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Radiative Improvement of the Lattice NRQCD Action with applications to bottomonium hyperfine structure

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We describe how the background field method can be applied to Non-Relativistic QCD (NRQCD) on the lattice in order to determine the one-loop radiative corrections to the coefficients of the NRQCD action in a manifestly gauge-covariant manner. As a first application, we compute the shift of the hyperfine splitting of bottomonium stemming from the one-loop corrections to the coefficient of the $\boldsymbol{\sigma} \cdot B$ term in the NRQCD action and the spin-dependent four-fermion couplings arising at the one-loop level. This is found to bring the lattice predictions in line with experiment as well as greatly reducing $O(a^2)$ dependences. We report a preliminary estimate for the hyperfine splitting, calculated on a fine lattice, of 70(6)MeV.

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1. Introduction

Non-Relativistic QCD (NRQCD) [1] is an effective field theory that has been applied with considerable success to the description of hadrons containing heavy quarks [2]. However, the currently used NRQCD actions do not include radiative improvement (with the exception of tadpole improvement), and this will affect the precision with which crucial quantities such as the hyperfine splitting between the Υ and the η_b can be determined [3]. Radiative improvement is complicated by the nature of the non-abelian gauge interactions in QCD and NRQCD, which requires NRQCD to be implemented on a lattice and hence makes it necessary to retain the full $1/(ma)^n$ dependences. Moreover, IR divergences play a non-trivial rôle in QCD and NRQCD.

We report on the calculation of the full chromomagnetic moment and four-fermion spin dependent interaction [4]. This is used to radiatively correct NRQCD, allowing a more accurate determination of the hyperfine splitting. This is the first time the lattice NRQCD action has been corrected using the background field (BF) method.

2. The Background field method for lattice NRQCD

The BF method [5] is a well-established tool to compute the effective action in quantum field theory. The auxiliary gauge invariance of BFG amplitudes implies that the effective action contains only gauge-covariant operators. This leads to a set of Ward Identities in QCD that reduce the amount of calculation necessary to renormalize the theory. This property is also important for operators of dimension $D > 4$, where the loss of gauge-covariance would lead to a proliferation of additional operators, obscuring the underlying gauge symmetry and greatly complicating the theory and simulation. Moreover, an attempt to match without using BFG would lead to the appearance of ultraviolet logarithms, which would have to be cancelled by the contributions from additional non-gauge-covariant operators. As a consequence of BFG, we are free to use different regulators in QCD and NRQCD. In particular, we can calculate the QCD vertex analytically in the continuum using dimensional regularization, or on a fine lattice and taking the continuum limit, which is paticularly convenient for checking the gauge-parameter independence of the result. Although BFG does not guarantee that the coefficients in the effective action are independent of the gauge parameter [6], in our case we match between theories using on-shell quantities and we explicitly find that the coefficients are independent of the gauge parameter in both QCD and NRQCD.

In the following we denote the perturbative expansion for a generic parameter w as $w = \sum_{n=0} w^{(n)}\alpha^n$.

3. Matching the $\sigma \cdot B$ term

The effective action for continuum QCD contains the following terms involving the fermion fields:

$$\Gamma[\Psi,\overline{\Psi},A] = Z_2^{-1}\overline{\Psi}\,\not{D}\Psi + \delta Z_\sigma \overline{\Psi}\frac{\sigma^{\mu\nu}F_{\mu\nu}}{2m}\Psi + \dots \tag{3.1}$$

which after renormalization of the first term gives

$$\Gamma[\Psi_R,\overline{\Psi}_R,A] = \overline{\Psi}_R\,\not{D}\Psi_R + b_\sigma \overline{\Psi}_R\frac{\sigma^{\mu\nu}F_{\mu\nu}}{2m_R}\Psi_R + \dots \tag{3.2}$$

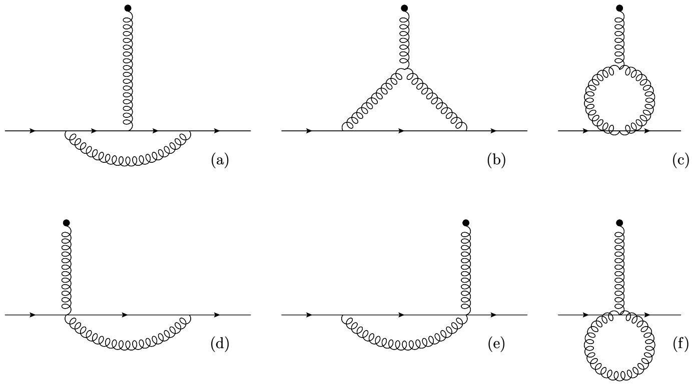

Figure 1: Feynman diagrams to be computed in both QCD and NRQCD for matching the $\sigma \cdot B$ term in the NRQCD action.

with

$$b_\sigma = \delta Z_\sigma Z_2 Z_m = \sum_{n=1} b_\sigma^{(n)} \alpha^n \,, \tag{3.3}$$

where the leading correction is of order $O(\alpha_s)$ and comes from δZ_σ alone. After performing the non-relativistic reduction by a Foldy-Wouthuysen-Tani (FWT) transformation, we find that the term relevant for the determination of the chromomagnetic moment of the quark is

$$(1+b_\sigma)\, \psi_R^\dagger \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2m_R} \psi_R \,. \tag{3.4}$$

A straightforward analytical calculation of the Feynman diagrams shown in figure 1 (a)–(b) gives

$$b_\sigma = \left(\frac{3}{2\pi} \log\frac{\mu}{m} + \frac{13}{6\pi} \right) \alpha \tag{3.5}$$

at the one-loop level, where μ is the infrared cutoff.

The effective action for NRQCD contains the spin-dependent term

$$\Gamma_\sigma[\psi, \psi^\dagger, A] = c_4 Z_\sigma^{\mathrm{NR}} \psi^\dagger \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} \psi \tag{3.6}$$

which after renormalization becomes

$$\Gamma_\sigma[\psi_R, \psi_R^\dagger, A] = c_4 Z_\sigma^{\mathrm{NR}} Z_2^{\mathrm{NR}} Z_m^{\mathrm{NR}} \psi_R^\dagger \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M_R} \psi_R \,. \tag{3.7}$$

We require that the anomalous chromomagnetic moment in QCD and NRQCD be equal and find the matching condition

$$c_4 Z_\sigma^{\mathrm{NR}} Z_2^{\mathrm{NR}} Z_m^{\mathrm{NR}} = 1 + b_\sigma \tag{3.8}$$

and at tree level and one-loop order we find

$$c_4^{(0)} = 1 \,, \qquad c_4^{(1)} = b_\sigma^{(1)} - \delta Z_\sigma^{\mathrm{NR},(1)} - \delta Z_2^{\mathrm{NR},(1)} - \delta Z_m^{\mathrm{NR},(1)} \,. \tag{3.9}$$

The NRQCD contribution to $c_4^{(1)}$ contains a logarithmic IR divergence $\frac{3\alpha}{2\pi}\log(\mu a)$, which combines with the IR logarithm from the QCD result above to yield an overall logarithmic contribution $-\frac{3\alpha}{2\pi}\log(Ma)$.

Besides the ordinary diagrammatic contributions calculated below, we also need to take into account the contributions from the mean-field improvement $U \mapsto U/u_0$, which affect $\delta Z_\sigma^{\mathrm{NR},(1)}$ and $\delta Z_m^{\mathrm{NR},(1)}$. Perturbatively, $u_0 = 1 - \alpha_s u_0^{(2)}$, and the contributions from inserting this expansion into the NRQCD action can be worked out algebraically. The final result for the one-loop correction to c_4 is then

$$c_4^{(1)} = \tfrac{13}{6\pi} - \delta\tilde{Z}_\sigma^{\mathrm{NR},(1)} - \delta\tilde{Z}_2^{\mathrm{NR},(1)} - \delta\tilde{Z}_m^{\mathrm{NR},(1)} - \delta Z_m^{\mathrm{tad},(1)} - \delta Z_\sigma^{\mathrm{tad},(1)} - \tfrac{3}{2\pi}\log Ma \tag{3.10}$$

where $\delta\tilde{Z}_X$ denotes a finite diagrammatic contribution. We expect the coefficient c_4 to be gauge-parameter independent for on-shell quarks, since it is directly related to the hyperfine splitting, which is a physical quantity.

4. The four-fermion spin-spin interaction

In NRQCD the hyperfine splitting in the $b\bar{b}$ system also receives a contribution from the spin-dependent four-fermion operators generated by $Q\bar{Q} \to Q\bar{Q}$ scattering in the colour singlet channel. It is conventional to write these contributions using a Fierz transformation [1, 7]

$$S_{4f} = d_1 \frac{\alpha^2}{M^2}(\psi^\dagger\chi^*)(\chi^T\psi) + d_2\frac{\alpha^2}{M^2}(\psi^\dagger\boldsymbol{\sigma}\chi^*)\cdot(\chi^T\boldsymbol{\sigma}\psi)\,, \tag{4.1}$$

where ψ and χ are the quark and anti-quark fields, respectively, treated as different particle species with corresponding representations of their spin and colour algebras. The spin-independent contributions to d_1 and d_2 from $Q\bar{Q}$ scattering are not included as they do not influence the hyperfine structure. In QCD the two continuum diagrams are shown in figures 2(a) and 2(b), and in NRQCD all diagrams in figure 2 need to be calculated. The one-loop contributions to the renormalization constants for the operators in eqn. (4.1) take the form, respectively,

$$Z_{f1} = \alpha^2\left(A_{f1} - \log\tfrac{\mu}{m} - \tfrac{16\pi}{27}\tfrac{m}{\mu}\right)\,, \qquad Z_{f2} = -\tfrac{1}{3}Z_{f1}. \tag{4.2}$$

The last term in both expressions is the Coulomb singularity arising from the Coulomb gluon exchange in figure 2(a). For QCD these expressions were verified numerically and for both QCD and NRQCD were shown to be gauge-parameter independent; there are two independent colour trace combinations, each of which is separately gauge independent. In the numerical calculations we used IR subtraction functions to analytically remove both IR and Coulomb divergences; this greatly improved convergence. For QCD we find

$$A_{f1}^R = \frac{8}{27}\,. \tag{4.3}$$

The matching parameters for the term in the NRQCD action, including the two-gluon annihilation contribution to d_1 [7], are then

$$d_1 = -3d_2 - \frac{2}{9}(2 - 2\log 2)\,, \qquad d_2 = \frac{8}{81} - \frac{1}{3}A_{f1}^{NR} + \frac{1}{3}\log Ma\,. \tag{4.4}$$

Ma	1.9	2.65	3.4
$c_4^{(1)}$	0.683(13)	0.776(19)	0.817(27)
d_1	0.049(1)	-0.527(6)	-1.186(8)
d_2	-0.146(1)	0.130(2)	0.350(2)

Table 1: Renormalization parameters of the $\sigma \cdot B$ and the four-fermion terms.

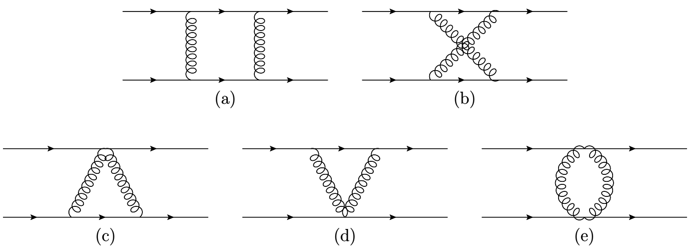

Figure 2: Feynman diagrams to be computed in both QCD and NRQCD for matching the four-fermion terms in the NRQCD action. There are two diagrams with the topology of (c).

5. Implementation and results

To perform the calculation in NRQCD, we employ the HiPPy and HPsrc packages for automated lattice perturbation theory [8, 9], which we extended to deal with the modifications of the usual Feynman rules engendered by the use of BFG [10, 11]. For further implementation details the reader is referred to [12].

For the $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ operator matching we compute the diagrams in figures 1 (a)–(f) and for the four-fermion operator matching we compute the diagrams in figure 2. We carried out a number of checks of the calculation. Firstly, we replicate the known IR logs correctly. We find that the coefficients of these logs are gauge-parameter independent and, since this is not true of the contributions from individual diagrams, it provides a strong check. Second, we check that the non-logarithmic part of the result is similarly gauge-parameter independent where the individual contributions are not. For matching the four-fermion terms it is vital to employ IR subtraction functions to remove logarithmic and Coulomb IR singularities. For NRQCD, we used the action from [13] with stability parameter $n = 4$, and we used the Symanzik improved gluon action [14], which were also used by the MILC collaboration [15] whose configurations were used in [13]. We find

$$\delta Z_m^{\text{tad},(1)} = -\left(\frac{2}{3} + \frac{3}{(Ma)^2}\right) \alpha_s u_0^{(2)} \tag{5.1}$$

The tadpole contribution to $\delta Z_\sigma^{\text{NR},(1)}$ comes from the mean-field improvement of the improved field-strength tensor and from the cross-multiplication of the tree-level $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ term with the tadpole corrections terms in H_0 [13]. The overall result is

$$\delta Z_\sigma^{\text{tad},(1)} = \left(\frac{13}{3} + \frac{13}{4Ma} - \frac{3}{8n(Ma)^2} - \frac{3}{4(Ma)^3}\right) .u_0^{(2)} . \tag{5.2}$$

We chose the Landau mean link to be $u_0^{(2)} = 0.750$ [16]. Our results are shown in table 1. The radiative correction to the $g\sigma \cdot B$ term can easily be included in simulations. The correction to the four-fermion operator can be included by noting that both operators give a contribution to the

		hfs (MeV)		Correction	hfs (MeV)
Ma	$\alpha_V(q^*)$	$c_4 = 1$	improved c_4	4-fermion	corrected
1.9	0.225	56.1(1)	72.1(1)	-1.7(1)	70.4(1)(28)(56)
2.65	0.253	50.5(1)	69.8(1)	+4.5(1)	74.3(1)(32)(50)
3.4	0.275	45.6(1)	65.6(1)	+10.3(1)	75.9(1)(34)(46)

Table 2: Corrections to the bottomonium hyperfine splitting results of [13] arising from the radiative improvement of the action. In the last column the errors are statistical, $O(\alpha^2)$, and relativistic corrections.

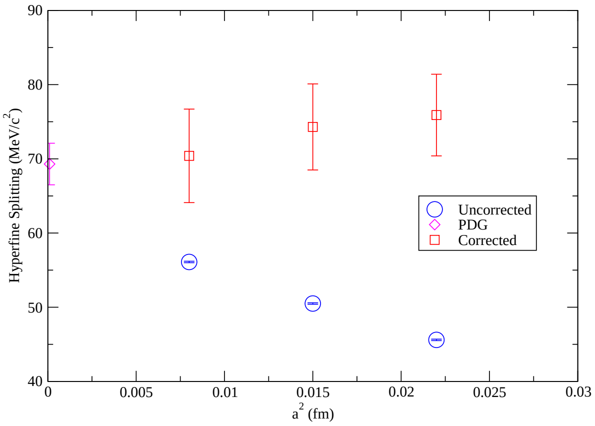

Figure 3: Corrected and uncorrected bottomonium hyperfine splitting results. Note that total error is displayed with the corrected results, whereas uncorrected results contain purely statistical errors; $O(\alpha)$ errors would be too large on this scale. PDG data from [17].

hyperfine splitting that is dominated by a contact term. A reasonable estimate for the multiplicative correction to the tree-level prediction for the hyperfine splitting is then

$$1 + \alpha_V(q^*)\left(-\frac{27}{16\pi}(d_1 - d_2)\right), \tag{5.3}$$

where we chose $q^* = \pi/a$. Our results have been applied to the hyperfine splitting of bottomonium [13] , where we find the corrections given in table 2 and summarised in figure 3.

6. Conclusion

We have presented the first application of the BF method to lattice NRQCD and have computed the one-loop radiative correction to the coefficient, c_4, of the $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ operator and the one-loop radiative contribution to the coefficients, d_1 and d_2 of the four-fermion contact operators that affect the hyperfine structure of heavy quark mesons. The gauge independence of our calculation was explicitly checked by carrying out both relativistic and non-relativistic calculations in the lattice theory. This is possible because in BFG all calculations are UV finite. Our results are summarized in table 1 and in eqns. (3.10) and (4.4). Corrections to the $g\sigma \cdot B$ term have been included in simulations[13], and we have given an estimate for the contribution of four-fermion corrections to $\Upsilon - \eta_b$ hyperfine splitting. The result is to reduce the lattice spacing dependence to within errors and to give an estimate for this hyperfine splitting of $70.4(1)(28)(56)$MeV to be compared with the experimental value of $69.3(2.8)$MeV [17]. The errors shown are statistical, $O(\alpha^2)$, and due to relativistic corrections, respectively. The elimination of $O(\alpha a^2)$ errors and the agreement with experiment gives us confidence that the calculations are robust. Our results have been included in a recent paper on the Upsilon spectrum [18].

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