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# Critical properties of $2D$ $Z(N)$ vector models for $N > 4$


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We investigate the critical properties of two-dimensional $Z(N)$ vector models for $N$ larger than 4. In particular, critical points of the two phase transitions are located and some critical indices are determined. We study also the behavior of the helicity modulus and the dependence of the critical points on $N$.



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## 1. Introduction

The Berezinskii-Kosterlitz-Thouless (BKT) phase transition was originally discovered in the two-dimensional ($2D$) $XY$ model in the first half of the seventies [1]. Since then it was realized that this type of phase transition takes place in a number of other models including discrete $2D$ $Z(N)$ models for large enough $N$ and even $3D$ gauge models at finite temperature (see [2, 3] for a recent study of the deconfinement transition in $3D$ $U(1)$ lattice gauge theory). Here we are interested in the phase structure of $2D$ $Z(N)$ vector models. On a $2D$ lattice $\Lambda = L^2$ with linear extension $L$ and periodic boundary conditions, the partition function of the model can be written as

$$Z(\Lambda,\beta) = \left[\prod_{x\in\Lambda} \frac{1}{N} \sum_{s(x)=0}^{N-1}\right] \exp\left[\sum_{x\in\Lambda} \sum_{n=1,2} \beta \, \cos\frac{2\pi}{N}(s(x)-s(x+e_n))\right] . \tag{1.1}$$

The BKT transition is of infinite order and is characterized by the essential singularity, *i.e.* the exponential divergence of the correlation length. The low-temperature or BKT phase is a massless phase with a power-law decay of the two-point correlation function governed by a critical index $\eta$. The $Z(N)$ spin model in the Villain formulation has been studied analytically in Refs. [4]. It was shown that the model has at least two BKT-like phase transitions when $N \geq 5$. The critical index $\eta$ has been estimated both from the renormalization group (RG) approach of the Kosterlitz-Thouless type and from the weak-coupling series for the susceptibility. It turns out that $\eta(\beta_c^{(1)}) = 1/4$ at the transition point from the strong coupling (high-temperature) phase to the massless phase, *i.e.* the behavior is similar to that of the $XY$ model. At the transition point $\beta_c^{(2)}$ from the massless phase to the ordered low-temperature phase, one has $\eta(\beta_c^{(2)}) = 4/N^2$. A rigorous proof that the BKT phase transition does take place, and so that the massless phase exists, has been constructed in Ref. [5] for both Villain and standard formulations. Monte Carlo simulations of the standard version with $N = 6, 8, 12$ were performed in Ref. [6]. Results for the critical index $\eta$ agree well with the analytical predictions obtained from the Villain formulation of the model.

In Refs. [7, 8] we have started a detailed numerical investigation of the BKT transition in $2D$ $Z(N)$ models for $N = 5$ which is the lowest number where this transition can occur. Our findings support the scenario of two BKT transitions with conventional critical indices. Here we continue our study with investigation of models for $N$=7 and 17. We want to locate the transition points and to compute some critical indices. Such results could serve as checking point of universality in our studies of the BKT transitions in $3D$ gauge models at finite temperature. Our second goal is to use the available data on the position of critical points to deduce phenomenological scaling of these points with $N$.

## 2. Numerical results

We simulated the model defined by Eq. (1.1) using the same cluster Monte Carlo algorithm adopted in the Refs. [7, 8] for the case $N = 5$. We used several different observables to probe the two expected phase transitions. In order to detect the first transition (*i.e.* the one from the disordered to the massless phase) we used the absolute value $|M_L|$ of the complex magnetization,

$$M_L = \frac{1}{L^2}\sum_i \exp\left(i\frac{2\pi}{N}s_i\right) \equiv |M_L|e^{i\psi}, \tag{2.1}$$

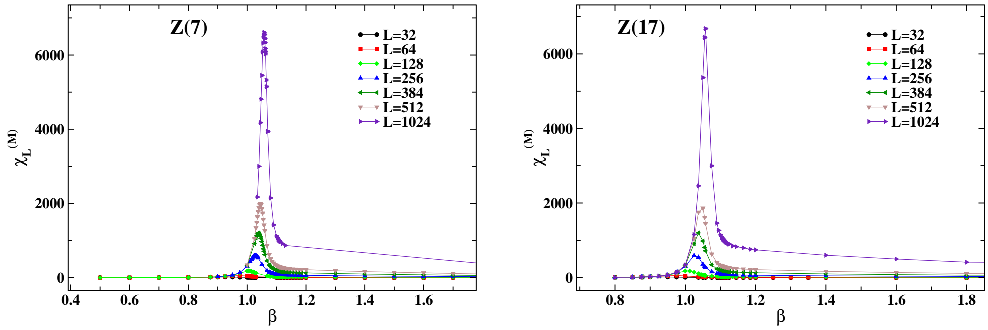


**Figure 1:** Susceptibility $\chi_L^{(M)}$ versus $\beta$ in $Z(7)$ (left) and $Z(17)$ (right) on lattices with several values of $L$.

and the *helicity modulus* [9, 10]

$$\Upsilon = \langle e \rangle - L^2 \beta \left\langle s^2 \right\rangle , \tag{2.2}$$

where $e \equiv \frac{1}{L^2} \sum_{<ij>_x} \cos\left(\theta_i - \theta_j\right)$ and $s \equiv \frac{1}{L^2} \sum_{<ij>_x} \sin(\theta_i - \theta_j)$. For the second transition (*i.e.* the one from the massless to the ordered phase) we adopted the real part of the "rotated" magnetization,

$$M_R = |M_L| \cos(N\psi) ,$$

and the order parameter

$$m_\psi = \cos(N\psi)$$

introduced in Ref. [11], where $\psi$ is the phase of the complex magnetization defined in Eq. (2.1). In this work, both for $N = 7$ and $N = 17$, we collected typically 100k measurements for each value of the coupling $\beta$, with 10 updating sweeps between each configuration. To ensure thermalization we discarded for each run the first 10k configurations. The jackknife method over bins at different blocking levels was used for the data analysis.

In Fig. 1 we show the behavior of the susceptibility $\chi_L^{(M)} \equiv L^2(\langle |M_L|^2 \rangle - \langle |M_L| \rangle^2)$ of the absolute value of the complex magnetization, which exhibits, for each volume considered, a clear peak signalling the first phase transition. The position of the peak in the thermodynamic limit defines the first critical coupling, $\beta_c^{(1)}$. Fig. 2 shows instead the behavior of $m_\psi$ versus $\beta$ on various lattice sizes; here the second critical coupling $\beta_c^{(2)}$ is identified by the crossing point (in the thermodynamic limit) of the curves formed by the data on different lattice sizes.

To determine the first critical coupling $\beta_c^{(1)}$, we could extrapolate to infinite volume the pseudo-critical couplings given by the position of the peaks of $\chi_L^{(M)}$. However, since the approach to the thermodynamic limit is rather slow (powers of $\log L$), we adopted a different method, based on the use of the "*reduced fourth-order*" Binder cumulant

$$U_L^{(M)} = 1 - \frac{\langle |M_L|^4 \rangle}{3\langle |M_L|^2 \rangle^2} , \tag{2.3}$$

the cumulant $B_4^{(M_R)}$ defined as

$$B_4^{(M_R)} = \frac{\langle |M_R - \langle M_R \rangle|^4 \rangle}{\langle |M_R - \langle M_R \rangle|^2 \rangle^2} , \tag{2.4}$$

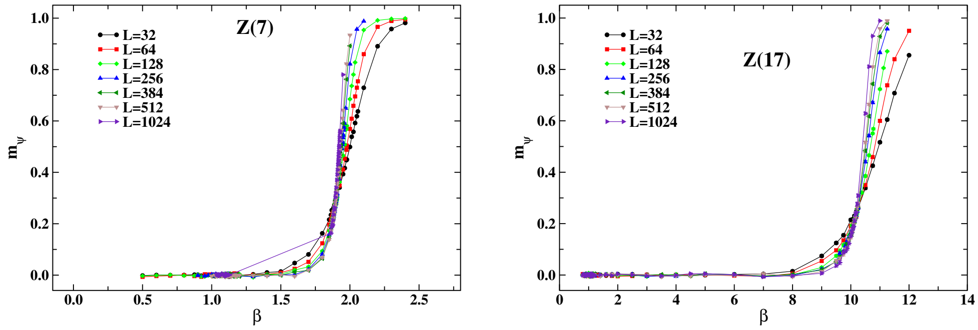


**Figure 2:** Behavior of $m_\psi$ with $\beta$ in $Z(7)$ (left) and $Z(17)$ (right) on lattices with several values of $L$.

and the helicity modulus $\Upsilon$. We estimated $\beta_c^{(1)}$ by looking for (i) the crossing point of the curves, obtained on different volumes, giving a Binder cumulant versus $\beta$ and (ii) the optimal overlap of the same curves after plotting them versus $(\beta-\beta_c)(\log L)^{1/\nu}$, with $\nu$ fixed at 1/2. The method (ii) has been applied also to the helicity modulus $\Upsilon$. Our best values for $\beta_c^{(1)}$ are

$$N=7: \quad \beta_c^{(1)} = 1.1113(13)\,, \qquad\qquad N=17: \quad \beta_c^{(1)} = 1.11375(250)\,.$$

Then, we performed the finite size scaling (FSS) analysis of the magnetization $|M_L|$ and the susceptibility $\chi_L^{(M)}$ at $\beta_c^{(1)}$ using the following laws:

$$|M_L|(\beta_c^{(1)}) = AL^{-\beta/\nu}\,, \qquad\qquad \chi_L^{(M)}(\beta_c^{(1)}) = BL^{\gamma/\nu}\,, \tag{2.5}$$

where $\gamma/\nu = 2-\eta$ and $\eta$ is the *magnetic critical index*. Results are summarized in Tables 1 and 2. We observe that the hyperscaling relation $\gamma/\nu + 2\beta/\nu = d = 2$ is nicely satisfied within statistical errors in both models.

**Table 1:** Results of the fit to the data of $|M_L|(\beta_c^{(1)})$ with the scaling law (2.5-left) on $L^2$ lattices with $L \geqslant L_{\rm min}$, for $N=7$ and $N=17$.

| $N=7$ | | | |
|---|---|---|---|
| $L_{\rm min}$ | $A$ | $\beta/\nu$ | $\chi^2$/d.o.f. |
| 32 | 1.00653(48) | 0.12210(08) | 5.5 |
| 64 | 1.00858(70) | 0.12243(12) | 3.7 |
| 128 | 1.01074(94) | 0.12277(15) | 2.0 |
| 256 | 1.0146(16) | 0.12336(26) | 0.40 |
| 384 | 1.0162(22) | 0.12359(34) | 0.19 |
| 512 | 1.0177(38) | 0.12381(56) | 0.16 |
| 640 | 1.0185(57) | 0.12393(84) | 0.28 |

| $N=17$ | | | |
|---|---|---|---|
| $L_{\rm min}$ | $A$ | $\beta/\nu$ | $\chi^2$/d.o.f. |
| 32 | 1.00388(51) | 0.12111(09) | 7.98 |
| 64 | 1.00620(69) | 0.12149(12) | 3.58 |
| 128 | 1.0089(11) | 0.12191(18) | 1.54 |
| 256 | 1.0107(15) | 0.12219(24) | 0.74 |
| 384 | 1.0113(24) | 0.12228(36) | 1.36 |

We can cross-check our determination of the critical exponent $\eta$ by an independent method, which does not rely on the prior knowledge of the critical coupling, but is based on the construction of a suitable universal quantity [12, 8]. The idea is to plot $\chi_L^{(M_R)} L^{\eta-2}$ versus $B_4^{(M_R)}$ and to look for the value of $\eta$ which optimizes the overlap of curves from different volumes. We found that, both in $Z(7)$ and $Z(17)$, $\eta = 1/4$ is this optimal value, since it gives the best overlap of these curves in

**Table 2:** Results of the fit to the data of $\chi_L^{(M)}(\beta_c^{(1)})$ with the scaling law (2.5-right) on $L^2$ lattices with $L \geqslant L_{\min}$, for $N = 7$ and $N = 17$.

| $N=7$ | | | |
|---|---|---|---|
| $L_{\min}$ | $B$ | $\gamma/\nu$ | $\chi^2$/d.o.f. |
| 32 | 0.00558(07) | 1.7402(23) | 2.38 |
| 64 | 0.00540(09) | 1.7457(29) | 1.32 |
| 128 | 0.00522(12) | 1.7508(38) | 0.68 |
| 256 | 0.00518(20) | 1.7520(61) | 0.84 |
| 384 | 0.00546(33) | 1.7443(93) | 0.70 |
| 512 | 0.00489(52) | 1.760(16) | 0.28 |
| 640 | 0.00444(76) | 1.775(25) | 0.0066 |

| $N=17$ | | | |
|---|---|---|---|
| $L_{\min}$ | $B$ | $\gamma/\nu$ | $\chi^2$/d.o.f. |
| 32 | 0.00559(08) | 1.7372(26) | 2.7 |
| 64 | 0.00532(11) | 1.7453(35) | 0.39 |
| 128 | 0.00521(15) | 1.7484(46) | 0.16 |
| 256 | 0.00514(23) | 1.7507(71) | 0.15 |
| 384 | 0.00522(31) | 1.7483(92) | 0.13 |

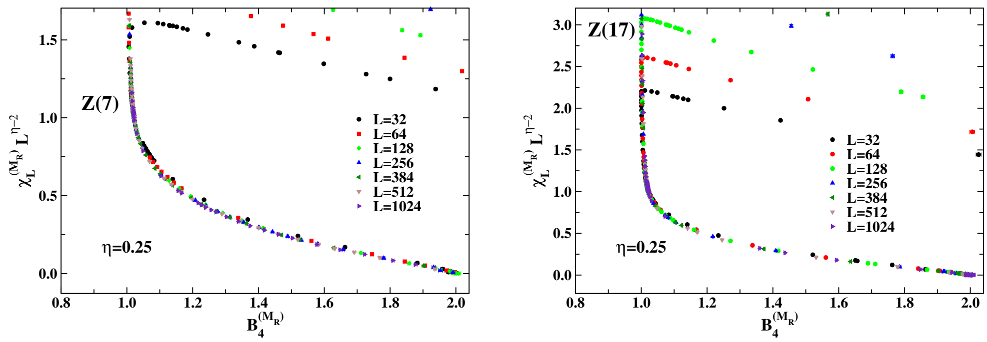


**Figure 3:** Correlation between $\chi_L^{(M_R)} L^{\eta-2}$ and the Binder cumulant $B_4^{(M_R)}$ for $\eta = 0.25$ in $Z(7)$ (left) and $Z(17)$ (right) on lattices with $L$ ranging from 128 to 1024.

the region of values corresponding to the first phase transition, *i.e.* the lower branch of the curves of Fig. 3. This result for $\eta$ agrees with the determinations $\eta = 2-\gamma/\nu$ from the FSS analysis.

As for the second critical coupling $\beta_c^{(2)}$, we used the same method adopted for $\beta_c^{(1)}$, but applied now to $B_4^{(M_R)}$ and $m_\psi$. Our best estimates are

$$N=7: \quad \beta_c^{(2)} = 1.8775(75)\,, \qquad\qquad N=17: \quad \beta_c^{(2)} = 10.13(12)\,.$$

The standard FSS analysis applied to the susceptibility $\chi_L^{(M_R)}$ of the rotated magnetization $M_R$ at $\beta_c^{(2)}$ leads to the result for the critical indices $\gamma/\nu$ given in Table 3 [1].

Also in this case the critical index $\eta$ can be determined by an independent method, irrespectively of the knowledge of $\beta_c^{(2)}$: $M_R L^{\eta/2}$ is plotted versus $m_\psi$ and the value of $\eta$ is searched for, which optimizes the overlap of data points coming from different volumes. The results we found for $\eta$ in $Z(7)$ and $Z(17)$ are in perfect agreement with the theoretical prediction $\eta^{(2)} = 4/N^2$ (see Fig. 4).

Finally, in Fig. 5 we present the behavior with $\beta$ of the helicity modulus (2.2). This quantity is constructed in such a way that it should exhibit a discontinuous jump (in the thermodynamic limit)

[1] We do not report in this work the determinations of $\beta/\nu$ by the FSS analysis of the rotated magnetization $M_R$, since they are affected by large statistical and systematic uncertainties.

**Table 3:** Results of the fit to the data of $\chi_L^{(M_R)}(\beta_c^{(2)})$ with the scaling law (2.5-right) on $L^2$ lattices with $L \geqslant L_{min}$, for $N = 7$ and $N = 17$.

| $N = 7$ | | | | $N = 17$ | | | |
|---|---|---|---|---|---|---|---|
| $L_{min}$ | $A$ | $\gamma/\nu$ | $\chi^2$/d.o.f. | $L_{min}$ | $B$ | $\gamma/\nu$ | $\chi^2$/d.o.f. |
| 32 | 0.8767(37) | 1.92340(71) | 2.02 | 32 | 0.9319(47) | 1.98933(89) | 1.65 |
| 64 | 0.8833(47) | 1.92219(87) | 1.41 | 64 | 0.9408(68) | 1.9878(12) | 1.38 |
| 128 | 0.8858(57) | 1.9217(11) | 1.57 | 128 | 0.9533(89) | 1.9857(16) | 0.67 |
| 256 | 0.8997(93) | 1.9193(16) | 1.02 | 256 | 0.954(16) | 1.9856(27) | 0.83 |
| 384 | 0.916(15) | 1.9166(25) | 0.68 | 384 | 0.950(24) | 1.9861(40) | 1.09 |
| 512 | 0.921(24) | 1.9158(39) | 0.98 | 512 | 0.931(38) | 1.9892(62) | 1.44 |
| 640 | 0.942(34) | 1.9124(54) | 1.07 | 640 | 0.911(59) | 1.9925(98) | 2.67 |

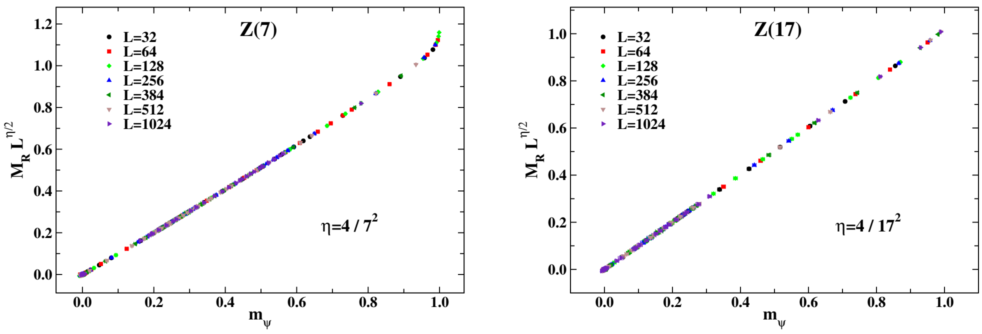


**Figure 4:** Correlation between $M_R L^{\eta/2}$ and $m_\psi$ for $\eta = 4/7^2$ in $Z(7)$ (left) and $\eta = 4/17^2$ in $Z(17)$ (right), on lattices with L ranging from 128 to 1024.

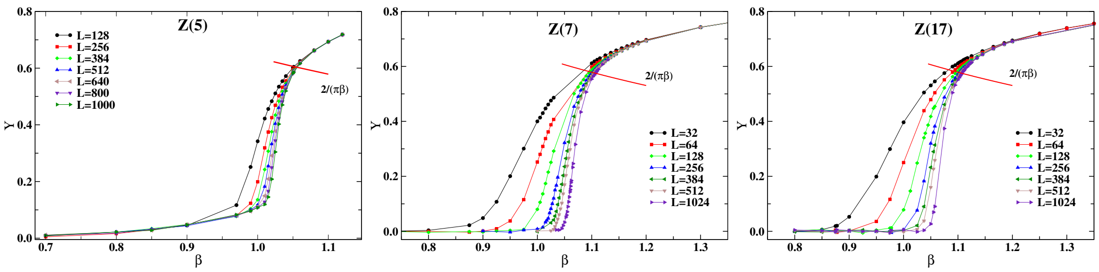


**Figure 5:** Helicity modulus versus $\beta$ in $Z(5)$, $Z(7)$ and $Z(17)$ on lattices with various sizes.

at the critical temperature separating the disordered phase from the massless one, if the transition is of infinite order (BKT). Since the Kosterlitz-Thouless RG equations for the $XY$ model [1, 13, 14] lead to the prediction that the helicity modulus $\Upsilon$ jumps from the value $2/(\pi\beta)$ to zero at the critical temperature, one can check if the same occurs for vector Potts models. In Fig. 5 we plot a red line, representing the function $2/(\pi\beta)$; the crossing between this line and the curves formed by data points of $\Upsilon$ approaches indeed $\beta_c^{(1)}$ when the lattice size increases.

We studied also the specific heat at the two transitions, finding that, in contrast to the case of first- and second-order phase transitions, it does not reflect any nonanalytical critical properties at the critical temperatures, thus confirming that only BKT transitions are at work here.

## 3. Summary

We have determined the two critical couplings of the $2D$ $Z(N = 7, 17)$ vector models and given estimates of the critical indices $\eta$ at both transitions. Our findings support for all $N \geq 5$ the standard scenario of three phases: a disordered phase at high temperatures, a massless or BKT one at intermediate temperatures and an ordered phase, occurring at lower and lower temperatures as $N$ increases. This matches perfectly with the $N \to \infty$ limit, *i.e.* the $2D$ $XY$ model, where the ordered phase is absent or, equivalently, appears at $\beta \to \infty$. Considering the determinations of the critical coupling $\beta_c^{(1,2)}(N)$ obtained in this work for $N = 7$ and 17, together with those obtained for $N = 5$ in Refs. [7, 8] and those for $N = 6$, 8 and 12 of Ref. [6], one can verify that $\beta_c^{(1)}(N)$ approaches the $2D$ $XY$ value, $\beta_c^{(1)} = 1.1199$, exponentially in $N$ or even faster, while $\beta_c^{(2)}(N)$ grows to infinity with $N^2$. A more detailed analysis of the $N$-behavior of critical couplings will be given elsewhere [15].

We have also found that the values of the critical index $\eta$ at the two transitions are compatible with the theoretical expectations.

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