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Noncommutative frames, an insight to the geometry of the Grosse-Wulkenhaar model

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In order to argue physical relevance of the noncommutative geometry and understand its relations to the loop quantum gravity we review the noncommutative frame approach. We then discuss a number of examples, the most elaborate of them being the background space of the Grosse-Wulkenhaar model.

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1. Foreword

A common point of noncommutative gravity with the loop quantum gravity is the belief that by representing coordinates and gravitational field by operators it is possible to reduce or remove singularities of the classical solutions. Whether such representations can be obtained only in the process of quantization is perhaps unclear: in noncommutative geometry we usually assume that noncommutativity of coordinates is present already at the classical level.

There is a variety of approaches to noncommutative gravity which differ in initial physical premises, but remarkably, they often give identical or very similar formulae. One of the current ideas discussed at the School is that gravity is an 'emergent phenomenon', [1]. Starting from the matter which consists of Yang-Mills fields in a matrix model, one recognizes that a part of gauge symmetry describes in fact the gravitational degrees of freedom. As the basic objects in this scheme are matrices, the description of gravity is noncommutative. In the 'twisted gravity' approach [2] the basic idea is to deform the usual group of diffeomorphisms by a twist into a Hopf algebra, thus extending the action of symmetries to noncommutative spaces. The metric and the connection in this framework are defined by an appropriate generalization of the Einstein-Hilbert action. In the proposal which we study, noncommutative gravity is identified with noncommutative geometry through the 'noncommutative frame formalism', [3]. Thus from the point of view of physics gravity is treated as background and nondynamical (at this step); objects which characterize noncommutative space like the metric or the curvature are inherent and purely geometric. We will try here to explain, in short strokes and through simple examples, how is this idea realized.

2. Introduction

One aspect of noncommutative geometry is discretization and fragmentation of spacetime. They are both obtained as a result of representation by operators: the eigenvalues of coordinates give specific discretization and the uncertainty relations arising from $[x^\mu, x^\nu] = i\bar{k}J^{\mu\nu}$ give fragmentation into Planck cells. In particular, an important class of noncommutative spaces is provided by finite matrix algebras which have very interesting discrete geometries. Our aim is to show that on every algebraic structure one can define differential geometry in consistent and natural way, thus giving it a notion of smoothness necessary to introduce physics.

2.1 Basics of differential geometry in the coordinate basis

We start with a brief list of concepts in differential geometry which are of importance for gravity [4, 5]. The basic notion is that of a *manifold* or spacetime $\mathscr{M}$. It is a topological space which is homeomorphic to $\mathbb{R}^n$ locally, but may be different from $\mathbb{R}^n$ globally. This means that on $\mathscr{M}$ we can introduce local coordinates x^μ, $\mu = 1,\dots,n$; the dimension of space is n. Significance of manifolds in part lays in the fact that one can use de Rham differential calculus developed on $\mathbb{R}^n$; its main elements are as follows.

The basic *algebraic structure* on $\mathscr{M}$ is given by functions of coordinates $f(x^\mu)$. The algebra of functions can even describe geometric notions as points: a point $a^\mu \in \mathscr{M}$ is identified with the δ-function $\delta(x^\mu - a^\mu)$. A *curve* $c(t)$ is a mapping $c : (a,b) \to \mathscr{M}$, that is a set of one-parameter functions $x^\mu = f^\mu(t)$, $t \in (a,b)$.

Vectors X are defined as tangent vectors to curves. A tangent vector of a function f along a curve $c(t)$ is $df(c(t))/dt$, so a vector field is a map X: functions $\rightarrow$ functions. The set of all vector fields is a linear space in which as basis we can choose partial derivatives ∂_μ. Expanding in this basis we have $X = X^\mu \partial_\mu$. Vector fields obey the Leibniz rule,

$$X(fh) = (Xf)h + f(Xh). \tag{2.1}$$

Linear space dual to the tangent space is the cotangent space: it consists of differential *1-forms* $\chi, \omega \dots$ which map χ : vectors $\rightarrow$ functions. The basis of 1-forms dual to ∂_μ is dx^μ: $dx^\mu(\partial_\nu) = \delta^\mu_\nu$. Dimensions of both tangent and cotangent spaces are n, equal to the dimension of $\mathscr{M}$. By making direct products of tangent and cotangent spaces and imposing linearity we obtain *tensors*. In the coordinate basis a (q,r) tensor T is $T = T^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_r} \partial_{\mu_1} \otimes \dots \partial_{\mu_q} \otimes dx^{\nu_1} \otimes \dots dx^{\nu_r}$.

Differential forms of higher rank can be defined by exterior multiplication of 1-forms. Exterior or *wedge product* of two 1-forms is a 2-form: the space of 2-forms is linear, so the exterior product can be defined by linearity and by the action in the basis:

$$dx^\mu \wedge dx^\nu = P^{\mu\nu}{}_{\rho\sigma} dx^\rho \otimes dx^\sigma = \frac{1}{2}(dx^\mu dx^\nu - dx^\nu dx^\mu), \tag{2.2}$$

where $P^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$. In the following we will often omit the sign $\wedge$, writing for instance $dx^\mu dx^\nu = -dx^\nu dx^\mu$. In analogy, the space of r-forms, $r \leq n$, is a linear space with a basis consisting of totally antisymmetric products of dx^μ. There is only one linearly independent form of the highest rank n which is called the volume form. *Exterior algebra* is the linear space of differential forms of arbitrary rank, including functions which are 0-forms.

Exterior derivative is a linear mapping d : r-forms $\rightarrow$ $(r+1)$-forms defined as

$$d\omega = \frac{1}{r!}\, \partial_\nu \omega_{\mu_1 \dots \mu_r}\, dx^\nu dx^{\nu_1} \dots dx^{\nu_r}. \tag{2.3}$$

In particular, $df = (\partial_\mu f) dx^\mu$. One can show that $d^2 = 0$, and that the differential obeys the Leibniz rule, $d(fh) = df\, h + f\, dh$.

In order to define the length of vectors and to compare vector and tensor fields at different points of the manifold one introduces additional structures. The length is defined by the *metric* g which is a (0,2) tensor field, g : vector $\otimes$ vector $\rightarrow$ function; in the coordinate basis, $g(\partial_\rho \otimes \partial_\sigma) = g_{\rho\sigma}$. The metric is bilinear, which means that when we multiply a vector by a number (possibly different at different points), its length is multiplied by the same factor, $g(f\partial_\rho \otimes \partial_\sigma) = f g_{\rho\sigma}$ and $g(\partial_\rho \otimes f\partial_\sigma) = f g_{\rho\sigma}$. One can also define the inverse metric, $g(dx^\mu \otimes dx^\nu) = g^{\mu\nu}$, $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$. In the common notation metric is written as the line element, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where now of course the product $dx^\mu dx^\nu$ is not the exterior product.

Assume that we have a vector field Y given on $\mathscr{M}$. To differentiate it along a curve $c(t)$ we have first to 'translate' Y from point $c(t+dt)$ to $c(t)$ and then to subtract the correspoding values. *Affine connection* or covariant derivative ∇ is a mapping which paralelly transports a vector field Y along the curve $c(t)$ (defined by its tangent X), to another vector field $\nabla_X Y$. Since the connection maps ∇: vector $\times$ vector $\rightarrow$ vector, it is a 1-form. In the coordinate basis we have $\nabla_X Y = X^\mu \,(\frac{\partial Y^\lambda}{\partial x^\mu} + Y^\nu \Gamma^\lambda{}_{\mu\nu})\, \partial_\lambda$, in components, $\nabla_\mu Y^\lambda = \frac{\partial Y^\lambda}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu} Y^\nu$. The covariant derivative can be extended to all tensor fields, for example for the metric we have $\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\rho{}_{\nu\lambda} g_{\rho\mu} - \Gamma^\rho{}_{\nu\mu} g_{\lambda\rho}$. If the last expression is zero, we say that the connection is compatible with the metric.

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2.2 Cartan moving frames

Physical quantities (fields) are scalars, vectors, forms, etc., and their components are defined by the choice of basis in tangent and cotangent spaces. But physics is insensitive to the change of basis, and certainly more natural than the coordinate bases ∂_μ and dx^μ (which are of mathematical convenience) is the basis in which spacetime is locally flat. It is called the Cartan *moving frame* or *vielbein*, or tetrad in four dimensions. In tangent space basis vectors are denoted by $e_\alpha = e^\mu_\alpha \partial_\mu$. If we choose the basis to be locally orthonormal then

$$g(e_\alpha \otimes e_\beta) = \eta_{\alpha\beta} = e^\mu_\alpha e^\nu_\beta g_{\mu\nu}, \tag{2.4}$$

where $\eta_{\alpha\beta}$ is the constant Minkowski metric. The 1-forms dual to e_α are θ^α: $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$. In components, $\theta^\alpha = \theta^\alpha_\mu dx^\mu$ and $\theta^\alpha_\mu e^\mu_\beta = \delta^\alpha_\beta$. Thus the line element is written as

$$ds^2 = \eta_{\alpha\beta}\theta^\alpha\theta^\beta \tag{2.5}$$

and $g_{\mu\nu} = \theta^\alpha_\mu \theta^\beta_\nu \eta_{\alpha\beta}$. The fact that the moving frame depends on the point

$$d\theta^\alpha = (d\theta^\alpha_\mu)dx^\mu = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma \neq 0 \tag{2.6}$$

indicates that the space is curved. As all other 1-forms, θ^α anticommute.

The simplest example is the moving frame of the Minkowski space, $ds^2 = -dt^2 + (dx^i)^2$, $i = 1,2,3$. It is given by

$$\theta^0 = dt, \quad \theta^a = \delta^a_i dx^i. \tag{2.7}$$

Obviously from $d^2 = 0$ we see that $d\theta^a = 0$ and the space is flat. Of course we can write the line element in polar coordinates too: $ds^2 = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2$, and then

$$\theta^0 = dt, \quad \theta^1 = dr, \quad \theta^2 = r\,d\theta, \quad \theta^3 = r\sin\theta\, d\varphi. \tag{2.8}$$

The latter is a special case of the Schwarzschild frame for $m = 0$,

$$\theta^0 = \sqrt{1-\frac{2m}{r}}\,dt, \quad \theta^1 = \frac{1}{\sqrt{1-\frac{2m}{r}}}\,dr, \quad \theta^2 = r\,d\theta, \quad \theta^3 = r\sin\theta\, d\varphi. \tag{2.9}$$

For de Sitter space, if we choose the coordinates such that $ds^2 = -dt^2 + e^{2\alpha t}(dx^i)^2$, the moving frame is given by

$$\theta^0 = dt, \quad \theta^a = e^{\alpha t}\,\delta^a_i dx^i. \tag{2.10}$$

The connection 1-form can of course also be expanded in the moving frame basis, $\omega^\alpha{}_\beta = \omega^\alpha{}_{\gamma\beta}\theta^\gamma$, and the condition that it is compatible with metric is its antisymmetry $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. The frame and the connection define the *torsion* $T^\alpha = \frac{1}{2}T^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma$ and the *curvature* $\Omega^\alpha{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\theta^\gamma\theta^\delta$ through the Cartan structure equations

$$T^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta\theta^\beta, \quad \Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\omega^\gamma{}_\beta. \tag{2.11}$$

If the space is torsion-free, the connection is related in a simple way to the Ricci rotation coefficients $C^\alpha{}_{\beta\gamma}$: $\omega_{\alpha\beta\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta})$.

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Geometry of spacetime, that is kinematics of gravity is described by the vielbein and the connection which give the metric, curvature and torsion. In order to speak of dynamics of matter on a given space we need the action for the matter fields and the corresponding variational principle: that is, besides the differential we need the integration. The integral on a manifold is also defined basically on $\mathbb{R}^n$, except that one has to be careful to use the invariant volume element given by the *volume form* Θ,

$$\Theta = \sqrt{|g|}\, dx^1 \dots dx^n = \theta^1 \dots \theta^n. \tag{2.12}$$

Physical fields which we typically have are scalars ϕ, spinors ψ and gauge fields A. The vector potential A is a connection in the gauge group and a 1-form, $A = A_\mu dx^\mu = A_\alpha \theta^\alpha$; the corresponding field strength is a 2-form, $F = dA + A^2 = \frac{1}{2} F_{\alpha\beta}\theta^\alpha\theta^\beta$. It is perhaps instructive to write its components in the frame basis: $F_{\alpha\beta} = e_{[\alpha}A_{\beta]} - A_\gamma C^\gamma{}_{\alpha\beta} + [A_\alpha, A_\beta]$.

3. Noncommutative geometry

3.1 Noncommutative space

We now wish to generalize the structure just described to the case when the spacetime is a noncommutative algebra $\mathscr{A}$. We assume that $\mathscr{A}$ is generated by elements (operators, matrices) x^μ, coordinates[1]; of course we cannot map $\mathscr{A} \to \mathbb{R}^n$ now. It is even nontrivial to determine 'dimensionality' because we do not consider $\mathscr{A}$ as a linear space but as an algebra. A manifold is usually defined by a rule which its coordinates satisfy ('the set of all points that...'); likewise, a noncommutative space is defined by commutation relations

$$[x^\mu, x^\nu] = iJ^{\mu\nu}(x). \tag{3.1}$$

By definition we have the basic algebraic structure of *functions* $f(x^\mu)$ on $\mathscr{A}$. We can therefore define vector fields or *derivations* as before: as mappings X: functions $\to$ functions which satisfy the Leibniz rule, $X(fh) = (Xf)h + f(Xh)$. However the lack of commutativity implies that the space of derivations is not linear, that is, a product of function g with vector field X is not a vector field anymore because the Leibniz rule does not hold,

$$gX(fh) \neq gX(f)\,h + f\,gX(h) \tag{3.2}$$

as $gf \neq fg$. Similarly, Xg is not a vector field either. An important kind of derivations on noncommutative spaces are *inner derivations* defined as commutators with the elements of the algebra,

$$Xf = [p, f]; \tag{3.3}$$

they are derivations because $[p, fh] = [p, f]h + f[p, h]$. One can notice two things: first, in the case of commutative manifolds partial derivatives ∂_μ do not belong to $\mathscr{M}$, they are 'outer derivations'. The opposite situation, when all vector fields are inner is also possible: on a space generated by finite matrices all derivations are inner.

[1] We do *not* distinguish the noncommuting objects by special notation like $\hat{x}^\mu$.

Since the set of vector fields is not closed under multiplication with functions one preferably uses the dual space of *1-forms* which is a linear space. Naturally no rule of multiplication remains the same as in the commutative case, and in general we have for two 1-forms χ and ω we have

$$f\omega \neq \omega f, \quad \chi \wedge \omega \neq -\omega \wedge \chi. \tag{3.4}$$

Next we need the differential calculus. One can define a *differential* d as a linear mapping d: r-forms $\to$ $(r+1)$-forms which obeys additional requirements of associativity, Leibniz rule and $d^2 = 0$. Differential should also preserve constraints in the algebra, that is

$$d[x^\mu, x^\nu] = [dx^\mu, x^\nu] + [x^\mu, dx^\nu] = i\bar{k}\, dJ^{\mu\nu}. \tag{3.5}$$

For example for constant noncommutativity

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu} = \text{const} \tag{3.6}$$

it is consistent to assume that

$$[dx^\mu, x^\nu] = 0 \tag{3.7}$$

because differential of a constant is zero. The last relation fully defines the calculus on the corresponding noncommutative space (which as we shall see shortly is not the only one consistent with (3.6)). On the other hand if noncommutativity is of the Lie-algebra type,

$$[x^\mu, x^\nu] = i f^{\mu\nu}{}_\rho x^\rho, \tag{3.8}$$

$f^{\mu\nu}{}_\rho =$const, it is easy to see that the choice (3.7) is inconsistent with (3.8). In general for a fixed set of commutation relations (a fixed noncommutative space), differential is not given uniquely and consequently the multiplication rules in the exterior algebra are in principle a subject of an additional (though not completely independent) definition. The question then is, how to choose the differential? Should one go from example to example, or attempt to define some kind of a 'canonical' calculus, significant for physics for example, or for gravity?

3.2 Noncommutative frames

One possibility is to suppose that the calculus is flat locally and the same as one which we defined for the Heisenberg algebra (3.6-3.7). The obstacle is, what is exactly the meaning of 'locally' in noncommutative geometry, what does it mean to get 'close' to a point (of the spectrum?) and expand? In order to do that, we should know representations of a given noncommutative space.

Proposal given in [3] is based on the two main ideas: to adapt the differential calculus to general relativity and to keep linearity. The differential calculus is defined in close analogy to the Cartan moving frame formalism. Moving frames have special role in gravity because they give a basis in which the equivalence principle is realized. In noncommutative geometry this role is extended: we assume that commutators between functions and the frame 1-forms θ^α vanish,

$$[f, \theta^\alpha] = 0 \tag{3.9}$$

and this is the additional relation which we need to define the exterior algebra. Relation (3.9) has of course to be consistent with (3.1); there are further consistency constraints, [6].

When we have frame 1-forms θ^α we can determine *dual derivations* e_α from $\theta^\alpha(e_\beta)=\delta^\alpha_\beta$. The *differential* d is defined as

$$df(e_\alpha)=e_\alpha f, \qquad d=\theta^\alpha e_\alpha. \tag{3.10}$$

Definition (3.9) is motivated by the requirement that components of the metric are constant in the frame basis, $g(\theta^\alpha\otimes\theta^\beta)=\eta^{\alpha\beta}$. In fact, if we impose (left and right) linearity of the metric g, requirement that components $g_{\alpha\beta}$ be constant (or in the center of the algebra) implies relation (3.9). Now we can establish the analogy with the formulae from the usual differential geometry. Differentiating coordinates, $dx^\mu=(e_\alpha x^\mu)\theta^\alpha$, we get $e^\mu_\alpha=e_\alpha x^\mu$; by linearity of the metric then

$$g(dx^\mu\otimes dx^\nu)=g(e^\mu_\alpha\theta^\alpha\otimes e^\nu_\beta\theta^\beta)=e^\mu_\alpha e^\nu_\beta\eta^{\alpha\beta}=g^{\mu\nu}. \tag{3.11}$$

In comparison with the symplectic structure on commutative manifolds, there is a new pattern in noncommutative geometry: one can identify in some cases the phase space with the position space. This happens when derivations e_α are inner, given by the *momenta* $p_\alpha\in\mathscr{A}$: $e_\alpha f=[p_\alpha,f]$. In particular, $e^\mu_\alpha=[p_\alpha,x^\mu]$ then.

As in commutative geometry, the affine connection can be chosen freely. The torsion and the curvature are defined through the Cartan structure equations, but conditions of linearity and hermiticity are stronger here and constrain the system nontrivially, [3].

4. Examples

4.1 Flat space, de Sitter space

The simplest example is the flat space: the basic commutators in the algebra are constant,

$$[x^\mu,x^\nu]=i\bar{k}J^{\mu\nu}=\mathrm{const} \tag{4.1}$$

and we assume usually that matrix $J^{\mu\nu}$ is nondegenerate. If we take as momenta

$$p_\mu=(i\bar{k}J)^{-1}_{\mu\nu}x^\nu \tag{4.2}$$

we obtain $e^\rho_\mu=[p_\mu,x^\rho]=\delta^\rho_\mu$ and the frame is flat:

$$\theta^\alpha=\delta^\alpha_\mu dx^\mu, \qquad g_{\mu\nu}=e^\alpha_\mu e^\beta_\nu\eta_{\alpha\beta}=\eta_{\mu\nu}. \tag{4.3}$$

From (3.9) we see that the exterior algebra is (3.7). We can notice further that the commutative limit $\bar{k}\to 0$ is singular as it means that the momenta tend to infinity: in fact, inner derivatives become outer.

There are other possible choices of momenta which are consistent with commutator (4.1): let us mention one in two dimensions. Consistently with $[t,x]=i\bar{k}$ we can also set

$$p_0=\frac{i}{\bar{k}}x, \qquad p_1=\frac{i}{\alpha\bar{k}}e^{-\alpha t}. \tag{4.4}$$

The vielbein is

$$e^0_0=[p_0,t]=1, \qquad e^1_1=[p_1,x]=e^{-\alpha t}, \tag{4.5}$$

and the off-diagonal components are zero. This corresponds to the frame of de Sitter space (2.10). From $\theta^0=dt$, $\theta^1=e^{-\alpha t}dx$ we obtain that the only nonvanishing Ricci rotation coefficient is ${C^1}_{01}=-\alpha=\mathrm{const}$: the space has constant curvature.

4.2 Fuzzy sphere

Coordinates x^m, $m = 1,2,3$ of the fuzzy sphere are proportional to generators J^m of the SU(2) group in irreducible representation by $n \times n$ matrices,

$$[x^m, x^n] = \frac{i\bar{k}}{r}\,\varepsilon^{mn}{}_p x^p. \tag{4.6}$$

For large r (and for large n) $x^m = \frac{\bar{k}}{r} J^m$ approximately. This space is two-dimensional because Casimir relation $(J^m)^2 = (n^2-1)/4$ can be interpreted as the condition that coordinates lay on the sphere, $(x^m)^2 = r^2$.

What is the differential? We have seen already that, as we are in a Lie algebra, dx^m is not a frame. We can define θ^a by the following choice of the momenta p_a,

$$p_a = \frac{1}{i\bar{k}}\,\delta_{am} x^m. \tag{4.7}$$

The corresponding vector fields e_a are generators of the spherical symmetry,

$$[e_a, e_b] = C^c{}_{ab} e_c, \qquad C^c{}_{ab} = \frac{1}{r}\,\varepsilon_{cab}. \tag{4.8}$$

We obtain easily for the vielbein components $e^m_a = [p_a, x^m] = -\frac{1}{r}\,\varepsilon_{mab} x^b$, that is

$$dx^m = \frac{1}{r}\,\varepsilon_{mnb} x^n \theta^b. \tag{4.9}$$

There is a peculiar fact: we have three basis 1-forms and three basis derivations. From $[x^m, \theta^a] = 0$ one can check for example that $dx^m x^m = -x^m dx^m$, which just means that the Casimir relation is stable under differentiation, $dx^m x^m + x^m dx^m = 0$. To invert (4.9) and to find θ^a_m we need to use the special 1-form called the 'Dirac operator', $-p_a\theta^a$

$$-p_a\theta^a = \frac{i}{\bar{k}} x_a \theta^a = \frac{r^2}{\bar{k}^2} x_m\, dx^m, \tag{4.10}$$

and after some algebra we obtain

$$\theta^a_m = \frac{1}{r}\,\varepsilon^{am}{}_c x^c + \frac{i\bar{k}}{r^2}\,\delta^{am} - \frac{i}{\bar{k}} x^a x^m. \tag{4.11}$$

Therefore the coordinate components of the metric (assuming that it is euclidean, $g^{ab} = \delta^{ab}$), are

$$g^{mn} = g(C^m{}_{pa} x^p \theta^a \otimes C^n{}_{qb} x^q \theta^b) = \frac{1}{r^2}(r^2\delta^{mn} - \frac{1}{2}\{x^m, x^n\} - \frac{1}{2}[x^m, x^n]). \tag{4.12}$$

The metric is, as are the tangent and cotangent spaces, three-dimensional. In the commutative limit $\bar{k} \to 0$ we recover the metric of the sphere, a two-dimensional object. Thus we see that though the position algebra is two-dimensional, the form algebra is of one dimension higher: the sphere is, also in this sense, fuzzy.

We shall see later that for the Lie-algebra noncommutativity it is consistent to assume that the frame 1-forms anticommute, $\{\theta^a, \theta^b\} = 0$. Further we have that $d\theta^a = -\frac{1}{2} C^a{}_{bc}\,\theta^b\theta^c$, with $C^a{}_{bc}$ given by (4.8). If we define the connection so that the torsion vanish, $\omega_{acb} = -\frac{1}{2} C_{abc}$, we obtain that the curvature is constant,

$$R^a{}_{bcd} = \frac{1}{4r^2}(\delta^a_c \delta_{bd} - \delta^a_d \delta_{bc}), \quad R = \frac{3}{2r^2}. \tag{4.13}$$

4.3 Quantum line

Quantum or q-euclidean spaces of dimension n are algebras generated by coordinates x^i, $i = 1 \ldots n$, which satisfy a quadratic relation of the form

$$P^{ij}{}_{lm} x^l x^m = 0 \tag{4.14}$$

and are covariant under the coaction of the quantum group $SO_q(n)$, [7]. We are interested in the quantum line $\mathbb{R}^1_q$. In general $\mathbb{R}^n_q$ can be extended by an operator Λ which satisfies

$$x^i \Lambda = q \Lambda x^i, \tag{4.15}$$

in our case $x\Lambda = q\Lambda x$ and x is hermitian, Λ unitary. When x and Λ are both unitary operators (4.15) is the Weyl algebra; as noncommutative space it is called the fuzzy torus. In the special case $q^n = 1$, the Weyl algebra has a finite-dimensional representation

$$x = \begin{pmatrix} 1 & 0 & 0 & . & . & . \\ 0 & q & 0 & . & . & . \\ 0 & 0 & q^2 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & q^{n-1} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 0 & . & . \\ 0 & 0 & 0 & 1 & . & . \\ . & . & . & . & . & . \\ 1 & . & . & . & . & 0 \end{pmatrix}. \tag{4.16}$$

On the quantum line parameter q is real and $q > 1$, but we can still represent the algebra on a Hilbert space in a similar way, [8]

$$x|k\rangle = q^k |k\rangle, \qquad \Lambda|k\rangle = |k+1\rangle; \tag{4.17}$$

$|k\rangle$ is the infinite-dimensional version of the basis of (4.16). The explicit wave-function representation of (4.17), called the Bohr compactification of the real line, [9], is given by

$$\langle \lambda | k \rangle = e^{-i\lambda k}, \tag{4.18}$$

where $\langle \lambda | k \rangle$ is the momentum-space wave function which is obviously of the same form as the Fourier transformation of the δ-function. This representation differs from the ordinary one in the scalar product which is rescaled or 'compactified':

$$\langle \psi | \chi \rangle = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \psi^*(\lambda) \chi(\lambda)\, d\lambda. \tag{4.19}$$

Therefore the basis $|k\rangle$ is properly normalized, $\langle k|l\rangle = \delta_{kl}$, for all $k, l \in \mathbb{R}$. Instead of x we can introduce coordinate y such that $q^y = x$; then we have

$$y|k\rangle = k|k\rangle, \qquad \Lambda^{-1} y \Lambda = y + 1. \tag{4.20}$$

One differenial calculus on $\mathbb{R}^1_q$ can be defined choosing the momentum to be $p_1 = \frac{q}{1-q}\Lambda$. Then

$$e_1 x = [p_1, x] = q\Lambda x, \qquad e_1 \Lambda = 0, \qquad e_1 y = \frac{q}{q-1}\Lambda. \tag{4.21}$$

The 1-form dual to e_1 is $\theta^1 = \Lambda^{-1}x^{-1}dx$. It is not straightforward to interpret $\mathbb{R}^1_q$ with the differential $d = \theta^1 e_1$ as a one-dimensional space because its metric depends on the phase-space variable Λ, $g_{11} = q^3\Lambda^2 x^2$; we can either enlarge the space including Λ, or alter the calculus, for further details see [8]. But this differential has very interesting action on functions $f(y)$ in the eigenbasis $|k\rangle$. If we define that θ^1 is on the space of states represented by identity, we obtain

$$df(y)|k\rangle = e_1 f(y)|k\rangle = \frac{q}{q-1}\big(f(k+1) - f(k)\big)\,|k+1\rangle, \tag{4.22}$$

so in this basis d acts as a finite-difference operator. The laplacian $\Delta = e_1 e^1$ is then given by

$$\Delta f(y)|k\rangle = \frac{q^2}{(q-1)^2}\big(f(k+2) - 2f(k+1) + f(k)\big)\,|k+2\rangle. \tag{4.23}$$

We see also easily that for other choices like $p_1' = (\Lambda)^\mu$, where μ is a real number, we have $\Lambda^{-\mu} y \Lambda^\mu = y + \mu$ and therefore $df|k\rangle \sim \big(f(k+\mu) - f(k)\big)|k+\mu\rangle$.

The algebra $\mathbb{R}^1_q$ and the calculus (4.22) are important in the loop quantum cosmology. Coordinate y is in fact the variable p which describes the scale factor $|p| = a^2 l_0^2/4$ and the volume V of the universe, while Λ^μ correspond to the holonomies. The connection c, roughly equal to the $\log\Lambda$, is in fact not a well defined operator, much the same as it was not defined in the finite representation (4.16). The gravitational part of the Hamiltonian constraint that is a part of the equations of motion is written in terms of the laplacian Δ, [10, 11].

5. Truncated Heisenberg space

5.1 Exterior algebra

The guiding idea of the construction of the truncated Heisenberg algebra [12] was to obtain a matrix approximation to the Heisenberg algebra (4.1). This can be achieved most naturally in the harmonic oscillator representation, by truncation of infinite matrices which represent coordinate x and momentum p (now, the second coordinate y) to finite $n \times n$ matrices

$$\mu x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & \sqrt{2} & \cdot & \cdot & \cdot \\ 0 & \sqrt{2} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \sqrt{n-1} \\ \cdot & \cdot & \cdot & \cdot & \sqrt{n-1} & 0 \end{pmatrix}, \quad \mu y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & -\sqrt{2} & \cdot & \cdot & \cdot \\ 0 & \sqrt{2} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & -\sqrt{n-1} \\ \cdot & \cdot & \cdot & \cdot & \sqrt{n-1} & 0 \end{pmatrix} \tag{5.1}$$

The third generator is needed to close the algebra; we take

$$\mu' z = n\begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Then the truncated Heisenberg algebra is defined by the following quadratic relations

$$[\mu x, \mu y] = i\varepsilon(1-\mu' z), \ \ [\mu x, \mu' z] = i\varepsilon(\mu y\mu' z + \mu' z\mu y), \ \ [\mu y, \mu' z] = -i\varepsilon(\mu x\mu' z + \mu' z\mu x). \tag{5.3}$$

Constant ε in (5.3) is a dimensionless parameter introduced to generalize relations (5.1-5.2), and to render at least formally the commutative limit. The μ and μ' have dimension of the inverse length and they describe 'magnitudes' of the x-y and z directions respectively. Clearly for $\varepsilon = 1$ the algebra reduces to the algebra of finite matrices, in other words the truncated Heisenberg algebra has finite-dimensional representations for $\varepsilon = 1$. On the other hand, the Heisenberg algebra $[x,y] = i\bar{k}$ ($\bar{k} = \varepsilon\mu^{-2}$) is obtained as contraction $\mu' \to 0$ of (5.3); we refer to this contraction as to subspace $z = 0$ of the truncated Heisenberg algebra.

What we wish to explore is the geometry of (5.3), that is its differential geometry. In the frame formalism we can define differential by choosing momenta; however, not every choice leads to an acceptable d, [3]. In particular, restriction $d^2 = 0$ gives the constraint

$$[p_\alpha, p_\beta] = \frac{1}{i\varepsilon}K_{\alpha\beta} + F^\gamma{}_{\alpha\beta}p_\gamma - 2i\varepsilon Q^{\gamma\delta}{}_{\alpha\beta}p_\gamma p_\delta. \tag{5.4}$$

Constants $K_{\alpha\beta}$, $F^\gamma{}_{\alpha\beta}$ and $Q^{\gamma\delta}{}_{\alpha\beta}$ are called the *structure coefficients*. It turns out that $Q^{\gamma\delta}{}_{\alpha\beta}$ determine also the exterior algebra. If we define the exterior multiplication of two frame 1-forms by $\theta^\gamma\theta^\delta = P^{\gamma\delta}{}_{\alpha\beta}\theta^\alpha\otimes\theta^\beta$, the consistency of the wedge product with the differential implies that

$$P^{\gamma\delta}{}_{\alpha\beta} = \frac{1}{2}(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\gamma_\beta\delta^\delta_\alpha) + i\varepsilon Q^{\gamma\delta}{}_{\alpha\beta}. \tag{5.5}$$

The last relation is a commutation rule, $\theta^\gamma\theta^\delta + \theta^\delta\theta^\gamma = 2i\varepsilon Q^{\gamma\delta}{}_{\alpha\beta}\theta^\alpha\theta^\beta$, and we see that in the case of quadratic algebras the frame 1-forms do *not anticommute*.

The truncated Heisenberg algebra is quadratic in its generators so the momenta can be introduced as

$$\varepsilon p_1 = i\mu^2 y, \quad \varepsilon p_2 = -i\mu^2 x, \quad \varepsilon p_3 = i\mu(\mu z - \frac{1}{2}). \tag{5.6}$$

For $z = 0$, p_1 and p_2 reduce to the flat-space momenta (4.2). The momentum algebra is

$$[p_1, p_2] = \frac{\mu^2}{2i\varepsilon} + \mu p_3, \ [p_2, p_3] = \mu p_1 - i\varepsilon(p_1p_3 + p_3p_1), \ [p_3, p_1] = \mu p_2 - i\varepsilon(p_2p_3 + p_3p_2) \tag{5.7}$$

and we can identify the structure coefficients:

$$K_{12} = \frac{\mu^2}{2}, \quad F^1{}_{23} = \mu, \quad Q^{13}{}_{23} = \frac{1}{2}, \quad Q^{23}{}_{31} = \frac{1}{2}. \tag{5.8}$$

As we have seen, exterior multiplication is completely given by $Q^{\gamma\delta}{}_{\alpha\beta}$: we have

$$\begin{aligned} &(\theta^1)^2 = 0, \quad (\theta^2)^2 = 0, \quad (\theta^3)^2 = 0, \quad \{\theta^1, \theta^2\} = 0, \\ &\{\theta^1, \theta^3\} = i\varepsilon(\theta^2\theta^3 - \theta^3\theta^2), \quad \{\theta^2, \theta^3\} = i\varepsilon(\theta^3\theta^1 - \theta^1\theta^3). \end{aligned} \tag{5.9}$$

Due to quadratic terms in (5.4) 1-forms do not anticommute, but the linear space of 2-forms is three-dimensional as in commutative geometry. We can extend the exterior algebra to 3-forms using associativity, [13]:

$$\theta^1\theta^3\theta^1 = \theta^2\theta^3\theta^2, \qquad \theta^3\theta^1\theta^3 = 0, \qquad \theta^3\theta^2\theta^3 = 0. \tag{5.10}$$
$$\theta^1\theta^2\theta^3 = -\theta^2\theta^1\theta^3 = \theta^3\theta^1\theta^2 = -\theta^3\theta^2\theta^1 = i\,\frac{\varepsilon^2-1}{2\varepsilon}\,\theta^2\theta^3\theta^2,$$
$$\theta^1\theta^3\theta^2 = -\theta^2\theta^3\theta^1 = i\,\frac{\varepsilon^2+1}{2\varepsilon}\,\theta^2\theta^3\theta^2.$$

Again, relations are unusual but perfectly consistent: they show that there is just one linearly independent 3-form, which implies further that the volume is unambiguously defined. We choose the volume form as $\Theta = -\dfrac{i}{2\varepsilon}\,\theta^2\theta^3\theta^2$.

5.2 Metric and connection

The metric structure of the space is completely fixed once the frame is defined. From the vielbein $e^\mu_\alpha = [p_\alpha, x^\mu]$ we obtain the differentials $dx^\mu = e^\mu_\alpha\theta^\alpha$,

$$dx = (1-\mu z)\theta^1 + \mu^2(yz+zy)\theta^3, \tag{5.11}$$
$$dy = (1-\mu z)\theta^2 - \mu^2(xz+zx)\theta^3,$$
$$dz = \mu^2(xz+zx)\theta^1 + \mu^2(yz+zy)\theta^2,$$

so the coordinate components of the metric $g^{\mu\nu} = e^\mu_\alpha e^\nu_\beta g^{\alpha\beta}$ are given by

$$g^{\mu\nu} = \begin{pmatrix} (1-\mu z)^2+\mu^4(yz+zy)^2 & -\mu^4(yz+zy)(xz+zx) & (1-\mu z)\mu^2(xz+zx) \\ -\mu^4(xz+zx)(yz+zy) & (1-\mu z)^2+\mu^4(xz+zx)^2 & (1-\mu z)\mu^2(yz+zy) \\ \mu^2(xz+zx)(1-\mu z) & \mu^2(yz+zy)(1-\mu z) & \mu^4(xz+zx)^2+\mu^4(yz+zy)^2 \end{pmatrix} \tag{5.12}$$

The metric is hermitian and reduces to diag $(1,1,0)$ on the subspace $z=0$ as it should. For the Ricci rotation coefficients we find

$$C^1{}_{23} = -C^1{}_{32} = 2\mu^2 z, \quad C^2{}_{31} = -C^2{}_{31} = 2\mu^2 z, \quad C^3{}_{12} = -C^3{}_{21} = \mu, \tag{5.13}$$
$$C^3{}_{13} = -C^3{}_{31} = 2\mu^2 x, \quad C^3{}_{23} = -C^3{}_{32} = 2\mu^2 y;$$

they are nonzero along the third direction even for $z=0$, which means that this subspace has extrinsic curvature. If we define the connection as $\omega_{\alpha\beta\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta})$, we have

$$\omega_{12} = -\omega_{21} = (-\frac{\mu}{2} + 2i\varepsilon p_3)\theta^3 = \mu\,(\frac{1}{2} - 2\mu z)\theta^3, \tag{5.14}$$
$$\omega_{13} = -\omega_{31} = \frac{\mu}{2}\,\theta^2 + 2i\varepsilon p_2\theta^3 = \frac{\mu}{2}\,\theta^2 + 2\mu^2 x\,\theta^3,$$
$$\omega_{23} = -\omega_{32} = -\frac{\mu}{2}\,\theta^1 - 2i\varepsilon p_1\theta^3 = -\frac{\mu}{2}\,\theta^1 + 2\mu^2 y\,\theta^3.$$

This connection gives the torsion and the Riemann curvature by equations (2.11); both quantities are quadratic in p_α. From the Riemann curvature tensor we can calculate the Ricci curvature, $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$, and the curvature scalar, $R = \eta^{\alpha\beta}R_{\alpha\beta}$. For R for we obtain

$$R = \frac{11}{4}\mu^2 - 2\mu^2(\mu z - \frac{1}{2}) - 4\mu^4(x^2+y^2), \tag{5.15}$$

or on $z = 0$,

$$R = \frac{15}{4}\mu^2 - 4\mu^4(x^2 + y^2). \tag{5.16}$$

5.3 Fields

Once we have the scalar curvature (5.16) it is not difficult to recognize the relation between the Grosse-Wulkenhaar action [14]

$$S = \int \frac{1}{2}(1 - \frac{\Omega^2}{2})\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 + \frac{\Omega^2}{2}\tilde{x}^\mu\tilde{x}_\mu\varphi\varphi + \frac{\lambda}{4!}\varphi^4 \tag{5.17}$$

and the action for the scalar field on a curved space,

$$S' = \int \frac{1}{2}(e_\alpha\varphi)(e^\alpha\varphi) + \frac{M^2}{2}\varphi^2 - \frac{\xi}{2}R\varphi^2 + \frac{\Lambda}{4!}\varphi^4. \tag{5.18}$$

Reducing the action (5.18) from three to two dimensions, to subspace $z = 0$, we have $e_{1,2} = \partial_{1,2}$ and $e_3 = 0$. Thus the action (5.18) becomes equal to (5.17) up to an overall rescaling $S = \kappa S'$ and

$$1 - \frac{\Omega^2}{2} = \kappa, \qquad m^2 = \kappa(M^2 - \xi a), \qquad \frac{\Omega^2\mu^4}{\varepsilon^2} = \kappa\xi b, \qquad \lambda = \kappa\Lambda, \tag{5.19}$$

with $a = 15\mu^2/2$, $b = 8\mu^4$. Therefore we see that the Grosse-Wulkenhaar action describes in fact a scalar field moving on a specific curved noncommutative background.

Similarly, performing the dimensional reduction in the Yang-Mills action for the U(1) gauge field we obtain a Kaluza-Klein model. As the truncated Heisenberg space is curved, the gauge field couples to the connection (in the frame basis). In the noncommutative setting the formula for the components of the field strength generalizes to

$$F_{\zeta\eta} = e_{[\zeta}A_{\eta]} - A_\alpha C^\alpha{}_{\zeta\eta} + [A_\zeta, A_\eta] + 2i\varepsilon(e_\beta A_\gamma)Q^{\beta\gamma}{}_{\zeta\eta} + 2i\varepsilon A_\beta A_\gamma Q^{\beta\gamma}{}_{\zeta\eta} \tag{5.20}$$

so again we have an explicit dependence on coordinates in the action, in accordance with (5.13). Reducing to $z = 0$ the third component of the gauge potential becomes a scalar field, $A_3 = \phi$, while the field strength becomes

$$F_{12} = \mathscr{F}_{12} - \mu\phi, \quad F_{13} = D_1\phi - i\varepsilon\{p_2 + A_2, \phi\}, \quad F_{23} = D_2\phi + i\varepsilon\{p_1 + A_1, \phi\}, \tag{5.21}$$

where $\mathscr{F}_{12} = e_1A_2 - e_2A_1 + [A_1, A_2]$. After the insertion of $F_{\alpha\beta}$ into the Yang-Mills action we obtain, [13]

$$S_{YM} = \frac{1}{2}\int ((1-\varepsilon^2)(\mathscr{F}_{12})^2 - 2(1-\varepsilon^2)\mu\mathscr{F}_{12}\phi + (5-\varepsilon^2)\mu^2\phi^2 + 4i\varepsilon\mathscr{F}_{12}\phi^2$$
$$+ (D_1\phi)^2 + (D_2\phi)^2 - \varepsilon^2\{p_1 + A_1, \phi\}^2 - \varepsilon^2\{p_2 + A_2, \phi\}^2).$$

This is the action for noncommutative U(1) Kaluza-Klein theory obtained by dimensional reduction from the truncated Heisenberg space to the Heisenberg algebra. It is indeed a challenging task to understand details of the classical dynamics and quantization of this model.

6. Conclusions

We defined in this lecture, more in an intuitive than in a precise way, a noncommutative generalization of the Cartan moving frame formalism, and we described geometries of some noncommutative spaces which found applications in physics. The simplest of them and the most frequently used is the Moyal or flat noncommutative space (4.1). Dynamics of fields on the fuzzy sphere (4.6) and various aspects of its geometry have been also understood in many details, while the quantum euclidean spaces (4.14) and quantum groups became a classical subject already in the nineties. We here put the emphasis on geometry of the truncated Heisenberg algebra as it nontrivially embodies many specific properties of noncommutative models. And of course because it gives an interesting geometric interpretation of the Grosse-Wulkenhaar model.

Although at the moment the noncommutative frame formalism does not give one preferred model or description of noncommutative geometry, it certainly opens new possibilities and introduces new kinds of behavior in geometry and in subsequent physics. One of its characteristic aspects is the capacity to define geometry of the matrix spaces, to give them a notion of smoothness. Spaces of matrices are potentially very interesting as, unlike lattice discretizations, they allow representations of symmetries; furthermore being finite they in principle provide with finite or renormalizable field theories (defined either directly or as a limit). In such geometries the phase space and the position space are identified, which is a new feature too. One could even conjecture that, because of the mathematical minimality of such models, they are preferred by the formalism. But we have seen on the other hand that there are important examples in which the representation space is infinite and the momenta remain 'outside': the corresponding models are neither minimal nor simple. Thus it is very important to investigate and understand this kind of structures, and in particular to find out whether they can support physically relevant geometries as for example noncommutative versions of the Schwarzschild or FRW spaces. But as we have seen, this task is far from being straightforward.

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