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Loop quantum cosmology: a brief introduction

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Quantum geometric effects, as understood in loop quantum gravity, lead to a resolution of singularities in various spacetimes in loop quantum cosmology. A generic prediction of several models investigated so far is the existence of bounce in the Planck regime. We outline the quantization procedure for the spatially flat isotropic homogeneous model with a massless scalar field, which has served as a template for the loop quantization of various models. Key properties of the bounce and physical implications are discussed.

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1. Introduction

An important expectation from any theory of quantum gravity, is whether it provides insights on the problem of classical singularities, which according to the theorems of Penrose, Hawking and Geroch are the generic features of general relativity (GR) [1]. A simple example of such a singularity, is the big bang singularity, which occurs for all matter satisfying weak energy condition (WEC) if we evolve an expanding branch of the Friedmann-Robertson-Walker (FRW) universe backward in time. As the singularity is approached, curvature invariants diverge and geodesic evolution breaks down. When spacetime curvature is in the Planck regime, one expects that quantum properties of spacetime would become significant leading to a resolution of space-like singularities. New physics from such a theory of quantum spacetime or quantum gravity is not only expected to shed insights on the generic resolution of singularities, but also provides an invaluable opportunity to answer various fundamental questions associated with the physics of the early universe, such as: Is the spacetime beyond the big bang foamy or classical? At what scales does the classical spacetime emerges? What are the implications for the probability for inflation to occur? What are the signature of new physics in the cosmic microwave background? and so on.

Loop quantum cosmology (LQC) is a background independent non-perturbative quantization of homogeneous cosmological spacetimes, based on loop quantum gravity (LQG) [2, 3, 4, 5] (See lectures by Giesel and Sahlmann for a detailed introduction to the methods of LQG in this proceedings [6]). LQC began with seminal works of Bojowald which indicated resolution of singularities at a kinematical level of the quantum theory [7, 8]. A rigorous development of these ideas commenced with the work of Ashtekar, Bojowald and Lewandowski [9]. A first complete quantization of a cosmological spacetime in LQC, in the sense of availability of physical Hilbert, a family of Dirac observables and detailed physical implications, was performed for the quantization of the flat ($k = 0$) isotropic model sourced with a massless scalar field [10, 11, 12]. As in LQG, the elementary variables in LQC are the holonomies of SU(2) connection and fluxes of triads (which due to homogeneity assumption turn out to be proportional to triads). The resulting quantum geometry in LQC, as in LQG, is discrete. This is in contrast to the Wheeler-DeWitt quantum cosmology, based on continuum differentiable geometry. Unlike, the classical theory where all the solutions in this model are singular, and the Wheeler-DeWitt quantum cosmology, which fails to resolve the big bang singularity, in LQC, evolution via the quantum Hamiltonian constraint is non-singular, leading to a bounce of the universe when energy density reaches a maximum value, $\rho = \rho_{\rm max} \approx 0.41 \rho_{\rm Planck}$. The robustness of bounce, first observed in various numerical simulations performed in Ref. [10, 11, 12], has been established by using an exactly soluble model (sLQC) [13], where the bounce is proved to occur for a dense set of states in the physical Hilbert space. The energy density at which bounce occurs in numerical simulations with states which correspond to macroscopic universes at late times, agrees with the supremum of the expectation values of the Dirac observable corresponding to energy density, in the physical Hilbert space of sLQC. Further, sLQC has also provided insights on the behavior of fluctuations across the bounce, which turn out to be tightly constrained, thus preserving semi-classicality, and showing that a semi-classical state before the bounce evolves to a semi-classical state after the bounce and vice versa [14, 15, 16]. sLQC has also led to insights on spin foam models using path integral approach [17, 18, 19] and on applications of consistent histories approach to quantization of spatially flat isotropic models

[20, 21, 22]. These developments have been supplemented with the important results on mathematical aspects of these models [23, 24, 25, 26].

Using the simplifications due to the underlying symmetries of the homogeneous spacetimes, the quantization program of LQG has been successfully carried out in LQC for various cosmological spacetimes in recent years by various groups, and important insights on answers to above questions have been obtained (for an up to date extensive review, see [27]). In particular, complete quantization has been performed for spatially flat isotropic models sourced with positive cosmological constant [29, 28], negative cosmological constant [30], inflationary potential [31], spatially closed [32, 33] and open [34, 35] models with a massless scalar field, and anisotropic models [36, 37, 38, 39]. In all these models, initial singularity is shown to be resolved, which is a direct ramification of the non-local nature of the field strength of the connection in the quantum theory. Interestingly, for states which evolve to a macroscopic universe at late times, the results of singularity resolution and the new physics at the Planck scale can be captured using an effective spacetime description resulting from effective Hamiltonian [40, 41, 42]. The effective dynamics captures the underlying quantum evolution to an excellent accuracy for various models in LQC [11, 12, 33, 28, 30, 43, 44], and has been used to study detailed physical implications of loop quantization of cosmological spacetimes. As an example, using effective spacetime description, singularity resolution in LQC can be understood occurring due to bounds on the growth of spacetime curvature, which plays a conjugate role to geometry [45, 46, 47], and insights on the generic resolution of strong singularities and geodesic extendability in flat isotropic [45, 48] and Bianchi-I models [46] have been obtained. Effective equations have also been used to gain insights on constraining quantization ambiguities [49, 50], and have been used to explore the physics of Gowdy models [51]. A lot of activity is devoted to applying effective equations to understand the signatures of quantum geometry in cosmological perturbations [52]. Going beyond the effective treatment, in this direction, promising progress has been made to understand the effects of quantum spacetime, using the analysis of [53], on primordial perturbations [54].

Due to space limitations, it is not possible here to discuss many of the above interesting results obtained by various authors. For a detailed discussion of various results, we refer the reader to the following reviews which cover various developments in the field [27, 55, 56, 57, 58]. The goal of this article is not to review above results, but rather to demonstrate basic techniques and results for the simplest model in LQC, a spatially flat model sourced with a massless scalar field as first performed in Refs. [10, 11, 12], and the corresponding exactly soluble model [13]. Since techniques developed for this model, have been used for various other models, the analysis and discussion in this manuscript provides a useful framework to understand the way loop quantization is performed for more general homogeneous cosmological models. In Sec. II, we discuss the classical theory and the loop quantization of the spatially flat model with a massless scalar field and discuss some of the physical implications using the volume representation. In Sec. III we discuss the the way, spatially flat isotropic model can be solved exactly in the b representation (conjugate to volume representation), and discuss the genericness of bounce and supremum of the energy density [13]. We summarize the main results with a discussion.

2. Loop quantum cosmology: Spatially flat isotropic model

The goal of this section is to discuss the quantization of cosmological spacetimes in LQC using spatially flat isotropic model with a massless scalar field as an example. It is based on the works in Ref. [12, 13]. This model provides a stage to understand various subtleties with loop quantization in detail, such as the way matter degree of freedom is successfully used as an internal clock, inner product and Dirac observables can be introduced, and physics at the Planck scale can be extracted. We start with the discussion of the classical phase space in Ashtekar variables. We then discuss the quantum kinematics and the way quantum difference equation emerges from the quantum constraint and summarize the main features of new physics. We conclude this section, with a brief discussion of the effective spacetime description obtained from this model.

2.1 Classical theory

We consider spatially flat $k = 0$ Friedmann-Robertson-Walker (FRW) model with a non-compact spatial manifold $\Sigma = \mathbb{R}^3$, with a spatial metric: $q_{ab} = a^2 \mathring{q}_{ab}$, and the spacetime metric given by

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + a^2(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)\ . \tag{2.1}$$

Since the spatial manifold is non-compact, in order to define the symplectic structure we need to introduce a fiducial cell $\mathscr{V}$. This cell will be chosen as a cubical one, with volume V_o with respect to the fiducial metric $\mathring{q}_{ab}$ on the spatial manifold. The physical volume of the cell is given by $V = V_o a^3$.

In geometrodynamics, the gravitational phase space variables are the scale factor a and its conjugate $p_a = -a\dot{a}$, where a 'dot' denotes derivative with respect to the proper time t. These variables satisfy $\{a, p_a\} = \frac{4\pi G}{3V_o}$. If the matter is considered as scalar fields, the phase space variables are ϕ and its momentum p_ϕ, which satisfy $\{\phi, p_{(\phi)}\} = 1$. On the other hand, in LQC, phase space variables for the gravitational sector are obtained from the symmetry reduction of the phase space variables in LQG: the SU(2) connection A_a^i and the triad E_i^a. Given the symmetries of FRW spacetime, it is possible to express A_a^i and E_i^a such that [9],

$$A_a^i = c V_o^{-1/3} \mathring{\omega}_a^i, \quad E_i^a = p \sqrt{\mathring{q}}\, V_o^{-2/3} \mathring{e}_i^a\ . \tag{2.2}$$

Here $\mathring{e}_i^a$ and $\mathring{\omega}_a^i$ are the fiducial triad and co-triad compatible with $\mathring{q}_{ab}$. The symmetry reduced connection and triad variables satisfy

$$\{c, p\} = \frac{8\pi\gamma G}{3}\,. \tag{2.3}$$

Here $\gamma \approx 0.2375$ denotes the Barbero-Immirzi parameter, whose value is fixed by the black hole thermodynamics in LQG. The triad is related kinematically to the the metric variables as $|p| = V_o^{2/3} a^2$, where the modulus sign arises due to two possible orientations of the triad. The relationship of the connection with the metric variables is dynamical and can be derived from the Hamiltonian constraint. If matter is chosen as a massless scalar field, the classical Hamiltonian constraint is given by

$$C_{\rm cl} = -\frac{3}{8\pi G\gamma^2} |p|^{\frac{1}{2}} c^2 + \frac{p_{(\phi)}^2}{2|p|^{\frac{3}{2}}} \approx 0 \tag{2.4}$$

Using the Hamilton's equation for p, one obtains: $c = \gamma V_o^{1/3}\,\dot{a}$ on the physical solutions of the classical theory. Substituting c in terms of time derivative of the scale factor and using the vanishing of the Hamiltonian constraint, along with the definition of energy density ρ (equal to $p_{(\phi)}^2/(2|p|^3)$), one obtains the classical Friedmann equation:

$$H^2 := \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\,\rho, \tag{2.5}$$

where H denotes the Hubble rate. In a similar way, using Hamilton's equation for c, we are led to the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3P) \tag{2.6}$$

where P denotes the pressure of the matter field (in the case of massless scalar it equals ρ). Friedmann and Raychaudhuri equation imply the following conservation law:

$$\dot{\rho} + 3H(\rho + P) = 0\,. \tag{2.7}$$

For matter with a fixed equation of state $w = P/\rho$, such as a massless scalar $(w = 1)$, this equation can be easily integrated and one obtains $\rho \propto a^{-3(1+w)}$. Thus, for the case of a massless scalar field, energy density diverges as $1/a^6$ as $a \to 0$, leading to a big bang singularity.

It turns out that by canonically transforming to another set of phase space variables, (b,v), the quantum theory and resulting physical implications becomes much simpler to analyze. These are defined as

$$\mathrm{b} := \frac{c}{|p|^{\frac{1}{2}}}, \quad \mathrm{v} := \frac{|p|^{\frac{3}{2}}}{2\pi G}\,\mathrm{sgn}\,p \tag{2.8}$$

which satisfy $\{\mathrm{b},\mathrm{v}\} = 2\gamma$. The variable v is related to the physical volume as $\mathrm{v} = \varepsilon V/(2\pi G)$ where $\varepsilon = \pm 1$ depending on the orientation of the triad. In terms of (b,v) variables, the classical Hamiltonian constraint, for $N = 1$, can be written as

$$C_{\mathrm{cl}} = -\frac{3}{4\gamma^2}\mathrm{b}^2|\mathrm{v}| + \frac{p_{(\phi)}^2}{4\pi G|\mathrm{v}|} \approx 0\,. \tag{2.9}$$

Hamilton's equation for v results in the relation $\mathrm{b} = \gamma\dot{a}/a$. Thus, in the classical theory, b plays the role of Hubble rate, and is a measure of spacetime curvature.

The variables (b, v) satisfy an important property – they are invariant under the rescaling of the fiducial cell $\mathcal{V}$.[1] This can be seen as follows. Under the freedom of the fiducial cell, $\mathcal{V} \to \mathcal{V}'$ such that $V_o' = \alpha^3 V_o$. The connection and triad variables (c, p) transform under this freedom as:

$$c \to \alpha c \;\text{ and }\; p \to \alpha^2 p. \tag{2.10}$$

It thus follow from (2.8), that under the rescaling of the fiducial cell: $\mathrm{b} \to \mathrm{b}$ and $\mathrm{v} \to \mathrm{v}$. This is important to note because physical predictions must remain invariant under the choice of fiducial cell. It turns out that for the isotropic model, the resulting physics is invariant under rescaling of fiducial cell, only when loop quantization is based on b and v variables [49]. Similar considerations have been applied to the Bianchi models, which have led to the insights on the viability of different allowed quantizations [43, 50].

[1]The phase space variables (b, v) as well as (c, p) are invariant under another freedom – the freedom to rescale the fiducial metric: $\mathring{q}_{ab} \to l^2 \mathring{q}_{ab}$. See, Ref. [27] for details.

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2.2 Quantum theory

The quantization of the classical theory proceeds as in LQG. The elementary variables for the quantization are the holonomies of the connection A_a^i and flux of the electric field E_i^a. Due to underlying symmetries, holonomies can be computed along the edges of the cell $\mathscr{V}$. The holonomy of the connection c along an edge $\mu \mathring{e}_k^a$ with length $\mu V_o^{1/3}$, becomes

$$h_k^{(\mu)} = \cos(\mu\, c/2)\mathbb{I} + 2\, \sin(\mu\, c/2)\tau_k \tag{2.11}$$

where $\tau_k = -i\sigma_k/2$, and σ_k are the Pauli spin matrices. The flux of the electric field is computed by smearing by constant test function across a square tangential to the $\mathring{e}_i^a$. It turns out to be proportional to p. The elementary variables for quantization, thus turn out to be p and $N_\mu := \exp(i\mu c/2)$, the elements of holonomies. The latter generate an algebra of almost periodic functions of c. Using Gel'fand, Naimark and Segal's construction [59], a representation of this algebra can be obtained.[2] The kinematical Hilbert space turns out to be $\mathscr{H}_{\rm Kin} = L^2(\mathbb{R}_{\rm Bohr}, d\mu_{\rm Bohr})$, where $\mathbb{R}_{\rm Bohr}$ is the Bohr compactification of the real line and $d\mu_{\rm Bohr}$ is the Haar measure. The almost periodic functions, $N_\mu := \exp(i\mu c/2)$ provide an orthonormal basis in $\mathscr{H}_{\rm Kin}$, and satisfy $\langle N_{\mu_1} | N_{\mu_2}\rangle = \delta_{\mu_1,\mu_2}$. Action of the elementary operators on the states $\Psi(c)$ constructed from the orthonormal basis $\exp(i\mu c/2)$ turns out to be,

$$\hat{N}_{(\alpha)}\Psi = \exp\frac{i\alpha c}{2}\Psi, \quad \text{and} \quad \hat{p}\,\Psi = -i\frac{8\pi\gamma G\hbar}{3}\frac{{\rm d}\Psi}{{\rm d}c}\,. \tag{2.12}$$

This action becomes simpler in the representation, in which the action of $\hat{p}$ is diagonal. We label this representation by μ. The action of $\hat{p}$ on the eigenstates $|\mu\rangle$, can be written as

$$\hat{p}|\mu\rangle = \frac{8\pi\gamma\ell_{\rm Pl}^2}{6}\mu|\mu\rangle \tag{2.13}$$

where $\ell_{\rm Pl} = (G\hbar)^{1/2}$ is the Planck length. In this representation, the action of $N_{(\alpha)}$ is as a shift operator: $\hat{N}_{(\alpha)}|\mu\rangle = |\mu+\alpha\rangle$. Using this, we can find the action of the holonomy operator, which turns out to be

$$\widehat{h}_k^{(\alpha)}|\mu\rangle = \frac{1}{2}\left(|\mu+\alpha\rangle + |\mu-\alpha\rangle\right)\mathbb{I} + \frac{1}{i}\left(|\mu+\alpha\rangle - |\mu-\alpha\rangle\right)\tau_k\,. \tag{2.14}$$

We now turn to the Hamiltonian constraint. For a continuity with the results in Sec. III, we would choose to work with lapse $N = a^3$. The reason for this choice is tied to the observation that the flat, isotropic model sourced with a massless scalar field in LQC with the choice of lapse $N = a^3$ can be solved exactly by going to the b representation [13], and robustness of bounce can be proved in a rigorous way for all the states in the physical Hilbert space. Also, this choice of lapse corresponds to the harmonic time (τ), satisfying $\Box\tau = 0$, which is naturally suited for the use of massless scalar field as an internal clock, as considered here. For the lapse $N = a^3$, using the elementary variables, the gravitational part of the Hamiltonian constraint,

$$C_{\rm grav} = -\gamma^{-2}\int_{\mathscr{V}} {\rm d}^3x\, N, \varepsilon_{ijk}\, \frac{E^{ai}E^{bj}}{\sqrt{|\det E|}}\, F_{ab}^i \tag{2.15}$$

[2] The procedure mimics the one in the full theory (see Ref. [6] in this proceedings), where the algebra of holonomy and flux operators leads to a unique representation [60, 61]. Recently, the uniqueness of representation has been shown in the context of the Bianchi-I model in LQC [62].

can be expressed as

$$C_{\rm grav} = -\gamma^{-2} V_o^{-\frac{1}{3}} \, \varepsilon^{ij}{}_k \, \mathring{e}^a_i \mathring{e}^b_j \, |p|^2 F_{ab}{}^k \qquad (2.16)$$

where $F_{ab}{}^i$ is the field strength of the connection A^i_a. It is expressed in terms of holonomies by considering a square loop $\Box_{ij}$ with sides of length $\lambda V_o^{1/3}$ in the $i-j$ plane of the fiducial cell:

$$F^k_{ab} = -2 \lim_{Ar_\Box \to 0} \mathrm{Tr} \left(\frac{h^{(\bar\mu)}_{\Box_{ij}} - 1}{\bar\mu^2 V_o^{2/3}} \right) \tau^k \, \mathring{\omega}^i_a \, \mathring{\omega}^j_b, \quad h^{(\bar\mu)}_{\Box_{ij}} = h^{(\bar\mu)}_i h^{(\bar\mu)}_j (h^{(\bar\mu)}_i)^{-1} (h^{(\bar\mu)}_j)^{-1} . \qquad (2.17)$$

However, the limit $Ar_\Box \to 0$ does not exist in the quantum theory. This is a direct consequence of the underlying quantum geometry. The loop can be shrunk only to a minimum area, which is given by the minimum eigenvalue of the area operator. This turns out to be $\Delta \ell_{\rm Pl}^2$ where $\Delta = 4\sqrt{3}\pi\gamma$ [63]. Equating this with the physical area of the loop: $\bar\mu^2 V_o^{2/3} a^2 = \bar\mu^2 |p|^2$, and using (2.13), we find that $\bar\mu$ is not a constant but satisfies the following relation: $\bar\mu = (3\sqrt{3}/|\mu|)^{1/2}$. Due to this functional dependence the action of $\exp i(\bar\mu c/2)$ on states $\Psi(\mu)$ is not a simple translation in the argument of the wavefunction, but to drag $\Psi(\mu)$ a unit affine parameter along the vector field $\bar\mu \mathrm{d}/\mathrm{d}\mu$ [12]. The action simplifies if one works in volume representation. To see this, let us define a parameter $\lambda := \Delta^{1/2} \ell_{\rm Pl}$, such that $\bar\mu c = \lambda \mathrm{b}$. One is then interested in the action of $\exp(i\lambda \mathrm{b}/2)$ on the states in the volume representation $\Psi(\nu)$ where we have defined $\nu = \mathrm{v}/\gamma\hbar$. Since, λ is a constant, this action turns out to be a simple translation: $\exp(i\lambda \mathrm{b}) \Psi(\nu) = \Psi(\nu - \lambda)$. In this representation, the volume operator acts by multiplication:

$$\hat V \Psi(\nu) = 2\pi\gamma\ell_{\rm Pl}^2 |\nu| \Psi(\nu) , \qquad (2.18)$$

and the action of the operator corresponding to the gravitational part of the Hamiltonian constraint turns out to be

$$\hat C_{\rm grav} \Psi(\nu) = -24\pi^2 G^2 \gamma^2 \hbar^2 \, |\nu| \frac{\sin \lambda \mathrm{b}}{\lambda} |\nu| \frac{\sin \lambda \mathrm{b}}{\lambda} \Psi(\nu). \qquad (2.19)$$

Before we discuss the action of the above quantum constraint, it is important to note that in the absence of fermions, a change in the orientation of triads corresponds to a large gauge transformation generated by a parity operator: $\hat\Pi \Psi(\mu) = \Psi(-\mu)$, which acts either in a symmetric or anti-symmetric way on the physical states, leading to a super-selection of symmetric and anti-symmetric sectors. It turns out that the qualitative features of physics are not affected by the choice of the either sector. As is customary in LQC, we choose physical states should be symmetric under the change of orientation of the triad: $\Pi \Psi(\nu, \phi) := \Psi(-\nu, \phi) = \Psi(\nu, \phi)$. For such states, the action of total Hamiltonian constraint $C_{\rm H} = C_{\rm grav} + 16\pi G C_{\rm matt}$ is given by

$$\partial_\phi^2 \Psi(\nu, \phi) = -\Theta \Psi(\nu, \phi) , \qquad (2.20)$$

where $\Theta(\nu)$ is a positive definite quantum difference operator in ν with step size of 4λ, defined as[12]:[3]

$$\Theta \Psi(\nu, \phi) := -\frac{3\pi G}{4\lambda^2} \nu \left[(\nu + 2\lambda) \Psi(\nu + 4\lambda) - 2\nu \Psi(\nu, \phi) + (\nu - 2\lambda) \Psi(\nu - 4\lambda) \right] . \qquad (2.21)$$

[3]Since Θ is a difference operator, the space of physical states is divided in sectors, labeled by ε, which are preserved under evolution. This leads to a super-selection. In the following, we restrict ourselves to the sector $\varepsilon = 0$.

The quantum constraint, for the massless scalar field model, turns out to be of the form of the massless Klein-Gordon equation in a static space-time, where Θ plays the role of spatial Laplace operator and ϕ plays the role of time. Due to this reason, it becomes useful to treat ϕ as an internal clock or the emergent time in the quantum theory. The scalar field ϕ allows us to use the notion of relational dynamics, measuring the variation of volume (and similarly energy density) in 'time' ϕ, via the operator $\hat{V}|_\phi$. Apart from the volume observable, a natural choice for this model is the momentum of the scale field $p_{(\phi)}$ which is a constant in classical as well as the quantum theory.

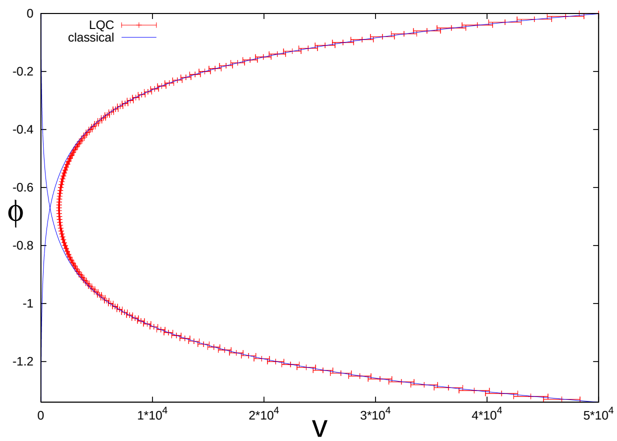

Figure 1: Expectation values of the volume observable in LQC are plotted. Comparison with trajectory obtained from GR shows that is an excellent approximation to LQC till the quantum state approaches Planck scale. The bounce of the volume occurs when expectation value of energy density observable reaches $\rho_{\rm max} \approx 0.41\rho_{\rm Planck}$.

The physical Hilbert space can be found by the group averaging method [64, 65, 66] (see also Ref. [26] in the context of LQC). This procedure group involves finding a rigging map $\eta : \Omega \to \Omega^*$ where Ω is a dense subspace of the auxiliary Hilbert space. The physical states can then be found by evaluating $\int d\xi \, \langle \exp(-i\xi\hat{C})\Psi'|$ where $\hat{C}$ is the self-adjoint quantum Hamiltonian constraint and $|\Psi\rangle \in \Omega$. The physical Hilbert space which consists of the positive frequency solutions of the quantum constraint:

$$-\, i\,\partial_\phi \,\Psi(v,\phi) = \sqrt{\Theta(v)}\Psi(v,\phi) \ . \tag{2.22}$$

satisfies the physical inner product

$$(\Psi_1,\Psi_2)_{\rm phys} = \sum_v \bar{\Psi}_1(v,\phi_o)\,\frac{1}{|v|}\,\Psi_2(v,\phi_o) \ . \tag{2.23}$$

An alternative way to find the above physical inner product is by demanding that the action of the Dirac observables, $\hat{p}_{(\phi)}$ and $\hat{V}|_\phi$ be self-adjoint on the physical Hilbert space. The Dirac observables of interest are $\hat{p}_\phi$ and $\hat{v}|_{\phi_o}$: given by

$$\hat{p}_{(\phi)}\Psi(v,\phi) = -i\hbar\frac{\partial\Psi(v,\phi)}{\partial\phi}, \quad |\hat{v}|_{\phi_o}\Psi(v,\phi) = e^{i\sqrt{\Theta}(\phi-\phi_o)}|v|\,\Psi(v,\phi_o) \ . \tag{2.24}$$

which are self-adjoint with respect to the physical inner product (2.23).

Using this structure we can extract predictions from the theory. In the volume representation, one has to rely on numerical simulations.[4] In the numerical simulations, one considers a semi-classical state peaked at a classical trajectory, $p_\phi = p_\phi^*$ and $v|_{\phi_o} = v^*$, in a large universe at large ϕ (late time). The state is then evolved backwards in time using quantum constraint equation (2.22). One can them compare the expectation values of the Dirac observables with the trajectories obtained from GR. A similar analysis can be performed using quantum constraint in the Wheeler-DeWitt theory. It turns out that the states in Wheeler-DeWitt theory remain peaked on the classical trajectory through out the evolution and the initial singularity is not resolved. On the other hand, results in LQC, turns out to be strikingly different. Numerical simulations with states which are semi-classical at late times, reveal that the big bang singularity is avoided (see Fig. 1 for an illustrative example of such a simulation). Instead a quantum bounce occurs when energy density reaches a maximum value, given by $\rho_{\rm max} \approx 0.41 \rho_{\rm Planck}$. Unlike the classical GR, and the Wheeler-DeWitt theory, the loop quantum evolution turns out to be non-singular with quantum bounce joining expanding branch of the universe with a contracting branch (with the same value of $p_{(\phi)}$). The ultra-violet problem of GR is solved by the underlying quantum geometric effects which manifest themselves in the quantum difference equation (2.22) via the non-local nature of the field strength operator (2.17). Now let us consider the infra-red limit. At small spacetime curvature, (i.e. large volume for a fixed value of $p_{(\phi)}$) the quantum difference equation, can be approximated by a differential operator

$$\partial_\phi^2 \Psi(v,\phi) = 12\pi G \, v \partial_v \, v \partial_v \Psi(v,\phi)\,. \tag{2.25}$$

This corresponds to the Wheeler-DeWitt equation for this model. Thus, not surprisingly, states in LQC are peaked on classical trajectories at small spacetime curvature and loop quantum evolution agrees with the predictions of GR when gravity is weak.

We now briefly discuss extensions of these results to other models. Results of Ref.[12], have been generalized to include spatial curvature. A rigorous quantization and analysis of physics for the closed model with a massless scalar field was performed in Refs. [32, 33] and the open model with a massless scalar field was considered in Ref. [34, 35]. For the $k = 1$ model, evolution of states with the quantum constraint has been performed using extensive numerics and as in the case of the flat model, it was found that bounce occurs when $\rho = \rho_{\rm max}$, fluctuations remain small through out the evolution and effective dynamics is an excellent approximation to the underlying quantum dynamics.[5] It is interesting to note that in the closed model, where the evolution is cyclic, the change in relative fluctuations over a large number of cycles remains negligible [33]. A similar conclusion holds for another cyclic model – the flat, isotropic model with a negative cosmological constant – which was quantized in Ref. [30]. More recently, quantization of isotropic model has been performed for the positive cosmological constant [29, 28] and the ϕ^2 inflationary scenario [31] where numerical simulations reveal that the bounce and peakedness properties of the semi-classical states are robust features in LQC. These results have been supplemented with the analytical studies in Refs. [14, 15, 16].

[4] In the next section, we will show that by going to the b representation, the model can be solved exactly.

[5] It is to be pointed out that in comparison to the spatially flat model sourced with a massless scalar, the closed model provides a much rigorous test of the infra-red limit. For an earlier quantization of LQC, Green and Unruh found certain difficulties with the behavior of eigenfunctions at large volumes [67]. These difficulties were resolved in the $\bar{\mu}$ quantization [12].

We conclude this subsection by noting that though the procedure outlined above has been widely applied to different isotropic models, it is never guaranteed to be straightforward and sometime subtle difficulties need to be overcome. As an example, the existing quantization of $k = -1$ model, relies on holonomies of extrinsic curvature, rather than the connection, and quantum Hamiltonian constraint is not self-adjoint. One can however find physical states using FFT method [34]. Similarly, for the case of the positive cosmological constant, the quantum difference equation turns out to be not essentially self-adjoint. However, one can choose a self-adjoint extension to obtain quantum evolution, and the choice of the extension does not affect qualitative features of physics.

2.3 Effective dynamics

We conclude this section with a brief discussion of the effective dynamics for this model in LQC. Using the geometrical formulation of quantum mechanics [68], where one treats the Hilbert space as an infinite dimensional phase space, it is possible to obtain an effective Hamiltonian up to well controlled approximations. This has been accomplished using two strategies in LQC. In the embedding method [40, 41, 42], one seeks a faithful embedding of the infinite dimensional quantum phase space in to finite dimensional classical phase space with a a judicious choice of states, and the truncation method [69, 70], which requires a careful and systematic truncation of terms arising in an order by order expansion to obtain a self-consistent set of dynamical equations without which reliable physical predictions can not be obtained. The embedding approach, from a physicist's perspective, serves as a useful tool because it works extremely well for states such as coherent states which lead to a macroscopic classical universe at the late times. The evidence of the reliability of the effective Hamiltonian derived using embedding approach comes from comparing the effective dynamics with the analytical models [13] as well as several numerical simulations performed in Refs. [10, 11, 12, 28, 30, 31, 33, 34, 43, 44]. For this reason, it has been widely used in literature to extract physical predictions (see Sec. V of Ref. [27] for a review).

The effective Hamiltonian constraint for $k = 0$ isotropic model, with lapse $N = 1$, is given by [41, 42]

$$-\frac{3}{\gamma^2}\frac{\sin^2(\lambda \mathrm{b})}{\lambda^2}V + 8\pi G H_{\mathrm{matt}} \approx 0, \tag{2.26}$$

which using Hamilton's equation for volume, leads to the modified Friedmann equation [71, 12]

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) , \tag{2.27}$$

Using the Hamilton's equations for connection and matter variables, modified Raychaudhuri and conservation laws can also be obtained. These equations imply, that at the maximum of energy density, Hubble rate vanishes, and $\ddot{a} > 0$, causing the universe to bounce. It is rather remarkable that the density at which bounce occurs and the effective trajectory agrees to an excellent precision with the underlying quantum theory. For $\rho \ll \rho_{\max}$, the modified Friedman equations reduce to the classical Friedman equation (2.5). Modified dynamical equations give important insights on the new physics at the Planck scale. To cite a few examples, the phase of super-inflation $(\dot{H} > 0)$ near bounce [71], has important implications for the probability for inflation to occur. For a quadratic potential, it has been shown that if initial conditions are provided at the bounce surface $(\rho = \rho_{\max})$ and probability for inflation to occur is evaluated using Liouville measure on the phase space, a

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phase of inflation with more than 65 e-foldings is almost guaranteed in LQC [72, 73, 74][6]. Another important implication results in the form of bounds on physical quantities which appear in geodesic and Raychaudhuri equations. It has been shown that energy density and expansion and shear scalars are bounded for isotropic [45] and anisotropic models [50, 47] which has important implications to understand general resolution of singularities. This turns out to be true for isotropic models where it has been demonstrated that there is indeed a generic resolution of strong singularities [45, 48]. The analysis have been recently extended for Bianchi-I model for the case of matter with a vanishing anisotropic stress [46]. These results are expected to shed important insights on a non-singularity theorem in LQC/LQG.

3. Exactly soluble LQC

In the previous section, we discussed the quantization of spatially flat homogeneous and isotropic model sourced with a massless scalar field in LQC in the volume representation. In this section, we consider the same model in the conjugate b representation. As remarked earlier, with the lapse $N = a^3$, this model becomes exactly soluble in the b representation, allowing the analysis of results on bounce at a purely analytical level. In particular, it allows to establish genericness of bounce for arbitrary states in the physical Hilbert space. Further, a notable merit of sLQC is that the underlying quantum theory shares similar features with the Wheeler-DeWitt quantization, yet due to an interplay of volume observable with physical states, the physical predictions are strikingly different. In the following, we summarize and contrast the main features of the quantization for both frameworks. (For details, we refer the reader to Ref. [13, 27]).

In the b representation, the classical Hamiltonian constraint for lapse $N = a^3$ can be written as

$$-3\pi G\, \mathrm{v}^2 \mathrm{b}^2 + p_{(\phi)}^2 \approx 0 \ . \tag{3.1}$$

In this representation, we consider the action of the corresponding Wheeler-DeWitt and sLQC quantum constraints on states $\underline{\chi}(\mathrm{b},\phi)$ and $\chi(\mathrm{b},\phi)$ respectively. In the Wheeler-DeWitt theory, $\mathrm{b} \in (-\infty,\infty)$, and due to the symmetry under the change of orientation of the triads, states satisfy $\underline{\chi}(\mathrm{b},\phi) = -\underline{\chi}(-\mathrm{b},\phi)$. Imposing this symmetry, we restrict to the states which have support on the positive b-half line. On these states, the quantum constraint in Wheeler-DeWitt theory has the following action:

$$\partial_\phi^2\, \underline{\chi}(\mathrm{b},\phi) = 12\pi G (\mathrm{b}\,\partial_b)^2\, \underline{\chi}(\mathrm{b},\phi) \ . \tag{3.2}$$

On the other hand, in sLQC, $\mathrm{b} \in (0, \pi/\lambda)$ and the action of the quantum constraint is given by

$$\partial_\phi^2\, \chi(\mathrm{b},\phi) = 12\pi G \left(\frac{\sin\lambda \mathrm{b}}{\lambda}\partial_b\right)^2 \chi(\mathrm{b},\phi) \ . \tag{3.3}$$

Interestingly, the quantum constraints (3.2) and (3.3), can be written in a simple form of a 2-dimensional Klein-Gordon equation by a simple change of variables. To see this, we introduce variable y for the Wheeler-DeWitt theory:

$$y := \frac{1}{(12\pi G)^{1/2}} \ln \frac{\mathrm{b}}{\mathrm{b}_o} \tag{3.4}$$

[6]The attractor properties for inflationary trajectories in LQC [75, 73, 74], play an important role in these results.

where b_o is constant. The resulting quantum constraint (3.2), then becomes

$$\partial_\phi^2 \underline{\chi}(y,\phi) = -\underline{\Theta}\,\underline{\chi}(y,\phi), \quad \text{where } \underline{\Theta} := -\partial_y^2 \ . \tag{3.5}$$

Similarly, for sLQC, we introduce x variable:

$$x = \frac{1}{\sqrt{12\pi G}} \ln\left(\tan\frac{\lambda \mathrm{b}}{2}\right) \ . \tag{3.6}$$

Due to this change of variable, the resulting action of the quantum constraint (3.3) takes the similar form as (3.5):

$$\partial_\phi^2\, \chi(x,\phi) = -\Theta\, \chi(x,\phi), \ \ \Theta := -\partial_x^2 \ . \tag{3.7}$$

Solutions of both the constraints (3.5) and (3.7)) are superselected in to the positive and negative frequency sub-spaces, and as in the previous section, one can restrict to the positive frequency sub-space. The physical inner product can be found using the group averaging procedure [64, 65, 66]. In the Wheeler-DeWitt theory, the inner product take the following Klein-Gordon form[7]:

$$(\underline{\chi}_1, \underline{\chi}_2)_{\rm phy} = 2\int_{-\infty}^{\infty} \mathrm{d}k |k|\, \bar{\underline{\tilde{\chi}}}_1(k) \underline{\tilde{\chi}}_2(k) \ . \tag{3.8}$$

Here $\tilde{\chi}$ is the Fourier transform of the Wheeler-DeWitt state $\underline{\chi}$. The action of the Dirac observables, $\hat{p}_{(\phi)}$ and $\hat{V}|_\phi$, is self-adjoint with respect to the inner product. It is given by

$$\hat{p}_\phi\, \underline{\chi}(y,\phi) = \hbar\sqrt{\underline{\Theta}}\,\underline{\chi}(y,\phi) \ . \tag{3.9}$$

and

$$\hat{V}|_{\phi_o}\, \underline{\chi}(y,\phi) = e^{i\sqrt{\underline{\Theta}}(\phi-\phi_o)}\, (2\pi\gamma\ell_{\rm Pl}^2 |\hat{\nu}|)\, \underline{\chi}(y,\phi) \ . \tag{3.10}$$

where ϕ_o is the slice at which initial datum is specified, and ν is defined as $\nu := \mathrm{v}/\hbar$, whose corresponding operator is given by,

$$\hat{\nu} = -\frac{2}{\sqrt{12\pi G}\mathrm{b}_o}\left(P_R(e^{\sqrt{12\pi G}y} i\,\partial_y)P_R + P_L(e^{\sqrt{12\pi G}y} i\,\partial_y)P_L\right) \ . \tag{3.11}$$

where P_L and P_R are projectors on the left and right moving sectors of the Schrödinger Hilbert space. The left moving states correspond to expanding universe, and the right moving states correspond to the contracting universe. The Dirac observables, $\hat{p}_\phi$ and $\hat{V}|_{\phi_o}$, which form a complete set, preserve these sectors. Hence, one can restrict to just one of these sectors to analyze physical implications. In the following analysis of the Wheeler-DeWitt theory, we will consider only left moving modes.

Despite similarities with the Wheeler-DeWitt theory, the situation in sLQC turns out to be sharply different. The first difference is rooted in the symmetry requirement of the orientation of triads. It turns out that $\chi(-x,\phi) = -\chi(x,\phi)$ [13]. Thus, a generic state $\chi(x,\phi)$ has support on both left and right moving sectors and unlike the WDW theory there is no super-selection of these

[7] Note that a general initial datum for the physical state specified at time $\phi = \phi_o$ is of the form $\underline{\chi}(y,\phi_o) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk e^{-iky}\underline{\tilde{\chi}}(k)$, and under time evolution one obtains $\underline{\chi}(y,\phi) = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} \mathrm{d}k e^{-ik(\phi+y)} e^{ik\phi_o}\underline{\tilde{\chi}}(k) + \int_0^\infty \mathrm{d}k e^{ik(\phi-y)} e^{-ik\phi_o}\underline{\tilde{\chi}}(k)\right)$. As in the Klein-Gordon case $\phi+y$ and $\phi-y$ can be identified as the left and right moving components respectively.

sectors. In particular, any solution of the quantum constraint (3.7), satisfies the following relation in terms of right moving x_+ or left moving parts x_-: $\chi(x,\phi) = \frac{1}{\sqrt{2}}(F(x_+) - F(x_-))$. Using this, the inner product (3.8), can then be expressed in terms of the right moving or left moving solutions, and in terms of the latter it becomes

$$(\chi_1, \chi_2)_{\rm phys} = -2i \int_{-\infty}^{\infty} {\rm d}x \bar{F_1}(x_+) \partial_x F_2(x_+) \ . \tag{3.12}$$

The second difference between Wheeler-DeWitt theory and sLQC arises, in the action of the volume operator, which is given by

$$\hat{v} = -\frac{2\lambda}{\sqrt{12\pi G}} \left(P_R (\cosh(\sqrt{12\pi G} x) i\, \partial_x) P_R + P_L (\cosh(\sqrt{12\pi G} x) i\, \partial_x) P_L \right) \ . \tag{3.13}$$

Due to the differences in the action of the volume observable, behavior of the corresponding expectation values in Wheeler-DeWitt theory and sLQC is qualitative distinct. In the Wheeler-DeWitt theory, for the left moving states, they are given by

$$(\underline{\chi}_L, \hat{V}|_\phi \underline{\chi}_L)_{\rm phy} = 2\pi\gamma\ell_{\rm Pl}^2 (\underline{\chi}_L, \hat{|v|}_\phi \underline{\chi}_L)_{\rm phy} = V_* e^{\sqrt{12\pi G}\phi} \tag{3.14}$$

where V_* is a positive definite constant determined by the initial data. In contrast, the expectation values $\langle \hat{V}|_\phi \rangle$ in sLQC become,

$$(\chi, \hat{V}|_\phi \chi)_{\rm phy} = 2\pi\gamma\ell_{\rm Pl}^2 (\chi, \hat{|v|}_\phi \chi)_{\rm phy} = V_+ e^{\sqrt{12\pi G}\phi} + V_- e^{-\sqrt{12\pi G}\phi} \tag{3.15}$$

where $V_\pm$ are positive constants determined by the initial data [13].

Let us compare the behavior of expectation values of the volume observable in Wheeler-DeWitt theory and sLQC. From (3.11), we find that in the Wheeler-DeWitt theory, for any given left-moving state (which corresponds to an expanding universe), the expectation values of the volume observable becomes infinite as $\phi \to \infty$ and becomes zero when $\phi \to -\infty$. That is, in the past evolution, an expanding Wheeler-DeWitt universe encounters a big bang singularity irrespective of the choice of state. Similar, conclusion arises for the right-moving states in Wheeler-DeWitt theory, which correspond to a contracting universe. The expectation values $\langle \hat{V}|_\phi \rangle$ become infinite as $\phi \to -\infty$ and vanish when $\phi \to \infty$. Thus, a flat isotropic Wheeler-DeWitt universe sourced with a massless scalar field, inevitably encounters a singularity, independent of the choice of state. Going beyond the analysis of expectation values, above conclusion is also reached using the consistent histories framework [76, 77], by computing the consistent probabilities for the singularity to occur [20, 21]. Interestingly, a careful analysis of consistent histories leads to even a stronger result, that arbitrary superpositions of left and right moving modes do not lead to avoidance of singularity, whose probability remains unity (for details, we refer the reader to [20, 21]).

Let us now consider the case of sLQC. From (3.15), we find that the expectation values $\langle \hat{V}|_\phi \rangle$ become infinite when $\phi \to \pm\infty$, attaining a minimum volume $V_{\rm min} = 2\sqrt{V_+ V_-}/||\chi||^2$ at $\phi_{\rm bounce} = \frac{1}{(2\sqrt{12\pi G})} \log(V_-/V_+)$. Further, the expectation values $\langle \hat{V}|_\phi \rangle$ are symmetric across the bounce time $\phi_{\rm bounce}$. Thus, in contrast to the Wheeler-DeWitt evolution, the expectation value of the volume observable never vanishes in sLQC.[8] Starting from the expanding branch, a backward evolution

[8] Computation of consistent probabilities shows that the probability for an sLQC universe to bounce is unity [22].

of an arbitrary state in sLQC leads to a bounce at the minimum value. Thus, generic states in sLQC lead to the resolution of the big bang/crunch singularities of the classical theory. Defining the energy density observable, one can compute the value at which bounce occurs. This provides a strong analytical test of results obtained from the numerical simulations. Let us define the Dirac observable for energy density as,

$$\hat{\rho}|_\phi = \frac{1}{2}\hat{A}|_\phi^2 \quad \text{where} \quad \hat{A}|_\phi = (\hat{V}|_\phi)^{-1/2}\,\hat{p}_\phi(\hat{V}|_\phi)^{-1/2}\,. \tag{3.16}$$

The expectation values of $\langle \hat{A}|_\phi \rangle$ are given by

$$\langle \hat{A}|_{\phi_o} \rangle = \frac{(\chi, \hat{p}_\phi \chi)_{\rm phy}}{(\chi, \hat{V}|_{\phi_o}\chi)_{\rm phy}} = \left(\frac{3}{4\pi\gamma^2 G}\right)^{1/2} \frac{1}{\lambda} \frac{\int_{-\infty}^{\infty} \mathrm{d}x |\partial_x F|^2}{\int_{-\infty}^{\infty} \mathrm{d}x |\partial_x F|^2 \cosh(\sqrt{12\pi G}x)}\,. \tag{3.17}$$

It turns out that these are bounded, with a maximum given by[9]

$$\rho_{\rm sup} = \frac{3}{8\pi\gamma^2 G\lambda^2} = \frac{\sqrt{3}}{32\pi^2\gamma^3 G^2\hbar} \approx 0.41\rho_{\rm Pl}\,, \tag{3.18}$$

which agrees with the value of energy density at which bounce occurs in the numerical simulations for spatially flat isotropic model with a massless scalar field [12]. Note that if quantum discreteness vanishes, i.e. if λ is set to zero, there is no maximum value of the energy density and the minimum allowed value of volume observable vanishes. The classical singularity is recovered in the absence of quantum discreteness.

4. Summary

Loop quantization of cosmological models provides us a new paradigm of the very early universe in which initial singularity of the classical theory is replaced by a quantum bounce when the spacetime curvature reaches Planck regime [27]. Non-singular evolution resulting from non-local properties of the field strength of connection, has turned out to be a general feature of various models, in isotropic and anisotropic spacetimes. We illustrated key aspects of quantization for the spatially flat isotropic universe sourced with a massless scalar field. We discussed the way physical Hilbert space is found, the role of scalar field as internal clock, the action of Dirac observables, and the way physics can be extracted using sophisticated numerical simulations [10, 11, 12]. Unlike the Wheeler-DeWitt quantization, the quantum constraint in LQC turns out to be discrete. The differential geometry of GR is replaced by quantum geometry. States which are semi-classical at late times, when evolved backward towards the big bang using quantum constraint, remain peaked at classical trajectories for small spacetime curvature, but result in a big bounce where energy density reaches a maximum value. Analysis of fluctuations show that they are tightly constrained in the evolution across the bounce [12, 33, 14, 15, 78]. In a striking contrast to Wheeler-DeWitt theory,

[9]Another way to obtain this bound is by considering the energy density observable for the states in the physical Hilbert space. For such states, using quantum Hamiltonian constraint, one finds $\hat{\rho}|_\phi = \frac{3}{8\pi G\gamma^2\lambda^2}\widehat{\sin^2(\lambda b)}$, which results in the same maximum of the energy density as in eq.(3.18).

big bang singularity is shown to be absent for a dense subspace of states using an exactly soluble model [13]. Effective dynamics, which captures the key features of underlying quantum geometry, provides an excellent approximation of the quantum evolution and has been extensively used to extract novel physical predictions, such as, on bounds on energy density, expansion and shear scalars [45, 47, 50], resolution of strong singularities [45, 46] and probability for inflation [72, 73, 74]. These have been also used to probe the physics beyond homogeneity approximation such as in Gowdy models [51] and imprint of quantum geometry on cosmological perturbations [52], which have also been derived using detailed properties of quantum spacetime [54]. It is hoped that the ongoing investigations on primordial perturbations will provide potential tests for LQC in near future astronomical experiments.

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References

[1] S. W. Hawking, G. F. R. Ellis, *The large scale structure of Space-Time*, Cambridge (1973)

[2] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Class. Quant. Grav. **21** R53-R152 (2004)

[3] C. Rovelli,*Quantum Gravity*. (Cambridge University Press, Cambridge (2004))

[4] T. Thiemann, *Introduction to Modern Canonical Quantum General Relativity.* (Cambridge University Press, Cambridge, (2007))

[5] R. Gambini, J. Pullin, *A first course in loop quantum gravity*, Oxford (2011).

[6] K. Giesel, H. Sahlmann, From Classical To Quantum Gravity: Introduction to Loop Quantum Gravity, arXiv:1203.2733 [gr-qc]

[7] M. Bojowald, Absence of singularity in loop quantum cosmology, Phys. Rev. Lett. **86**, 5667 (2001); M. Bojowald, Isotropic loop quantum cosmology, Class. Quant. Grav. **19** 2717 (2002).

[8] M. Bojowald, Homogeneous loop quantum cosmology, Class. Quant. Grav. **20**, 2595 (2003)

[9] A. Ashtekar, M. Bojowald and J. Lewandowski, Mathematical structure of loop quantum cosmology. Adv. Theo. Math. Phys. **7** 233–268 (2003)

[10] A. Ashtekar, T. Pawlowski and P. Singh, Quantum nature of the big bang, Phys. Rev. Lett. **96** 141301 (2006),

[11] A. Ashtekar, T. Pawlowski and P. Singh, Quantum nature of the big bang: An analytical and numerical investigation I, Phys. Rev. **D73** 124038 (2006)

[12] A. Ashtekar, T. Pawlowski and P. Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D**74** 084003 (2006)

[13] A. Ashtekar, A. Corichi, P. Singh, Robustness of key features of loop quantum cosmology, Phys. Rev. D **77**, 024046 (2008).

[14] A. Corichi and P. Singh, Quantum bounce and cosmic recall, Phys. Rev. Lett. **100** 209002 (2008)

[15] W. Kaminski and T. Pawlowski, Cosmic recall and the scattering picture of loop quantum cosmology, Phys. Rev. D**81** 084027 (2010)

[16] A. Corichi, E. Montoya, On the semiclassical limit of loop quantum cosmology, arXiv:1105.2804; Coherent semiclassical states for loop quantum cosmology, arXiv:1105.5081.

[17] A. Ashtekar, M. Campiglia and A. Henderson, Loop quantum cosmology and spin foams, Phys. Lett. **B681** 347-352 (2009)

[18] A. Ashtekar, M. Campiglia and A. Henderson, Casting Loop Quantum Cosmology in the spin foam paradigm, Class. Quant. Grav. **27** 135020 (2010)

[19] A. Ashtekar, M. Campiglia and A. Henderson, Path Integrals and the WKB approximation in loop quantum cosmology, Phys. Rev. D**82** 124043 (2010)

[20] D. Craig, P. Singh, Consistent histories in quantum cosmology, Found. Phys **41** 371 (2011)

[21] D. Craig, P. Singh, Consistent probabilities in Wheeler-DeWitt quantum cosmology, Phys. Rev. D**82** 123526 (2010)

[22] D. Craig, P. Singh, Consistent histories in loop quantum cosmology, In preparation (2012)

[23] W. Kaminski and J. Lewandowski, L. Szulc, The status of quantum geometry in the dynamical sector of loop quantum cosmology, Class. Quant. Grav. **25** 055003 (2008)

[24] W. Kaminski and J. Lewandowski, The flat FRW model in LQC: the self-adjointness, Class. Quant. Grav. **25** 035001 (2008)

[25] W. Kaminski, J. Lewandowski and T. Pawlowski, Physical time and other conceptual issues of QG on the example of LQC, Class. Quant. Grav. **26** 035012 (2009)

[26] W. Kaminski, J. Lewandowski and T. Pawlowski, Quantum constraints, Dirac observables and evolution: group averaging versus Schroedinger picture in LQC, Class. Quant. Grav. **26** 245016 (2009)

[27] A. Ashtekar, P. Singh, Loop quantum cosmology: a status report, Class. Quant. Grav. **28**, 213001 (2011).

[28] T. Pawlowski, A. Ashtekar, Positive cosmological constant in loop quantum cosmology, Phys. Rev. D **85**, 064001 (2012)

[29] W. Kaminski and T. Pawlowski, The LQC evolution operator of FRW universe with positive cosmological constant Phys. Rev. D**81** 024014 (2010)

[30] E. Bentivegna and T. Pawlowski, Anti-deSitter universe dynamics in LQC, Phys. Rev. D**77** 124025 (2008)

[31] A. Ashtekar, T. Pawlowski and P. Singh, Loop quantum cosmology and pre-inflationary epoch (In preparation).

[32] L. Szulc, W. Kaminski, J. Lewandowski, Closed FRW model in loop quantum cosmology, Class. Quant. Grav. **24** 2621 (2007)

[33] A. Ashtekar, T. Pawlowski, P. Singh and K. Vandersloot, Loop quantum cosmology of k=1 FRW models. Phys. Rev. D**75** 0240035 (2006)

PoS(QGQGS 2011)007

[34] K. Vandersloot, Loop quantum cosmology and the $k = -1$ FRW model, Phys. Rev. D **75** 023523 (2007)

[35] L. Szulc, Open FRW model in loop quantum cosmology, Class. Quant. Grav. **24** 6191 (2007)

[36] A. Ashtekar and E. Wilson-Ewing, Loop quantum cosmology of Bianchi type I models, Phys. Rev. D**79** 083535 (2009)

[37] M. MartÃ■n-Benito, L. J. Garay, G. A. Mena MarugÃạn, E. Wilson-Ewing, Loop quantum cosmology of the Bianchi I model: complete quantization, arXiv:1110.1941.

[38] A. Ashtekar and E. Wilson-Ewing, Loop quantum cosmology of Bianchi type II models, Phys. Rev. D**80** 123532 (2009)

[39] E. Wilson-Ewing, Loop quantum cosmology of Bianchi type IX models, Phys. Rev. D**82** 043508 (2010)

[40] J. Willis, On the low energy ramifications and a mathematical extension of loop quantum gravity. Ph.D. Dissertation, The Pennsylvaina State University (2004)

[41] V. Taveras, LQC corrections to the Friedmann equations for a universe with a free scalar field, Phys. Rev. **D78** 064072 (2008)

[42] P. Singh, V. Taveras, Effective equations for arbitrary matter in loop quantum cosmology, (In preparation).

[43] L. Szulc, Loop quantum cosmology of diagonal Bianchi type I model: Simplifications and scaling problems, Phys. Rev. D **78**, 064035 (2008)

[44] M. Martin-Benito, G. A. Mena Marugan and T. Pawlowski, Physical evolution in Loop Quantum Cosmology: The example of vacuum Bianchi I, Phys. Rev. D **80** 084038 (2009).

[45] P. Singh, Are loop quantum cosmologies never singular? Class. Quant. Grav. **26** 125005 (2009)

[46] P. Singh, Curvature invariants, geodesics and the strength of singularities in Bianchi-I loop quantum cosmology, Phys.Rev. D **85** (2012) 104011

[47] B. Gupt, P. Singh, Contrasting features of anisotropic loop quantum cosmologies: the role of spatial curvature, Phys.Rev. D **85** (2012) 044011

[48] P. Singh, F. Vidotto, Exotic singularities and spatially curved loop quantum cosmology, Phys. Rev. D **83** 064027 (2011).

[49] A. Corichi and P. Singh, Is loop quantization in cosmology unique? Phys. Rev. D**78** 024034 (2008)

[50] A. Corichi, P. Singh, A geometric perspective on singularity resolution and uniqueness in loop quantum cosmology, Phys. Rev. D **80**, 044024 (2009).

[51] M. Martin-Benito, L. J. Garay and G. A. Mena Marugan, Hybrid quantum Gowdy cosmology: combining loop and Fock quantizations, Phys. Rev. D**78** 083516 (2008);
L. J. Garay, M. Martn-Benito, G. A. Mena Marugan, Inhomogeneous loop quantum cosmology: Hybrid quantization of the Gowdy model, Phys. Rev. D**82** 044048 (2010); D. Brizuela, G. A. Mena Marugan and T. Pawlowski, Big bounce and inhomogeneities, Class. Quant. Grav. **27** 052001 (2010); M. Martin-Benito, G. A. Mena Marugan, E. Wilson-Ewing, Hybrid quantization: From Bianchi I to the Gowdy model, Phys. Rev. D**82** 084012 (2010); M. Martn-Benito, D. Martn-de Blas, G. A. Mena Marugan, Matter in inhomogeneous loop quantum cosmology: the Gowdy T^3 model, `arXiv:1012.2324`; D. Brizuela, G. A. Mena Marugan and T. Pawlowski, Effective dynamics of the hybrid quantization of the Gowdy T3 universe, `arXiv:1106.3793`

[52] See for eg., S. Tsujikawa, P. Singh, R. Maartens, Loop quantum gravity effects on inflation and the CMB, Class. Quant. Grav. **21** 5767 (2004); S. Hofmann and O. Winkler, The Spectrum of fluctuations in inflationary quantum cosmology, arXiv:astro-ph/0411124; G. M. Hossain, Primordial density perturbation in effective loop quantum cosmology, Class. Quant. Grav. **22** 2511 (2005); E. J. Copeland, D. J. Mulryne, N. J. Nunes and M. Shaeri, The gravitational wave background from super-inflation in Loop Quantum Cosmology, arXiv:0810.0104; E. J. Copeland, D. J. Mulryne, N. J. Nunes, and M. Shaeri, Superinflation in loop quantum cosmology, Phys. Rev. D **77** 023510 (2008) J. Magueijo, P. Singh, Thermal fluctuations in loop cosmology, Phys. Rev. D **76** 023510 (2007); M. Bojowald, H. H. Hernandez, M. Kagan, P. Singh, A. Skirzewski, Hamiltonian cosmological perturbation theory with loop quantum gravity corrections, Phys. Rev. D**74** 123512 (2006); M. Bojowald, H. H. Hernandez, M. Kagan, P. Singh, A. Skirzewski, Formation and Evolution of Structure in Loop Cosmology, Phys. Rev. Lett. **98** 031301 (2007); M. Artymowski, Z. Lalak, L. Szulc, Loop quantum cosmology corrections to inflationary models, JCAP **0901** 004 (2009); M. Bojowald and G. M. Hossain, Loop quantum gravity corrections to gravitational wave dispersion, Phys. Rev. D **77** 023508 (2008); J. Mielczarek, Possible observational effects of loop quantum cosmology, Phys. Rev. **D81** 063503 (2010); J. Grain and A. Barrau, Cosmological footprints of loop quantum gravity, Phys. Rev. Lett. **102** 081301 (2009); J. Mielczarek, T. Cailleteau, J. Grain and A. Barrau, Inflation in loop quantum cosmology: Dynamics and spectrum of gravitational waves, Phys. Rev. D**81** 104049 (2010); M. Shimano and T. Harada, Observational constraints on a power spectrum from super-inflation in Loop Quantum Cosmology, Phys. Rev. D **80** 063538 (2009); M. Bojowald, G. M. Hossain, M. Kagan, S. Shankaranarayanan, Phys. Rev. D**79** 043505 (2009); G. Calcagni, G. M Hossain, Loop quantum cosmology and tensor perturbations in the early universe, Adv. Sci. Lett. **2** 184 (2009); M. Bojowald and G. Calcagni, Inflationary observables in loop quantum cosmology, JCAP **1103** 032 (2011); M. Bojowald, G. Calcagni and S. Tsujikawa, Observational constraints on loop quantum cosmology, arXiv:1101.5391 [astro-ph.CO].

[53] A. Ashtekar, W. Kaminski and J. Lewandowski, Quantum field theory on a cosmological, quantum space-time, Physical Review D**79** 064030 (2009), arXiv:0901.0933

[54] I. Agullo, A. Ashtekar and W. Nelson, Perturbations in loop quantum cosmology, arXiv:1204.1288

[55] M. Bojowald, Loop quantum cosmology, Liv. Rev. Rel. **8**, 11 (2005).

[56] K. Banerjee, G. Calgani, M. Martin-Benito, Introduction to loop quantum cosmology, SIGMA **8**, 016 (2012)

[57] D. Brizuela, D. Cartin and G. Khanna, Numerical techniques in loop quantum cosmology, SIGMA **8**, 001 (2012)

[58] P. Singh, Numerical loop quantum cosmology: an overview, arXiv:1208.5456 [gr-qc].

[59] I. M. Gel'fand and M. A. Naimark, On the embedding of normed rings into the ring of operators in Hilbert space, Mat. Sobrn. **12, [54]** 197-217 (1943)
I. E. Segal, Postulates of general quantum mechanics, Ann. Math. **48** 930-948 (1947)

[60] J. Lewandowski, A. Okolow, H. Sahlmann, T. Thiemann, Uniqueness of the diffeomorphism invariant state on the quantum holonomy-flux algebras, Comm. Math. Phys. **267** 703 (2006)

[61] C. Fleishchack, Representations of the Weyl algebra in quantum geometry, Commun. Math. Phys. **285** 67 (2009)

[62] A. Ashtekar, M. Campiglia, On the Uniqueness of Kinematics of Loop Quantum Cosmology," arXiv:1209.4374 [gr-qc]

[63] A. Ashtekar and E. Wilson-Ewing, The covariant entropy bound and loop quantum cosmology, Phys. Rev. **D78** 06407 (2008)

[64] D. Marolf, Refined algebraic quantization: Systems with a single constraint. `arXive: gr-qc/9508015`;
Quantum observables and recollapsing dynamics. Class. Quant. Grav. **12** 1199–1220 (1994)

[65] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann, Quantization of diffeomorphism invariant theories of connections with local degrees of freedom. Jour. Math. Phys. **36** 6456–6493 (1995)

[66] A. Ashtekar, L. Bombelli and A. Corichi, Semiclassical states for constrained systems, Phys. Rev. **D72** 025008 (2005)

[67] D. Green and W. Unruh, Difficulties with recollapsing models in closed isotropic loop quantum cosmology, Phys. Rev. **D70** 103502 (2004)

[68] T. Schilling, Geomtery of quantum mechanics, Ph. D Dissertation, The Pennsylvania State University (1996); A. Ashtekar and T. A. Schilling, Geometrical formulation of quantum mechanics. In: *On Einstein's Path: Essays in Honor of Engelbert Schücking*, Harvey, A. (ed.) (Springer, New York (1999)), 23–65, `arXiv:gr-qc/9706069`

[69] M. Bojowald, A. Skirzewski, Effective theory for the cosmological generation of structure, Rev. Math. Phys. **18** 713 (2006);

[70] M. Bojowald, B. Sandhoefer, A. Skirzewski, A. Tsobanjan, Effective Constraints for Quantum Systems, Rev. Math. Phys. **21** 111 (2009)

[71] P. Singh, Loop cosmological dynamics and dualities with Randall-Sundrum braneworlds, Phys.Rev. D**73** (2006) 063508.

[72] A. Ashtekar and D. Sloan, Loop quantum cosmology and slow roll inflation, Phys. Lett. B**694** 108-112 (2010)

[73] A. Ashtekar and D. Sloan, Probability of inflation in loop quantum cosmology, `arXiv:1103.2475`

[74] A. Corichi and A. Karami, On the measure problem in slow roll inflation and loop quantum cosmology, `arXiv:1011.4249`

[75] P. Singh, K. Vandersloot, and G. V. Vereshchagin, Non-singular bouncing universes in loop quantum cosmology, Phys. Rev. D**74** 043510, (2006)

[76] J. B. Hartle, Spacetime quantum mechanics and the quantum mechanics of space-time, *Gravitation and Quantizations: Proceedings of the 1992 Les Houches Summer School*, ed. by B. Julia and J. Zinn-Justin, North Holland, Amsterdam (1995); arXiv:gr-qc/9304006.

[77] See for example, J. J. Halliwell, Somewhere in the Universe: Where is the information stored when histories decohere?, Phys. Rev. D **60** 105031 (1999); D. A. Craig, J. B. Hartle, Generalized quantum theory of recollapsing homogeneous cosmologies, Phys. Rev. D **69** 123525 (2004); J. J. Halliwell, P. Wallden, Invariant class operators in the decoherent histories analysis of timeless quantum theories, Phys. Rev. D **73** 024011 (2006); J. J. Halliwell, Probabilities in quantum cosmological models: a decoherent histories analysis using a complex potential, Phys. Rev. D **80** 124032 (2009); M. Gell-Mann, J. B. Hartle, Decoherent histories quantum mechanics with one 'real' fine-grained history, arXiv:1106.0767

[78] A. Corichi, A. Karami, Loop quantum cosmology of k=1 FRW: A tale of two bounces, arXiv:1105.3724