PROCEEDINGS OF SCIENCE

Reflection Positivity of $\mathcal{N}=1$ Wess-Zumino model on the lattice with exact U(1)$_R$ symmetry

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By using overlap Majorana fermions, the $\mathcal{N}=1$ chiral multiple can be formulated so that the supersymmetry is manifest and the vacuum energy is cancelled in the free limit, thanks to the bilinear nature of the free action. It is pointed out, however, that in this formulation the reflection positivity seems to be violated in the bosonic part of the action, although it is satisfied in the fermionic part. It is found that the positivity of the spectral density of the bosonic two-point correlation function is ensured only for the spacial momenta $a|p_k| \lesssim 1.84$ $(k=1,2,3)$. It is then argued that in formulating $\mathcal{N}=1$ Wess-Zumino model with the overlap Majorana fermion, one may adopt a simpler nearest-neighbor bosonic action, discarding the free limit manifest supersymmetry. The model still preserves the would-be U(1)$_R$ symmetry and satisfies the reflection positivity.

The XXIX International Symposium on Lattice Field Theory - Lattice 2011
July 10-16, 2011
Squaw Valley, Lake Tahoe, California

*Speaker.

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1. Introduction

The chiral multiplet of $\mathscr{N}=1$ supersymmetry [1] can be formulated on the lattice so that the supersymmetry is preserved and the vacuum energy is cancelled in the free limit, thanks to the bilinear nature of the free action. By using overlap (Majorana) fermion [2, 3, 4] for the fermionic component, species doublers [5, 6, 7] are successfully removed and $\mathrm{U}(1)_R$ symmetry can be maintained at the same time [8, 9, 10]. With this chiral multiplet, one may formulate lattice $\mathscr{N}=1$ Wess-Zumino model with exact $\mathrm{U}(1)_R$ symmetry [11, 12, 13, 14, 15, 16]. A numerical study of this lattice $\mathscr{N}=1$ Wess-Zumino model has recently been reported in [17].

The purpose of this short article is, however, to show that in this formulation of the chiral multiplet, the reflection positivity [18, 19, 20, 21, 22] seems to be violated in the bosonic part of the action, although it is satisfied in the fermionic part, as shown recently in [23]. We will also examine the spectral density of the bosonic two-point correlation function (cf. [24]). It is found that the positivity of the spectral density is ensured only for the momenta $a|p_k| \lesssim 1.84$ ($k=1,2,3$), and the mode with a negative density appears at the energy as low as $aE \simeq 0.69$ for the momenta $a\boldsymbol{p}=(\pi,0,0),(0,\pi,0),(0,0,\pi)$.

We will then argue that in formulating the lattice $\mathscr{N}=1$ Wess-Zumino model with the overlap (Majorana) fermion, one may adopt the simpler nearest-neighbor bosonic action, discarding the free limit manifest supersymmetry. The model so constructed still preserves the $\mathrm{U}(1)_R$ symmetry and satisfies the reflection positivity.

2. $\mathscr{N}=1$ chiral multiple with overlap Majorana fermion

The action of the free $\mathscr{N}=1$ chiral multiplet is given by

$$S_0 = a^4 \sum_x \left\{ \frac{1}{2}\chi^T C D_1 \chi + \phi^* D_1^2 \phi + F^* F + \frac{1}{2}\chi^T C D_2 \chi + F D_2 \phi + F^* D_2 \phi^* \right\}. \tag{2.1}$$

In this expression, we have used a decomposition of the overlap Dirac operator [2, 3], $D=D_1+D_2$, where

$$D_1 = \frac{1}{2}\gamma_\mu(\partial_\mu^* + \partial_\mu)(A^\dagger A)^{-1/2}, \tag{2.2}$$

$$D_2 = \frac{1}{a}\left\{1-(1+\frac{1}{2}a^2\partial_\mu^*\partial_\mu)(A^\dagger A)^{-1/2}\right\}, \tag{2.3}$$

and

$$A = 1 - aD_{\mathrm{w}}, \quad D_{\mathrm{w}} = \frac{1}{2}\left\{\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^*\partial_\mu\right\}. \tag{2.4}$$

Note that D_1 and D_2 have different spin structures with respect to spinor space. In particular, we have $\{\gamma_5, D_1\}=0$ and $[\gamma_5, D_2]=0$. In terms of this decomposition, the Ginsparg-Wilson relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$ [8] is expressed as

$$2D_2 = a(-D_1^2 + D_2^2), \tag{2.5}$$

and as a consequence, we have relations

$$\gamma_5(1-\frac{1}{2}aD)\gamma_5(1-\frac{1}{2}aD) = 1-\frac{1}{2}aD_2, \qquad \gamma_5(1-\frac{1}{2}aD)\gamma_5 D = D_1. \tag{2.6}$$

It is also understood that the 4×4 identity matrix in operators D_1^2 and D_2 is omitted when these operators are acting on bosonic fields.

It is straightforward to see that the above free action S_0 is invariant under "lattice $\mathscr{N}=1$ supersymmetry":

$$\begin{aligned}
&\delta_\varepsilon\chi = -\sqrt{2}P_+(D_1\phi+F)\varepsilon - \sqrt{2}P_-(D_1\phi^*+F^*)\varepsilon,\\
&\delta_\varepsilon\phi = \sqrt{2}\varepsilon^T CP_+\chi,\quad \delta_\varepsilon\phi^* = \sqrt{2}\varepsilon^T CP_-\chi,\\
&\delta_\varepsilon F = \sqrt{2}\varepsilon^T CD_1P_+\chi,\quad \delta_\varepsilon F^* = \sqrt{2}\varepsilon^T CD_1P_-\chi,
\end{aligned} \tag{2.7}$$

where ε is a 4 component Grassmann parameter. We also note that the free action S_0 possesses three types of $\mathrm{U}(1)$ symmetry [10]. The first is a rather trivial one acting only on bosonic fields and is defined by the transformation:

$$\begin{aligned}
&\delta_\alpha\chi = 0,\\
&\delta_\alpha\phi = i\alpha\phi,\\
&\delta_\alpha F = -i\alpha F,
\end{aligned} \tag{2.8}$$

where α is an infinitesimal real parameter. The second one is nothing but the chiral symmetry introduced by Lüscher,

$$\delta_\alpha\chi = i\alpha\gamma_5(1-\frac{1}{2}aD)\chi, \tag{2.9}$$

Thirdly, somewhat surprisingly, the *bosonic* sector of S_0 possesses a $\mathrm{U}(1)$ symmetry analogous to eq. (2.9):

$$\begin{aligned}
&\delta_\alpha\phi = +i\alpha\{(1-\frac{1}{2}aD_2)\phi - \frac{1}{2}aF^*\},\\
&\delta_\alpha F = +i\alpha\{(1-\frac{1}{2}aD_2)F - \frac{1}{2}aD_1^2\phi^*\}
\end{aligned} \tag{2.10}$$

due to the Ginsparg-Wilson relation. The lattice action S_0 is not invariant under a uniform rotation of the complex phase of bosonic fields, ϕ, F, due to the presence of terms $FD_2\phi$ and $F^*D_2\phi^*$. The above provides a lattice counterpart of this uniform phase rotation of bosonic fields under which the free action S_0 is invariant. Using a linear combination of the above three $\mathrm{U}(1)$ symmetries, it is possible to define the $\mathrm{U}(1)_R$ symmetry [10] in the interacting system.

$$\begin{aligned}
&\delta_\alpha\chi = +i\alpha\gamma_5(1-\frac{1}{2}aD)\chi,\\
&\delta_\alpha\phi = -3i\alpha\phi + i\alpha\{(1-\frac{1}{2}aD_2)\phi - \frac{1}{2}aF^*\},\\
&\delta_\alpha F = +3i\alpha F + i\alpha\{(1-\frac{1}{2}aD_2)F - \frac{1}{2}aD_1^2\phi^*\}.
\end{aligned} \tag{2.11}$$

3. Violation of the reflection positivity in the bosonic part

The reflection positivity is defined as the condition that any polynomial of positive time fields $\mathscr{F}$ fulfills the inequality

$$\langle\theta(\mathscr{F})\mathscr{F}\rangle \geq 0, \tag{3.1}$$

where $\langle\cdot\rangle$ means the expectation value of the theory defined through path integration as usual, and θ is an anti linear time-reflection operator given in [20]. The θ operator acts, on scalar field ϕ for instance, as

$$\theta\phi(t,\boldsymbol{x}) = \phi(-t+1,\boldsymbol{x})^*. \tag{3.2}$$

This is one of the fundamental conditions that lattice field theories have to fulfill because a lattice theory satisfying the reflection positivity condition corresponds to an acceptable quantum theory with unitary time evolution [18, 19, 20].

After integrating out the auxiliary field F, we have the 'overlap boson', which is characterized by the Klein-Gordon type operator $D^\dagger D$ with the overlap Dirac operator D. However, the reflection positivity of this overlap boson system seems to be violated. It can be seen by considering the spectral density function $\rho(E,\boldsymbol{p})$ in the Euclidean version of the Källén-Lehmann representation of the propagator. The spectral density $\rho(E,\boldsymbol{p})$ is expected to be a non-negative function if the lattice model indeed defines a quantum theory with physically acceptable evergy momentum spectrum, and is geven in the Källén-Lehmann representation by

$$\langle\phi(x)^*\phi(y)\rangle\Big|_{x_0>y_0} = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\int_0^\infty\frac{dE}{\pi}\,\mathrm{e}^{-E(x_0-y_0)}\mathrm{e}^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}\rho(E,\boldsymbol{p}). \tag{3.3}$$

In the present case, one can explicitly estimate the spectral density $\rho(E,\boldsymbol{p})$, and the result is [25]

$$\rho(E,\boldsymbol{p}) = (\text{singular part}) + \frac{(\cosh E - b(\boldsymbol{p}))\sqrt{2b(p)\cosh E - a(\boldsymbol{p})}}{\cosh^2 E - a(\boldsymbol{p}) + b(\boldsymbol{p})^2}\theta(E-E_1), \tag{3.4}$$

where $\cosh E_1 = a(\boldsymbol{p})/2b(\boldsymbol{p})$ and

$$a(\boldsymbol{p}) = 1+\sum_j\sin^2 p_j + b(\boldsymbol{p})^2, \qquad b(\boldsymbol{p}) = \sum_j(1-\cos p_j). \tag{3.5}$$

The second term, continuous spectrum, is not positive (non-negative) because of the factor $\cosh E - b(\boldsymbol{p})$. The fact that the positivity of the spectral density breaks down is an indirect but strong circumstantial evidence that the overlap boson system does not fulfill the reflection positivity condition. In fact, one can prove in mathematically rigorous manner [26] that a lattice theory satisfying the reflection positivity condition, in addition to some natural assumptions which are satisfied by the overlap boson system, must have a non-negative spectral density.

From the explicit form of the spectral density (3.4), we can find where in the Brillouin zone the reflection positivity is violated. Note that there is the region in the spacial Brillouin zone where the spectral density $\rho(E,\boldsymbol{p})$ can not become negative. The necessary and sufficient condition for spacial momenta $\boldsymbol{p}$ to avoid negative $\cosh E - b(\boldsymbol{p})$ is that

$$\cosh E - b(\boldsymbol{p}) \geq 0, \qquad \forall E \geq E_1, \tag{3.6}$$

or, equivalently,

$$1+\sum_k\sin^2 p_k - b(\boldsymbol{p})^2 \geq 0. \tag{3.7}$$

In order to give some concrete examples, we will consider the following two simple cases in $d=4$. First, we consider the case in which three components of spacial momenta are such that $p_1=p_2=p_3=:p$. In this case, the condition (3.7) becomes

$$-1.84 \lesssim p \lesssim 1.84. \tag{3.8}$$

Secondly, we consider another direction $p_1=p$, $p_2=p_3=0$. In this case, by the condition (3.7), p is restricted to

$$-2.23 \lesssim p \lesssim 2.23. \tag{3.9}$$

When the spacial momenta $\boldsymbol{p}$ does not satisfy (3.7), the spectral density $\rho(E,\boldsymbol{p})$ has to become negative on the energy interval $E_1 \leq E < E_c$, where E_1 and E_c are determined by

$$\cosh E_1 = \frac{a(\boldsymbol{p})}{2b(\boldsymbol{p})}, \qquad \cosh E_c = \boldsymbol{b}(\boldsymbol{p}), \tag{3.10}$$

since $\rho(E,\boldsymbol{p})<0$ is equivalent to $a(\boldsymbol{p})/2b(\boldsymbol{p}) \leq \cosh E < b(\boldsymbol{p})$ when $\boldsymbol{p}$ breaks (3.7). We will numerically estimate E_1 and E_c, the lower and upper bound of the energy interval on which the spectral density become negative. For instance, if $d=4$, these energy values are computed as shown in the following table:

$\boldsymbol{p}$	$b(\boldsymbol{p})$	$a(\boldsymbol{p})$	$a(\boldsymbol{p})/2b(\boldsymbol{p})$	E_1	E_c
(π,π,π)	6	37	37/12	1.79...	2.48...
$(\pi,\pi,0)$	4	17	17/8	1.39...	2.06...
$(\pi,0,0)$	2	5	5/4	0.69...	1.32...

Whether these values are large enough or not should depend on the physics one wants to see through the overlap boson.

4. Refletion positivity of lattice Wess-Zumino model

To remedy the violation of the reflection positivity, one may adopt the simpler nearest-neighbor action for the boson fields, ϕ and F as follows[1]:

$$S_0' = \sum_x \Big\{ -\frac{1}{2}\chi^T CD\chi + \phi^*(-\partial_\mu^*\partial_\mu)\phi + F^*F \Big\}. \tag{4.1}$$

This action still possesses three types of U(1) symmetry, Eq. (2.8), (2.9) and

$$\begin{aligned} \delta_\alpha \phi &= +i\alpha\phi, \\ \delta_\alpha F &= +i\alpha F, \end{aligned} \tag{4.2}$$

instead of Eq. (2.10).

[1]Here, we have changed the sign convention of the fermionic action by introducing new Majorana field $\chi' = i\chi$. Of course this does not change any physical results. It is simply because this convention has been used in the proof of the reflection positivity for the overlap fermions in our previous work [23].

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In this formulation of the chiral multiplet, the action of the lattice $\mathscr{N}=1$ Wess-Zumino model may be given as follows:

$$
\begin{aligned}
S = \sum_x \Big\{ & -\frac{1}{2}\chi^T C D\chi + \phi^*(-\partial_\mu^*\partial_\mu)\phi + F^*F + X^T C X \\
& - g\tilde{\chi}^T C\phi P_+\tilde{\chi} - g^*\tilde{\chi}^T C\phi^* P_-\tilde{\chi} + gF\phi^2 + g^*F^*\phi^{*2} \Big\},
\end{aligned} \tag{4.3}
$$

where $X(x)$ is an auxiliary Majorana fermion field and $\tilde{\chi}(x) = \chi(x) + X(x)$. Then one may define the $\mathrm{U}(1)_R$ symmetry as follows:

$$
\begin{aligned}
\delta_\alpha \chi &= +i\alpha\gamma_5(1-\frac{1}{2}D)\chi, \\
\delta_\alpha \phi &= -2i\alpha\phi, \\
\delta_\alpha F &= +4i\alpha F
\end{aligned} \tag{4.4}
$$

The reflection positivity is now satisfied in this formulation of the Wess-Zumino model [25].

5. Discussion

Preserving R symmetry exactly is a useful way in formulating supersymmetric field theories on the lattice. This point has been emphasized by Elliot, Giedt and Moore [27] in their formulation of four-dimensional $\mathscr{N}=4$ super Yang-Mills theory. The discrete R symmetry in the two-dimensional $\mathscr{N}=2$ Wess-Zumino model [28] has played an important role in the numerical study of the correspondence to $\mathscr{N}=2$ conformal field theories [29].

In formulating the exact R symmetry on the lattice, however, there is a freedom in the choice of the bosonic part of the action. When one can preserve some part of the extended supersymmetries in the theories with $\mathscr{N}\geq 2$ [28, 30], it seems useful to adopt the bosonic actions to preserve the supersymmetries, although one should take into care a possible effect of the violation of the reflection positivity. But, for the theories of $\mathscr{N}=1$, it seems difficult to preserve the supersymmetry in general [31]. The free limit supersymmetry may still help in the convergence to the supersymmetric limit in the interacting models. Otherwise, the reflection positivity condition may give a possible guideline to choose a bosonic action.

Acknowledgements

K.U. would like to thank Tsutomu T. Yanagida for continuous encouragement. This work was partly supported by Global COE Program “the Physical Science Frontier", MEXT, Japan. K.U. is supported by JSPS Research Fellowship for young scientists. Y.K. is supported in part by Grant-in-Aid for Scientific Research No. 21540258, 21105503. This work was supported by World Premier International Center Initiative (WPI Program), MEXT, Japan.

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