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Investigations of QCD at non-zero isospin density

Zhifeng Shi*
Department of Physics, College of William and Mary, Williamsburg, VA 23187, USA
Jefferson Laboratory, 12000 Jefferson Avenue, Newport News, VA 23606, USA
E-mail: zshi@email.wm.edu

William Detmold
Department of Physics, College of William and Mary, Williamsburg, VA 23187, USA
Jefferson Laboratory, 12000 Jefferson Avenue, Newport News, VA 23606, USA
E-mail: wdetmold@wm.edu

We investigate the QCD phase diagram as a function of isospin chemical potential at a fixed temperature by directly putting large numbers of π^+s into the system. Correlation functions of N π^+-systems involves $N!N!$ contractions, and become extremely expensive when N is large. In order to alleviate this problem, a recursion relation of correlation functions has been derived in Ref. [1] that substantially reduces the number of independent contractions needed and makes the study of many pions systems be possible. In this proceeding this method is investigated numerically. We have also constructed a new method that is even more efficient, enabling us to study systems of up to 72 π^+'s.

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1. Introduction

The QCD phase diagram has been studied with many methods, but there are still many questions waiting to be answered. At zero isospin density, there is a crossover (phase transition) from a confinement phase at low temperature to a decomfinement phase at higher temperature. From chiral perturbation theory (χPT), we expect that when the isospin chemical potential reaches the mass of a single pion, pions start to condense, and at higher isospin chemical potential there might be a crossover or a phase transition to another state [2]. Systems containing up to 12 π^+s have been studied by directly computing the required contractions in Ref. [5] [3], finding agreement with the expectations from χPT. In order to put more π^+s into the system, a second source is required because of the Pauli principle, and studying systems of more than 12 π^+ becomes almost impossible by directly computing all possible contractions.

In Ref. [1], a recursion relation of correlation functions of n-π^+ systems is constructed which significantly reduces the amount of work needed, enabling us to study systems up to 24 π^+'s. Recursion relations of M species systems have also been discussed in the same paper. Even with the recursion relation, studying a 3-source system (36 pions) still consumes a substantial amount of time. In order to overcome this problem we have constructed a new method which allows practical calculation of up to $72\pi^+$s.

In this proceeding, we first review the methodology of the recursion relation in position space and then extend its application to momentum space, after that we present studies of systems containing up to $24\pi^+$s by applying the recursion relation in momentum space. At last we present results from the new method to study large number of pion systems.

2. Methodology of the recursion relation

2.1 Recursion relation in position space

In order to explore systems containing up to $12M$ π^+'s, M different sources are required because of the Pauli principle. A correlation function of a system with n_i-π^+s in the i^{th} source is:

$$C_{(n_1\pi_1^+,\ldots,n_m\pi_m^+)}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x},t) \right)^{\bar{n}} \left(\pi^-(\mathbf{y_1},0)\right)^{n_1} \ldots \left(\pi^-(\mathbf{y_m},0)\right)^{n_m} \right\rangle , \qquad (2.1)$$

where $\bar{n} = \sum_{i=1}^{m} n_i$. Calculating this correlation function according to Wick's theorem involves $\bar{n}!\bar{n}!$ contractions, which make the study for a system of large number of π^+'s extremely time consuming. However the recursion relation of uncontracted correlation functions $Q_{(n_1,n_2,\ldots,n_m)}(t)$, makes the study of such systems feasible. The correlation function can be expanded as:

$$C_{(n_1\pi_1^+,\ldots,n_m\pi_m^+)}(t) = (-)^{\bar{n}} \left(\prod_i n_i! \right) \langle\, Q_{(n_1,n_2,\ldots,n_m)}(t) \,\rangle \quad , \qquad (2.2)$$

where$\langle\ \rangle$ denotes spin color trace, and $Q_{(n_1,n_2,\ldots,n_m)}(t)$ satisfies the following ascending recursion relation:

$$\begin{aligned} Q_{(n_1+1,n_2,\ldots,n_m)} &= \langle\, Q_{(n_1,n_2,\ldots,n_m)} \rangle\, P_1 \;-\; \bar{n}\, Q_{(n_1,n_2,\ldots,n_m)} P_1 \\ &\ldots + \langle\, Q_{(n_1+1,n_2,\ldots n_k-1,\ldots,n_m)} \rangle\, P_k \;-\; \bar{n}\, Q_{(n_1+1,n_2,\ldots n_k-1,\ldots,n_m)} P_k \\ &\ldots + \langle\, Q_{(n_1+1,n_2,\ldots,n_m-1)} \rangle\, P_m \;-\; \bar{n}\, Q_{(n_1+1,n_2,\ldots,n_m-1)} P_m \quad , \end{aligned} \qquad (2.3)$$

and the initial conditions are $Q_{(1,0,\ldots,0)} = P_1 = A_1, Q_{(0,1,\ldots,0)} = P_2 = A_2, \cdots$, where A_i's are uncontracted single pion correlators. Descending recursion relations can also be derived:

$$Q_{\mathbf{n}} = \sum_{k=1}^{M} \frac{1}{N+1-\bar{n}} \langle Q_{\mathbf{n}+\mathbf{1}_k} A^{-1} \left(P_k \cdot A^{-1}\right)\rangle \cdot I_N - Q_{\mathbf{n}+\mathbf{1}_k} A^{-1} \left(P_k \cdot A^{-1}\right) \tag{2.4}$$

where $\mathbf{n} = (n_1, n_2, \cdots, n_m)$, $\mathbf{1}_k = (0,0,\cdots,1,0,\cdots)$ with only the k^{th} nonvanishing unit element, and $Q_{12,\ldots,12}$, P_k and A are constructed in the following way:

$$\begin{aligned} Q_{12,\ldots,12} &= (N-1)!det(A)\cdot I_N \\ P_k &= \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline A_{k1}(t) & A_{k2}(t) & \ldots & A_{kM} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \quad A = \left(\begin{array}{c|c|c|c} A_{11}(t) & A_{12}(t) & \ldots & A_{1M} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline A_{k1}(t) & A_{k2}(t) & \ldots & A_{kM} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline A_{M1}(t) & A_{M2}(t) & \ldots & A_{MM} \end{array}\right) \end{aligned} \tag{2.5}$$

where $A_{i,j}(t)$ is defined as $A_{i,j}(t) = \sum_{\mathbf{x}} S(\mathbf{x}_i, \mathbf{x}) S^{+}(\mathbf{x}_j, \mathbf{x})$.

One way to construct $A_{i,j}$ is shown in the left plot of Fig. 1. Correlation functions of two species from multiple sources have similar recursion relations, for detail discussion about the recursion relations see Ref. [1].

2.2 The recursion relation in momentum space

A correlation function of a system having n_1-π^+s in the first source and n_2-π^+s in another source with total momentum $n_1\mathbf{p}_{f_1} + n_2\mathbf{p}_{f_2}$ is:

$$C_{n_1\pi^+, n_2\pi^+}(t) = \left\langle \prod_{i=1}^{2} \left(\sum_{\mathbf{x}_i, \mathbf{x}\prime_i} e^{-i\left(\mathbf{p}_1^i \mathbf{x}_i - \mathbf{p}_2^i \mathbf{x}\prime_i\right)} \overline{u}(\mathbf{x}_i, t)\, \gamma_5 d(\mathbf{x}\prime_i, t) \right)^{n_i} \cdot \prod_{j=1}^{\overline{n}} \left(\sum_{\mathbf{y}_j} e^{i\mathbf{p}_{f_j}\mathbf{y}_j} \overline{d}(\mathbf{y}_j, 0)\, \gamma_5 u(\mathbf{y}_j, 0) \right) \right\rangle$$

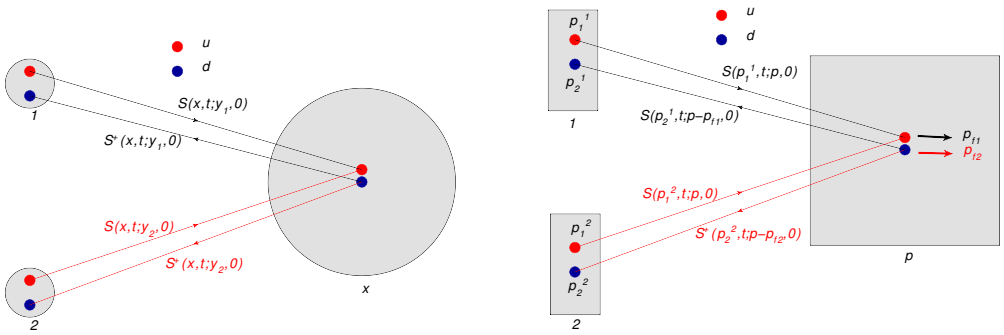
where $\overline{n} = n_1 + n_2$. Momentum conservation requires that $n_1\mathbf{p}_1^1 + n_2\mathbf{p}_1^2 - n_1\mathbf{p}_2^1 - n_2\mathbf{p}_2^2 = \sum_{j=1}^{\overline{n}} \mathbf{p}_{f_j}$ must be satisfied to get non-vanishing $C_{n_1\pi^+, n_2\pi^+}$. Each choice of $\mathbf{p}_j^i, i, j = 1,2$ satisfying this relation is a separate measurement. By replacing propagators in position space by propagators in momentum space, a similar recursion relation still holds. The only difference is the construction of uncontracted correlation functions $A_{i,j}$ defined as $A_{i,j}(t) = \sum_{\mathbf{p}} S(\mathbf{p}_i^1, \mathbf{p})\, S^{+}\left(\mathbf{p}_j^2, \mathbf{p} - \mathbf{p}_{f_j}\right)$, which are compared on Fig. 1.

3. Results

Because of the finiteness of the temporal extent and the factorisable nature of the multi-hadron systems being constructed, thermal effects are particularly important in multi-hadron systems and the correlation functions assume the form:

$$C_{n\pi^+}(t) = \sum_{m=0}^{n} \binom{n}{m} Z_m^n e^{-(E_m + E_{n-m})T/2} \cosh\left((E_m - E_{n-m}) \cdot (t - T/2)\right) + \cdots \tag{3.1}$$

where dots represent higher excitations, T is the temporal extent and E_n is the energy of a rest system of n-π^+s. The ground state comes from all π^+'s propagating in the same direction in time, and thermal states are from some π^+'s propagating in one direction while the rest propagate in the opposite direction.



(a) (b)

Figure 1: The left figure shows how to construct uncontracted correlators in spatial space and the right figure shows how to get its counterparts in momentum space. $A_{i,j}$ is constructed by following the line from source i to $x(p)$, returning to source j, multiplying the $S(S^\dagger)$ with respect to each line and summing over $x(p)$.

3.1 Verify the dispersion relationship

Calculations have been performed using anisotropic $\{16^3, 20^3, 24^3\} \times 128$ lattices with anisotropy $\xi = 3.5$ at a quark mass corresponding to a $m_\pi = 390$Mev. On the lattice, only discrete momentum $\frac{2\pi}{L}n$ are allowed. $E_{m\pi^+}$ of systems with total momentum $\mathbf{p}_t = m \cdot \mathbf{p}$, for $\mathbf{p} = (0,0,1),(0,1,1),(0,0,2)$, have been extracted for $m = 4,3,2$ respectively and are fitted into the dispersion relation: $\frac{E^2(m,\mathbf{p}_t)}{m^2} - (\frac{\mathbf{c}\cdot\mathbf{p}_t}{m})^2 = \frac{E^2(m,\mathbf{0})}{m^2}$, where $\mathbf{p}_t = m \cdot \mathbf{p}$ returning $|\mathbf{c}| = 1.015(32)$, which confirms that these many hadron systems are describable in special relativity (thermal states would have a different dependence).

3.2 One species from single source

For notational convenience, in the following $\mathbf{p}_1^1$ is denoted as $\mathbf{p}_1$, $\mathbf{p}_2^1$ as $\mathbf{p}_2$, $\mathbf{p}_1^2$ as $\mathbf{p}_3$ and $\mathbf{p}_2^2$ as $\mathbf{p}_4$. Azimuthal symmetry ensures many combinations of $\mathbf{p}_1$, $\mathbf{p}_2$ be separate measurements of the same physics, which provide more statistics. As $E_{n\pi^+}$ extracted from different choices of $\mathbf{p}_1$ and $\mathbf{p}_2$ agree with each other within errors, we choose $\mathbf{p}_1 = \mathbf{p}_2 = (1,1,1)$ for further discussions. Extracting $E_{n\pi^+}$ by fitting to functions with and without one excited state in addition to the thermal states discussed above from different time intervals produces consistent results, as is shown in Fig. 2.

Decomposing $C_{n\pi^+}(t)$ into different contributions gives insight into how much each state in Eq. 3.1 contributes. In Fig. 2, $C_{12\pi^+}(t)$ is shown. In this figure, the green line is from the first excited state, the blue one is the ground state and other lines are thermal states. It is remarkable that the zero temperature ground state is not dominant in any region of this correlator. Even for $C_{12\pi^+}(t)$ the ground state is strongly contaminated by both thermal states and excited states, and it becomes difficult to extract the ground state energy from $C_{n\pi^+}(t)$, $n > 12$, with the temporal extent of these configurations.

3.3 One species from two sources

By choosing $\mathbf{p}_1 = \mathbf{p}_2$, $\mathbf{p}_3 = \mathbf{p_4}$ but $\mathbf{p}_1 \neq \mathbf{p}_3$ correlation functions of systems having up to $24\pi^+$ have been computed with the same recursion relation, but $E_{n\pi^+}$ is hard to extract for $n > 12$. As there are more ways to construct a n-π^+ system, the recursion relation forces us to calculate all $Q_{n_1\pi^+,n_2\pi^+}$ for all pairs $n_1, n_2 = 1 \cdots 12$, before getting to $Q_{12\pi^+,12\pi^+}$, which costs $O(100)$ times

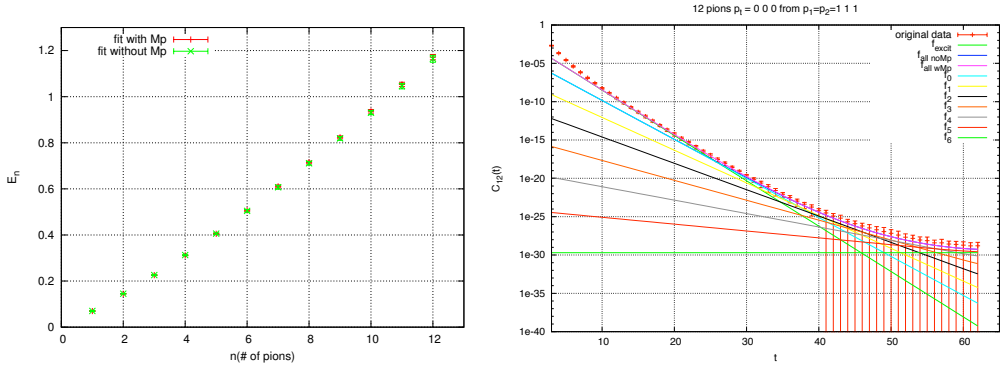


Figure 2: The left panel shows energies of a rest system of n-π^+($E_{n\pi^+}$) extracted from both methods and the right panel shows $C_{12\pi^+}(t)$, which is decomposed into contributions from excited state, the ground state and all thermal states.

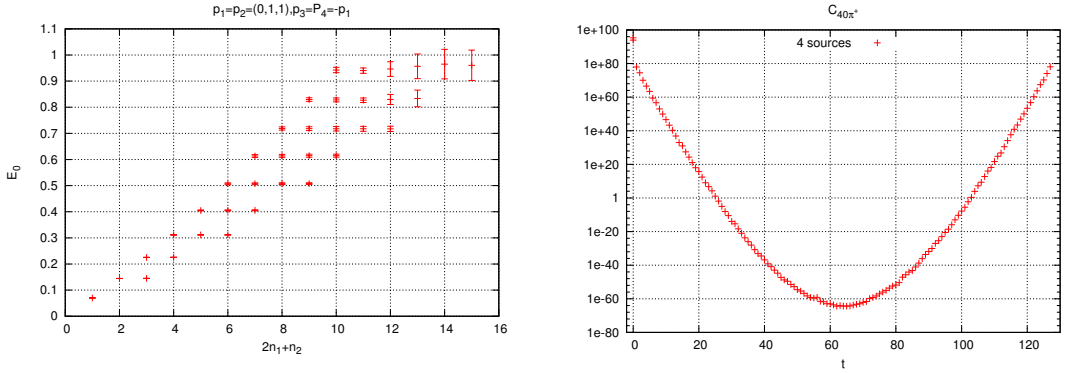


Figure 3: Left panel shows $E_{n_1\pi^+,n_2\pi^+}$ extracted from $C_{n_1\pi^+,n_2\pi^+}$ plotted against $x = 2n_1 + n_2$. Statistical error and systematic error are added up in quadrature. Right panel is $C_{40\pi^+}(t)$ calculated from the new method.

more than the one source case. Similarly studying system of 36 π^+'s requires a third source, and becomes $O(100)$ times more expensive again. In order to overcome this difficulty, we have developed a new method which is much faster than the recursion relation method and can easily study systems of at least 72 mesons [6]. This new method is based on the fact that the ground state energies extracted from all $C_{n_1,n_2}(t)$ for fixed $n_1 + n_2$ are the same as they correspond to the same spectrum but with different overlaps, shown in Fig. 3.

For systems containing large number of π^+'s, thermal states dominate the correlation function at later time slices, while excited states dominate earlier time slices, and extracting the ground state energy becomes extremely difficulty and systematic error is hard to control. In order to get better ground state signals we have also studied the system on $20^3 \times 256$ lattices (with other parameters unaltered) using the new method mentioned above.

3.4 Isospin chemical potential (μ_I) and pressure

The isospin chemical potential, the μ_I, is defined as $\mu_I(n) = \frac{dE}{dn}$, which is approximated by using a backward finite difference on the lattice. As the systematic errors of ground state energies

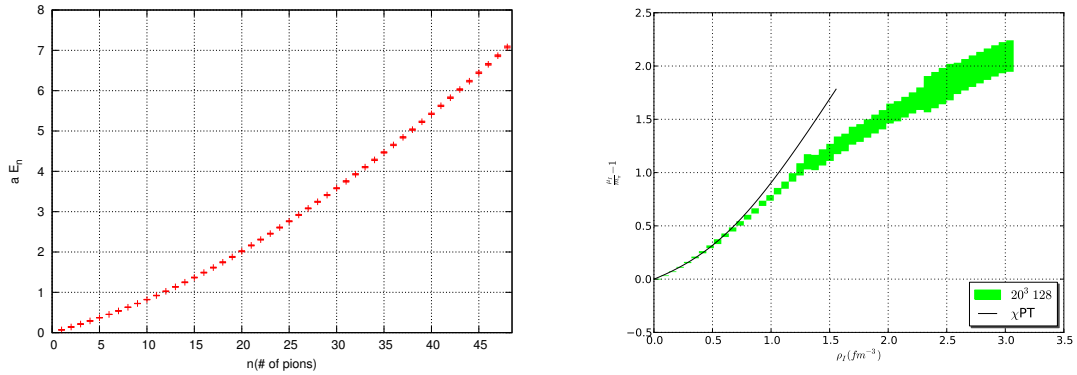


Figure 4: The left panel shows energies of a rest system of n-π^+($E_{n\pi^+}$) extracted from $20^3 \times 256$ lattice, and the right panel shows the isospin chemical potential as a functions of isospin density. Lattice space $a = 0.125fm$. The solid black line is from χPT [2]

extracted from the $T = 128$ lattices are large for systems with large number of π^+'s, we do not extract μ_I from these lattices. The $20^3 \times 256$ lattices give much better ground state signals, having an extremely long plateau in the effective mass plot, and a one exponential fit is enough to get the ground state energy. In Fig. 4, we show the ground state energies and isospin chemical potentials from the $20^3 \times 256$ lattice calculated with the newly constructed method mentioned above. μ_I is consistent with predictions from χPT [2] at small density and starts to deviate from them for larger densities.

Since calculations have been done on several lattices sizes $\{16^3, 20^3, 24^3\} \times 128$, the pressure ($P$) can be derived as function of density by utilizing the discrete version of $P = \frac{dE}{dV}$ for fixed number of π^+. One definition is the following:

$$P(\rho_I) = \frac{E_n^{L_i} - E_n^{L_j}}{(L_i \cdot a)^3 - (L_j \cdot a)^3}$$
$$\rho_I = \frac{n}{((L_i + L_j) \cdot a/2)^3} \tag{3.2}$$

where $L_i, L_j = 16, 20, 24$ and $a = 0.125fm$. Although we can not extract $E_{n\pi^+}$ reliably for large n on these ensambles, the pressure can still be studied as a function of isospin density from $E_{n\pi^+}$ for small n. The pressure is plotted as a function of isospin density in Fig. 5.

4. Conclusion

We have calculated correlation functions of systems up to $24\pi^+$s in momentum space by applying the recursion relation of Ref. [1], verified the dispersion relation, and extracted ground state energies of n-π^+ systems. Because of the contamination from thermal states in later time slices and excited states in earlier time slices, it is extremely difficult to see the ground state of systems having more than $12\pi^+$'s for $20^3 \times 128$ anisotropic lattices ($a_t \sim 0.04$) at this pion mass. In order to reduce contamination from thermal states, we also use $20^3 \times 256$ lattices. Correlation functions from these larger temporal extent lattices give clear ground state signal, and a one exponential fit is sufficient enough to reliably extract the ground state energy even for 72 mesons.

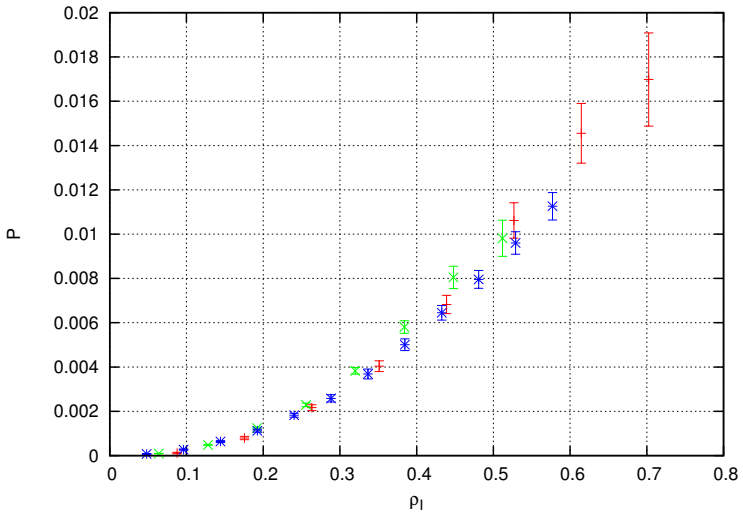


Figure 5: The pressure is computed from energy differences for fixed n on different lattice sizes. Red points are from 16^3-20^3 lattices, green points are from 16^3-24^3 lattices, and blue points are from 20^3-24^3 lattices.

Although the recursion relation requires much less time than direct contractions, it becomes $O(100)$ times more expensive with an additional source. Studying 3 sources system becomes quite expensive and studying 4 sources system becomes impractical. In order to overcome this difficulty, we have developed a new method. This new method is far more efficient than the recursion relation method, allowing us to study systems of up to 72 mesons [6].

5. Acknowledgement

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