_0^0)^2 + 27(a_1^1)^2 - 5(b_0^2)^2\right] . \tag{3.9}$$

To get the leading chiral log we need to input the threshold parameters at leading order in χPT . The explicit calculation shows that, although the individual terms in this combination are of order M^2, they cancel and leave as leading contribution something of $\mathscr{O}(M^4)$ and therefore beyond the accuracy of this calculation.

Neither the driving terms nor higher partial waves generate chiral logs of the order considered here. The subtraction term could in principle contain chiral logs, but the combination of scattering lengths $2a_0^0 - 5a_0^2$, which determines the coefficient β_1, does not have any [10].

3.1 Higher order $\pi\pi$ partial waves

To complete our argument we need to show that no chiral logs of the order we are interested in may appear in the integrands at any order. Since the $\pi\pi$ scattering amplitude at tree level and zero momentum vanishes linearly in the chiral limit it does not contain terms proportional to L, therefore

$$t^{(2)}(s) = \mathscr{O}(M^2) + \mathscr{O}(s) \ . \tag{3.10}$$

From unitarity, by applying chiral counting to Eq. (3.5) it follows that

$$\mathrm{Im} t^{(4)} = \sigma(s)|t^{(2)}|^2 \ . \tag{3.11}$$

Hence also the imaginary part of the partial wave to order p^4 contains no terms proportional to L. Using Roy equations (3.3) we can get the corresponding partial wave $t^{(4)}$ from the imaginary part. Since the explicit calculation shows that neither the integration nor the subtraction term produce any unwanted chiral logs, the partial wave to next-to-leading order has no terms proportional to sL. By induction one may reach the same conclusion to all chiral orders: for the partial waves expanded for $M^2 \ll s$:

$$t^{(2n)}(s) = \bar{t}^{(2n)}(s) + L\sum_{k=0}^{n-1} \beta_k s^k M^{2(n-1-k)} + \mathscr{O}(M^2) \tag{3.12}$$

the coefficient β_{n-1} vanishes to all orders.

4. Conclusions and a comment on inelastic contributions

In this talk we have shown that in the limit $M_\pi^2 \ll s$, the leading chiral log in the pion form factors factorizes to all orders in the chiral expansion if one considers only elastic contributions both in the form factor and in the $\pi\pi$ scattering amplitude. The proof is based on the dispersive representation of the form factors and crucially relies on the vanishing of the coefficients of the leading chiral logs in the $\pi\pi$ scattering amplitude. At $\mathscr{O}(p^4)$ these coefficients are shown to be zero by an explicit calculation — by induction we have shown that these vanish at all higher orders.

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In all this discussion we have always put inelastic contributions aside. These show up in the form factor at three-loop order (a dispersive contribution with a four-pion intermediate state). At first one might think that the four-particle phase space would give enough suppression in the threshold region so that chiral logs would not be generated. However, somewhat surprisingly we find that these inelastic contributions do generate chiral logs and that these violate factorization. Notice also that even if the chiral logs at this order were zero, since inelastic channels do give a contribution in the chiral limit, the violation of factorization would happen at one order higher (unless a miracle totally unrelated to the mechanism discussed here would happen). Inelastic channels had not been mentioned in the talk but will be discussed in detail in a subsequent publication [6].

We conclude that the factorization of chiral logs postulated in the H$\pi\chi$PT literature is only valid in an approximate sense: it holds only for a subclass of diagrams, and there are contributions which do not respect it, starting at three loops in χPT .

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