

Exact Amplitude-Based Resummation in Quantum Field Theory: Recent Results

B.F.L. Ward^{*†}

Baylor University

E-mail: bfl_ward@baylor.edu

S.K. Majhi

Indian Association for the Cultivation of Science

E-mail: tpskm@iacs.res.in

S.A. Yost[‡]

The Citadel

E-mail: scott.yost@citadel.edu

We present the current status of the application of our approach of exact amplitude-based resummation in quantum field theory to two areas of investigation: precision QCD calculations of all three of us as needed for LHC physics and the resummed quantum gravity realization by one of us (B.F.L.W.) of Feynman's formulation of Einstein's theory of general relativity. We discuss recent results as they relate to experimental observations. There is reason for optimism in the attendant comparison of theory and experiment.

10th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)

September 26-30, 2011

Mamallapuram, India

^{*}Speaker.

[†]Work supported in part by D.o.E. grant DE-FG02-09ER41600.

[‡]Work supported in part by D.o.E. grant DE-PS02-09ER09-01 and grants from The Citadel Foundation.

1. Introduction

With the start-up of the LHC the era of precision QCD, by which we mean predictions for QCD processes at the total precision tag of 1% or better, is upon us and the need for exact, amplitude-based resummation of large higher order effects is paramount. Such resummation allows one to have better than 1% precision as a realistic goal as we shall show in what follows, so that one can indeed distinguish new physics(NP) from higher order SM processes and can distinguish different models of new physics from one another as well. In a parallel development, the issue of the application of ordinary quantum field theoretic methods to Einstein's theory of general relativity lends itself as well to a resummation approach, provided again that the resummation is an exact amplitude-based one, as one of us(B.F.L.W.) has shown. In what follows, we present the status of these two applications of exact amplitude-based resummation theory in quantum field theory.

The two paradigms which we present are then as follows. First, in the next Section, we present an approach to precision LHC physics which is an amplitude-based QED \otimes QCD(\equiv QCD \otimes QED) exact resummation theory [1] realized by MC methods. The starting point is then the well-known fully differential representation

$$d\sigma = \sum_{i,j} \int dx_1 dx_2 F_i(x_1) F_j(x_2) d\hat{\sigma}_{\text{res}}(x_1 x_2 s) \quad (1.1)$$

of a hard LHC scattering process using a standard notation so that the $\{F_j\}$ and $d\hat{\sigma}_{\text{res}}$ are the respective parton densities and reduced hard differential cross section where we indicate the that latter has been resummed for all large EW and QCD higher order corrections in a manner consistent with achieving a total precision tag of 1% or better for the total theoretical precision of (1.1). The key issue to precision QCD theory is then the determination of the value of the total theoretical precision of (1.1), which we denote by $\Delta\sigma_{\text{th}}$. It can be decomposed as follows:

$$\Delta\sigma_{\text{th}} = \Delta F \oplus \Delta\hat{\sigma}_{\text{res}} \quad (1.2)$$

in an obvious notation where ΔA is the contribution of the uncertainty on A to $\Delta\sigma_{\text{th}}$. The theoretical precision $\Delta\sigma_{\text{th}}$ validates the application of a given theoretical prediction to precision experimental observations, for the discussion of the signals and the backgrounds for both SM and NP studies, and more specifically for the overall normalization of the cross sections in such studies. NP can be missed if a calculation with an unknown value of $\Delta\sigma_{\text{th}}$ is used for such studies. This point cannot be emphasized too much.

By our definition, $\Delta\sigma_{\text{th}}$ is the total theoretical uncertainty coming from the physical precision contribution and the technical precision contribution [2]: the physical precision contribution, $\Delta\sigma_{\text{th}}^{\text{phys}}$, arises from such sources as missing graphs, approximations to graphs, truncations,.....; the technical precision contribution, $\Delta\sigma_{\text{th}}^{\text{tech}}$, arises from such sources as bugs in codes, numerical rounding errors, convergence issues, etc. The total theoretical error is then given by

$$\Delta\sigma_{\text{th}} = \Delta\sigma_{\text{th}}^{\text{phys}} \oplus \Delta\sigma_{\text{th}}^{\text{tech}}. \quad (1.3)$$

The desired value for $\Delta\sigma_{\text{th}}$ depends on the specific requirements of the observations. As a general rule, one would like that $\Delta\sigma_{\text{th}} \leq f\Delta\sigma_{\text{expt}}$, where $\Delta\sigma_{\text{expt}}$ is the respective experimental error and

$f \lesssim \frac{1}{2}$ so that the theoretical uncertainty does not significantly affect the analysis of the data for physics studies in an adverse way.

With the goal of achieving such precision in a provable way, we have developed the QCD \otimes QED resummation theory in Refs. [1] for the reduced cross section in (1.1) and for the resummation of the evolution of the parton densities therein as well. In both cases, the starting point is the master formula

$$d\bar{\sigma}_{\text{res}} = e^{\text{SUM}_{\text{IR}}(\text{QCED})} \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int \prod_{j_1=1}^n \frac{d^3 k_{j_1}}{k_{j_1}} \prod_{j_2=1}^m \frac{d^3 k'_{j_2}}{k'_{j_2}} \int \frac{d^4 y}{(2\pi)^4} e^{iy \cdot (p_1 + q_1 - p_2 - q_2 - \sum k_{j_1} - \sum k'_{j_2}) + D_{\text{QCED}}} \tilde{\beta}_{n,m}(k_1, \dots, k_n; k'_1, \dots, k'_m) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0}, \quad (1.4)$$

where $d\bar{\sigma}_{\text{res}}$ is either the reduced cross section $d\hat{\sigma}_{\text{res}}$ or the differential rate associated to a DGLAP-CS [3, 4] kernel involved in the evolution of the $\{F_j\}$ and where the *new* (YFS-style [5]) *non-Abelian* residuals $\tilde{\beta}_{n,m}(k_1, \dots, k_n; k'_1, \dots, k'_m)$ have n hard gluons and m hard photons and we show the final state with two hard final partons with momenta p_2, q_2 specified for a generic $2f$ final state for definiteness. The infrared functions $\text{SUM}_{\text{IR}}(\text{QCED}), D_{\text{QCED}}$ are defined in Refs. [1, 6, 7]. This simultaneous resummation of QED and QCD large IR effects is exact. Moreover, the residuals $\tilde{\beta}_{n,m}$ allow a rigorous parton shower/ME matching via their shower-subtracted counterparts $\hat{\tilde{\beta}}_{n,m}$ [1].

The result in (1.4) also allows us an exact, amplitude-based resummation approach to Feynman's formulation of Einstein's theory, as one of us (B.F.L.W.) has shown in Refs. [8] via the following representation of the Feynman propagators in that theory:

$$\begin{aligned} i\Delta'_F(k) &= \frac{i}{(k^2 - m^2 - \Sigma_s + i\epsilon)} \\ &= \frac{ie^{B''_g(k)}}{(k^2 - m^2 - \Sigma'_s + i\epsilon)} \\ &\equiv i\Delta'_F(k)|_{\text{resummed}}. \end{aligned}$$

for scalar fields with an attendant generalization for spinning fields [8]. We stress that there are no approximations in (1.5). The formula for $B''_g(k)$ is given in Refs. [8] and is presented below. We now discuss the two paradigms opened by (1.4) for precision QCD for the LHC and for exact resummation of Einstein's theory in turn.

2. Precision QCD for the LHC

We first stress that the methods we employ for resummation of the QCD theory are fully consistent with the methods in Refs. [9, 10]. This can be seen by considering the application of the latter methods to the $2 \rightarrow n$ processes $[f]$ at hard scale Q , $f_1(p_1, r_1) + f_2(p_2, r_2) \rightarrow f_3(p_3, r_3) + f_4(p_4, r_4) + \dots + f_{n+2}(p_{n+2}, r_{n+2})$, where the p_i, r_i label 4-momenta and color indices respectively, by Akyat *et al.* in Ref. [11], where the respective amplitude is represented as

$$\begin{aligned} \mathcal{M}_{\{r_i\}}^{[f]} &= \sum_L^C \mathcal{M}_L^{[f]}(c_L)_{\{r_i\}} \\ &= J^{[f]} \sum_L^C S_{Ll} H_l^{[f]}(c_L)_{\{r_i\}}, \end{aligned} \quad (2.1)$$

where repeated indices are summed, and the functions $J^{[f]}$, S_{LI} , and $H_I^{[f]}$ are respectively the jet function, the soft function which describes the exchange of soft gluons between the external lines, and the hard coefficient function. The latter functions' infrared and collinear poles have been calculated to 2-loop order in Refs. [11]. To make contact between eqs.(1.4,2.1), identify in the specific application $\bar{Q}'Q \rightarrow \bar{Q}'''Q'' + m(G)$ in (1.4) $f_1 = Q, f_2 = \bar{Q}', f_3 = Q'', f_4 = \bar{Q}''', \{f_5, \dots, f_{n+2}\} = \{G_1, \dots, G_m\}$, in (2.1), where we use the obvious notation for the gluons here. This means that $n = m + 2$. Then, to use eq.(2.1) in eq.(1.4), one observes the following:

- I. By its definition in eq.(2.23) of Ref. [11], the anomalous dimension of the matrix S_{LI} does not contain any of the diagonal effects described by our infrared functions $\text{SUM}_{\text{IR}}(\text{QCD})$ and D_{QCD} , where

$$\begin{aligned} \text{SUM}_{\text{IR}}(\text{QCD}) &= 2\alpha_s \Re B_{\text{QCD}} + 2\alpha_s \tilde{B}_{\text{QCD}}(K_{\text{max}}), \\ 2\alpha_s \tilde{B}_{\text{QCD}}(K_{\text{max}}) &= \int \frac{d^3k}{k^0} \tilde{S}_{\text{QCD}}(k) \theta(K_{\text{max}} - k), \\ D_{\text{QCD}} &= \int \frac{d^3k}{k} \tilde{S}_{\text{QCD}}(k) \left[e^{-iy \cdot k} - \theta(K_{\text{max}} - k) \right], \end{aligned} \quad (2.2)$$

where the real IR emission function $\tilde{S}_{\text{QCD}}(k)$ and the virtual IR function $\Re B_{\text{QCD}}$ are defined eqs.(77,73) in Ref. [6]. Note that (1.4) is independent of K_{max} .

- II. By its definition in eqs.(2.5) and (2.7) of Ref. [11], the jet function $J^{[f]}$ contains the exponential of the virtual infrared function $\alpha_s \Re B_{\text{QCD}}$, so that we have to take care that we do not double count when we use (2.1) in (1.4) and in the equations in Refs. [1, 6, 7] that lead thereto.

In this way we get the following realization of our approach using the results in Ref. [11]: In our result in eq.(75) in Ref. [6] for the contribution to (1.4) of m -hard gluons for the process under study here,

$$\begin{aligned} d\hat{\sigma}^m &= \frac{e^{2\alpha_s \Re B_{\text{QCD}}}}{m!} \int \prod_{j=1}^m \frac{d^3k_j}{(k_j^2 + \lambda^2)^{1/2}} \delta(p_1 + q_1 - p_2 - q_2 - \sum_{i=1}^m k_i) \\ &\quad \bar{\rho}^{(m)}(p_1, q_1, p_2, q_2, k_1, \dots, k_m) \frac{d^3p_2 d^3q_2}{p_2^0 q_2^0}, \end{aligned} \quad (2.3)$$

we can identify the residual $\bar{\rho}^{(m)}$ as follows:

$$\begin{aligned} \bar{\rho}^{(m)}(p_1, q_1, p_2, q_2, k_1, \dots, k_m) &= \overline{\sum_{\text{colors, spin}} |\mathcal{M}_{\{r_i\}}^{[f]}|^2} \\ &\equiv \sum_{\text{spins, } \{r_i\}, \{r'_i\}} \mathfrak{h}_{\{r_i\}\{r'_i\}}^{cs} |\bar{J}^{[f]}|^2 \sum_{L=1}^C \sum_{L'=1}^C S_{LI}^{[f]} H_I^{[f]}(c_L)_{\{r_i\}} \left(S_{L'I'}^{[f]} H_{I'}^{[f]}(c_{L'})_{\{r'_i\}} \right)^\dagger, \end{aligned} \quad (2.4)$$

where here we defined $\bar{J}^{[f]} = e^{-\alpha_s \Re B_{\text{QCD}}} J^{[f]}$, and we introduced the color-spin density matrix for the initial state, \mathfrak{h}^{cs} , so that $\mathfrak{h}_{\{r_i\}\{r'_i\}}^{cs} = \mathfrak{h}_{\{r_1, r_2\}\{r'_1, r'_2\}}^{cs}$, suppressing the spin indices, i.e., \mathfrak{h}^{cs} only depends on the initial state colors and has the obvious normalization implied by (2.3). Proceeding then according to the steps in Ref. [6] leading from (2.3) to (1.4) restricted to QCD, we get the corresponding implementation of the results in Ref. [11] in our approach, without any double counting of effects. This proves that the new non-Abelian residuals $\tilde{\beta}_{m,n}$ in (1.4) transcend those of an Abelian massless gauge theory as introduced in Ref. [5].

As we have explained in Refs. [1], these new non-Abelian residuals allow rigorous shower/ME matching via their shower subtracted analogs:

$$\tilde{\tilde{\beta}}_{m,n} \rightarrow \hat{\hat{\beta}}_{m,n} \quad (2.5)$$

where the $\hat{\hat{\beta}}_{m,n}$ have had all effects in the showers associated to the $\{F_j\}$ removed from them.

When the formula in (1.4) is applied to the calculation of the kernels, P_{AB} , in the DGLAP-CS theory itself, we get an improvement of the IR limit of these kernels, an IR-improved DGLAP-CS theory [6, 7] in which large IR effects are resummed for the kernels themselves. The resulting new resummed kernels, P_{AB}^{exp} as given in Ref. [6, 7] and as illustrated below, yield a new resummed scheme for the PDF's and the reduced cross section:

$$F_j, \hat{\sigma} \rightarrow F'_j, \hat{\sigma}' \text{ for}$$

$$P_{gq}(z) \rightarrow P_{gq}^{exp}(z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2} \delta_q} \frac{1 + (1-z)^2}{z} z^{\gamma_q}, \text{ etc.},$$

with the same value for σ in (1.1) with improved MC stability as discussed in Ref. [12]. Here, the YFS [5] infrared factor is given by $F_{YFS}(a) = e^{-C_E a} / \Gamma(1+a)$ where C_E is Euler's constant and we refer the reader to Ref. [6, 7] for the definition of the infrared exponents γ_q , δ_q as well as for the complete set of equations for the new P_{AB}^{exp} . C_F is the quadratic Casimir invariant for the quark color representation.

The basic physical idea underlying the new kernels is illustrated in Fig. 2 as it was already shown by Bloch and Nordsieck [13]: an accelerated charge generates a coherent state of very

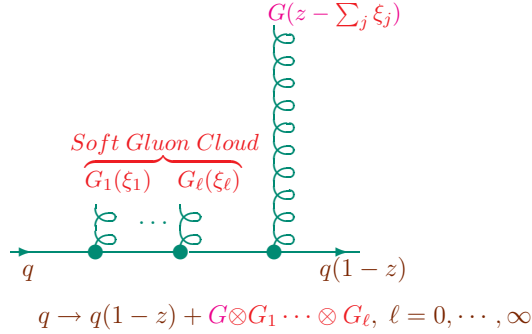


Figure 1: Bloch-Nordsieck soft quanta for an accelerated charge.

soft massless quanta of the respective gauge field so that one cannot know which of the infinity of possible states one has made in the splitting process $q(1) \rightarrow q(1-z) + G \otimes G_1 \cdots \otimes G_\ell, \ell = 0, \dots, \infty$ illustrated in Fig. 2. The new kernels take this effect into account.

The new MC Herwiri1.031 [12] gives the first realization of the new IR-improved kernels in the Herwig6.5 [14] environment. Here, we compare it with Herwig6.510, both with and without the MC@NLO [15] exact $\mathcal{O}(\alpha_s)$ correction, in Fig. 2 in relation to D0 data [16] on the Z boson p_T in single Z production and the CDF data [17] on the Z boson rapidity in the same process all at the Tevatron. We see [12] that the IR improvement improves the $\chi^2/d.o.f$ in comparison with the data in both cases for the soft p_T data and that for the rapidity data it improves the $\chi^2/d.o.f$

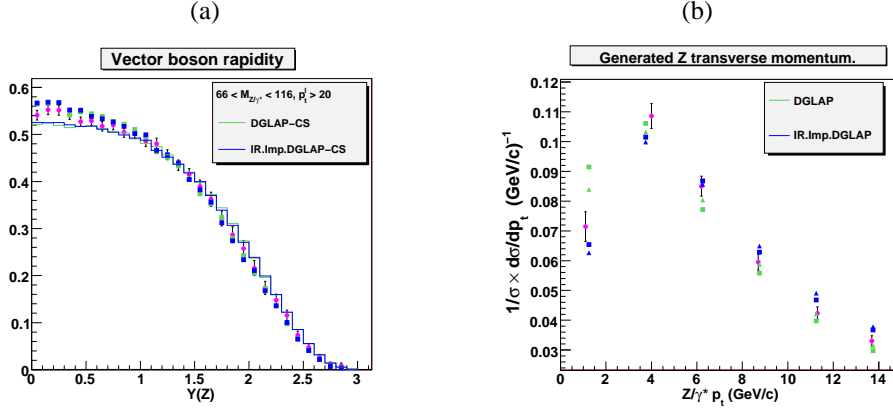


Figure 2: From Ref. [12], comparison with FNAL data: (a), CDF rapidity data on (Z/γ^*) production to e^+e^- pairs, the circular dots are the data, the light(dark) lines are HERWIG6.510(HERWIRI1.031); (b), D0 p_T spectrum data on (Z/γ^*) production to e^+e^- pairs, the circular dots are the data, the dark triangles are HERWIRI1.031, the light triangles are HERWIG6.510. In both (a) and (b) the dark squares are MC@NLO/HERWIRI1.031, and the light squares are MC@NLO/HERWIG6.510, where MC@NLO/X denotes the realization by MC@NLO of the exact $\mathcal{O}(\alpha_s)$ correction for the generator X. These are untuned theoretical results.

before the application of the MC@NLO exact $\mathcal{O}(\alpha_s)$ correction and that with the latter correction the $\chi^2/d.o.f$'s are statistically indistinguishable. More importantly, this theoretical paradigm can be systematically improved in principle to reach any desired $\Delta\sigma_{\text{th}}$. The suggested accuracy at the 10% level shows the need for the NNLO extension of MC@NLO, in view of our goals for this process. We are currently developing the analogous applications for the new kernels for Herwig++ [18], Herwiri++, for Pythia8 [19] and for Sherpa [20]. In addition we are currently analyzing recent LHC data using Herwiri1.031/MC@NLO, Herwiri++/Powheg [21] as we shall report elsewhere [22].

3. Resummed Quantum Gravity

One of us(B.F.L.W.) has recently continued his application of exact amplitude-based resummation theory to Feynman's formulation of Einstein's theory, as described in Refs. [8]. In particular, in Ref. [23], he has arrived at a first principles prediction of the cosmological constant that is close to the observed value [24, 25], $\rho_\Lambda \cong (2.368 \times 10^{-3} eV(1 \pm 0.023))^4$, as we now recapitulate.

In Ref. [23], using the deep UV result

$$B_g''(k) = \frac{\kappa^2 |k^2|}{8\pi^2} \ln \left(\frac{m^2}{m^2 + |k^2|} \right), \quad (3.1)$$

it is shown that the UV limit of Newton's constant, $G_N(k)$, is given by

$$g_* = \lim_{k^2 \rightarrow \infty} k^2 G_N(k^2) = \frac{360\pi}{c_{2,eff}} \cong 0.0442, \quad (3.2)$$

where [8, 23] $c_{2,eff} \cong 2.56 \times 10^4$ for the known world. In addition, it is shown that the contribution of a scalar field to Λ is

$$\begin{aligned}\Lambda_s &= -8\pi G_N \frac{\int d^4 k}{2(2\pi)^4} \frac{(2k_0^2) e^{-\lambda_c(k^2/(2m^2)) \ln(k^2/m^2+1)}}{k^2 + m^2} \\ &\cong -8\pi G_N \left[\frac{1}{G_N^2 64 \rho^2} \right],\end{aligned}\quad (3.3)$$

where $\rho = \ln \frac{2}{\lambda_c}$ and we have used the calculus of Refs. [8, 23]. We note that the standard equal-time (anti-)commutation relations algebra realizations then show that a Dirac fermion contributes -4 times Λ_s to Λ . The deep UV limit of Λ then becomes

$$\begin{aligned}\Lambda(k) &\xrightarrow{k^2 \rightarrow \infty} k^2 \lambda_*, \\ \lambda_* &= -\frac{c_{2,eff}}{2880} \sum_j (-1)^{F_j} n_j / \rho_j^2 \\ &\cong 0.0817\end{aligned}\quad (3.4)$$

where F_j is the fermion number of j and $\rho_j = \rho(\lambda_c(m_j))$. Our results for (g_*, λ_*) agree qualitatively with those in Refs. [26, 27].

For reference, we note that, if we restrict our resummed quantum gravity calculations above for g_*, λ_* to the pure gravity theory with no SM matter fields, we get the results

$$g_* = .0533, \lambda_* = -.000189.$$

We see that our results suggest that there is still significant cut-off effects in the results used for g_*, λ_*^1 in Refs. [26, 27], which already seem to include an effective matter contribution when viewed from our resummed quantum gravity perspective, as an artifact of the obvious gauge and cut-off dependencies of the results. Indeed, from a purely quantum field theoretic point of view, the cut-off action is

$$\Delta_k S(h, C, \bar{C}; \bar{g}) = \frac{1}{2} \langle h, \mathcal{R}_k^{\text{grav}} h \rangle + \langle \bar{C}, \mathcal{R}_k^{\text{gh}} C \rangle \quad (3.5)$$

where \bar{g} is the general background metric, which is the Minkowski space metric η here, and C, \bar{C} are the ghost fields and the operators $\mathcal{R}_k^{\text{grav}}, \mathcal{R}_k^{\text{gh}}$ implement the course graining as they satisfy the limits

$$\begin{aligned}\lim_{p^2/k^2 \rightarrow \infty} \mathcal{R}_k &= 0, \\ \lim_{p^2/k^2 \rightarrow 0} \mathcal{R}_k &\rightarrow \mathfrak{Z}_k k^2,\end{aligned}$$

for some \mathfrak{Z}_k [26]. Here, the inner product is that defined in the second paper in Refs. [26] in its Eqs.(2.14,2.15,2.19). The result is that the modes with $p \lesssim k$ have a shift of their vacuum energy by the cut-off operator. There is no disagreement in principle between our gauge invariant, cut-off independent results and the gauge dependent, cut-off dependent results in Refs. [26, 27].

¹In the first paper in Ref. [27], $(g_*, \lambda_*) \approx (0.27, 0.36)$.

3.1 An Estimate of Λ

To estimate the value of Λ today, we take the normal-ordered form of Einstein's equation,

$$:G_{\mu\nu}: + \Lambda :g_{\mu\nu}: = -8\pi G_N :T_{\mu\nu}: . \quad (3.6)$$

The coherent state representation of the thermal density matrix then gives the Einstein equation in the form of thermally averaged quantities with Λ given by our result above in lowest order. Taking the transition time between the Planck regime and the classical Friedmann-Robertson-Walker regime at $t_{tr} \sim 25t_{pl}$ from Refs. [27], we introduce

$$\begin{aligned} \rho_\Lambda(t_{tr}) &\equiv \frac{\Lambda(t_{tr})}{8\pi G_N(t_{tr})} \\ &= \frac{-M_{pl}^4(k_{tr})}{64} \sum_j \frac{(-1)^F n_j}{\rho_j^2} \end{aligned} \quad (3.7)$$

and use the arguments in Refs. [28] (t_{eq} is the time of matter-radiation equality) to get

$$\begin{aligned} \rho_\Lambda(t_0) &\cong \frac{-M_{pl}^4(1 + c_{2,eff}k_{tr}^2/(360\pi M_{pl}^2))^2}{64} \sum_j \frac{(-1)^F n_j}{\rho_j^2} \\ &\quad \times \left[\frac{t_{tr}^2}{t_{eq}^2} \times \left(\frac{t_{eq}^{2/3}}{t_0^{2/3}} \right)^3 \right] \\ &\cong \frac{-M_{pl}^2(1.0362)^2(-9.197 \times 10^{-3})(25)^2}{64} \frac{1}{t_0^2} \\ &\cong (2.400 \times 10^{-3} eV)^4. \end{aligned} \quad (3.8)$$

where we take the age of the universe to be $t_0 \cong 13.7 \times 10^9$ yrs. In the latter estimate, the first factor in the square bracket comes from the period from t_{tr} to t_{eq} (radiation dominated) and the second factor comes from the period from t_{eq} to t_0 (matter dominated)². This estimate should be compared with the experimental result [24, 25]³ $\rho_\Lambda(t_0)|_{\text{expt}} \cong (2.368 \times 10^{-3} eV(1 \pm 0.023))^4$. In closing, two of us (B.F.L.W., S.A.Y.) thank Prof. Ignatios Antoniadis for the support and kind hospitality of the CERN TH Unit while part of this work was completed.

References

- [1] C. Glosser, S. Jadach, B.F.L. Ward and S.A. Yost, *Mod. Phys. Lett. A* **19**(2004) 2113; B.F.L. Ward, C. Glosser, S. Jadach and S.A. Yost, in *Proc. DPF 2004*, *Int. J. Mod. Phys. A* **20** (2005) 3735; in *Proc. ICHEP04, vol. 1*, eds. H. Chen *et al.*, (World. Sci. Publ. Co., Singapore, 2005) p. 588; B.F.L. Ward and S. Yost, preprint BU-HEPP-05-05, in *Proc. HERA-LHC Workshop*, CERN-2005-014; in *Moscow 2006, ICHEP, vol. 1*, p. 505; *Acta Phys. Polon. B* **38** (2007) 2395; arXiv:0802.0724, PoS(RADCOR2007)(2007) 038; B.F.L. Ward *et al.*, arXiv:0810.0723, in *Proc. ICHEP08*; arXiv:0808.3133, in *Proc. 2008 HERA-LHC Workshop*, DESY-PROC-2009-02, eds. H. Jung and A. De Roeck, (DESY, Hamburg, 2009)pp. 180-186, and references therein.

²The method of the operator field forces the vacuum energies to follow the same scaling as the non-vacuum excitations.

³See also Ref. [29] for an analysis that suggests a value for $\rho_\Lambda(t_0)$ that is qualitatively similar to this experimental result.

- [2] See for example S. Jadach *et al.*, in *Physics at LEP2, vol. 2*, (CERN, Geneva, 1995) pp. 229-298.
- [3] G. Altarelli and G. Parisi, *Nucl. Phys.* **B126** (1977) 298; Yu. L. Dokshitzer, *Sov. Phys. JETP* **46** (1977) 641; L. N. Lipatov, *Yad. Fiz.* **20** (1974) 181; V. Gribov and L. Lipatov, *Sov. J. Nucl. Phys.* **15** (1972) 675, 938; see also J.C. Collins and J. Qiu, *Phys. Rev.* **D39** (1989) 1398.
- [4] C.G. Callan, Jr., *Phys. Rev.* **D2** (1970) 1541; K. Symanzik, *Commun. Math. Phys.* **18** (1970) 227, and in *Springer Tracts in Modern Physics*, **57**, ed. G. Hoehler (Springer, Berlin, 1971) p. 222; see also S. Weinberg, *Phys. Rev.* **D8** (1973) 3497.
- [5] D. R. Yennie, S. C. Frautschi, and H. Suura, *Ann. Phys.* **13** (1961) 379; see also K. T. Mahanthappa, *Phys. Rev.* **126** (1962) 329, for a related analysis.
- [6] B.F.L. Ward, *Adv. High Energy Phys.* **2008** (2008) 682312.
- [7] B.F.L. Ward, *Ann. Phys.* **323** (2008) 2147.
- [8] B.F.L. Ward, *Mod. Phys. Lett.* **A17** (2002) 2371; *Open Nucl. Part. Phys. J* **2** (2009) 1; J. Cos. *Astropart. Phys.* **0402** (2004) 011; *Mod. Phys. Lett.* **A23** (2008) 3299.
- [9] G. Sterman, *Nucl. Phys.* **B281**, 310 (1987); S. Catani and L. Trentadue, *Nucl. Phys.* **B327**, 323 (1989); *ibid.* **B353**, 183 (1991).
- [10] See for example C. W. Bauer, A.V. Manohar and M.B. Wise, *Phys. Rev. Lett.* **91** (2003) 122001; *Phys. Rev.* **D70** (2004) 034014; C. Lee and G. Sterman, *Phys. Rev. D* **75** (2007) 014022.
- [11] S.M. Abyat *et al.*, *Phys. Rev. D* **74** (2006) 074004.
- [12] S. Joseph *et al.*, *Phys. Lett.* **B685** (2010) 283; *Phys. Rev.* **D81** (2010) 076008.
- [13] F. Bloch and A. Nordsieck, *Phys. Rev.* **52** (1937) 54.
- [14] G. Corcella *et al.*, hep-ph/0210213; *J. High Energy Phys.* **0101** (2001) 010; G. Marchesini *et al.*, *Comput. Phys. Commun.* **67** (1992) 465.
- [15] S. Frixione and B. Webber, *J. High Energy Phys.* **0206** (2002) 029; S. Frixione *et al.*, arXiv:1010.0568.
- [16] V.M. Abasov *et al.*, *Phys. Rev. Lett.* **100** (2008) 102002.
- [17] C. Galea, in *Proc. DIS 2008*, London, 2008, <http://dx.doi.org/10.3360/dis.2008.55>.
- [18] M. Bahr *et al.*, arXiv:0812.0529 and references therein.
- [19] T. Sjostrand, S. Mrenna and P. Z. Skands, *Comput. Phys. Commun.* **178** (2008) 852-867.
- [20] T. Gleisberg *et al.*, *J. High Energy Phys.* **0902** (2009) 007.
- [21] P. Nason, *J. High Energy Phys.* **0411** (2004) 040.
- [22] A. Mukhopadhyay *et al.*, to appear.
- [23] B.F.L. Ward, arXiv:1008.1046; PoS(ICHEP10)(2010) 477.
- [24] A.G. Riess *et al.*, *Astron. Jour.* **116** (1998) 1009; S. Perlmutter *et al.*, *Astrophys. J.* **517** (1999) 565.
- [25] C. Amsler *et al.*, *Phys. Lett.* **B667** (2008) 1.
- [26] M. Reuter, *Phys. Rev.* **D57** (1998) 971; O. Lauscher and M. Reuter, *ibid.* **66** (2002) 025026; A. Bonanno and M. Reuter, *ibid.* **62** (2000) 043008, and references therein.
- [27] A. Bonanno and M. Reuter, *Phys. Rev.* **D65** (2002) 043508; *Jour. Phys. Conf. Ser.* **140** (2008) 012008, and references therein.
- [28] V. Branchina and D. Zappala, *G. R. Gravit.* **42** (2010) 141; arXiv:1005.3657, and references therein.
- [29] J. Sola, *J. Phys.* **A41** (2008) 164066.