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Infrared singularities in the high-energy limit

Lorenzo Magnea[*]
Dipartimento di Fisica, Università di Torino, and INFN, Sezione di Torino
E-mail: magnea@to.infn.it

Vittorio Del Duca
INFN, Laboratori Nazionali di Frascati
E-mail: delduca@lnf.infn.it

Claude Duhr
Institute for Particle Physics Phenomenology, University of Durham
E-mail: claude.duhr@durham.ac.uk

Einan Gardi
Tait Institute, School of Physics and Astronomy, University of Edinburgh
E-mail: Einan.Gardi@ed.ac.uk

Chris D. White
School of Physics and Astronomy, Scottish University Physics Alliance, University of Glasgow
E-mail: Christopher.White@glasgow.ac.uk

We use our current understanding of the all-order singularity structure of gauge theory amplitudes to probe their high-energy limit. Our starting point is the dipole formula, a compact ansatz for the soft anomalous dimension matrix of massless multi-particle amplitudes. In the high-energy limit, we find a simple and general expression for the infrared factor generating all soft and collinear singularities of the amplitude, which is valid to leading power in $|t|/s$ and to all logarithmic orders. This leads to a direct and general proof of leading-logarithmic Reggeization for infrared divergent contributions to the amplitude. Furthermore, we can prove explicitly that the simplest form of Reggeization, based on the absence of Regge cuts in the complex angular momentum plane, breaks down at the NNLL level. Finally, we note that the known features of the high-energy limit can be used to constrain possible corrections to the dipole formula, starting at the three-loop order.

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[*]Speaker.

1. The high-energy limit of gauge amplitudes

The high-energy limit of scattering amplitudes, defined as the limit in which the center-of-mass energy of the colliding particles, $\sqrt{s}$, is much larger than other relevant kinematic invariants, has been an interesting and lively subject of studies for several decades [1, 2, 3]. In renormalizable field theories, the high-energy limit is characterized by the presence of large logarithms, as is always the case for processes with two or more disparate energy scales. Such logarithms are of practical relevance, since they can spoil the convergence of perturbation theory, and they may need to be resummed; on the other hand, they are also theoretically interesting, since they are tied to the infrared, semiclassical regime: this means that they can be computed in principle to all orders in perturbation theory, providing a useful handle on the non-perturbative structure of the theory.

Studies of the high-energy limit began with non-relativistic potential models, and the first powerful tool that was brought to bear was the analytic continuation of amplitudes in the complex angular momentum plane [4]. The starting point for these studies is the well-known expansion of scattering amplitudes in partial waves. In the case of a four-point amplitude, one writes

$$A(s,t) \,=\, 16\pi \sum_{l=0}^{\infty} (2l+1)\, a_l(s)\, P_l\,(\cos\theta)\,, \tag{1.1}$$

where P_l are Legendre polynomials, θ is the center-of-mass scattering angle, in terms of which $t = -s(1-\cos\theta)/2$, and $a_l(s)$ are the s-channel partial wave amplitudes. Using the causality and analyticity of the S matrix as fundamental postulates, one can analytically continue the amplitude from the physical region ($s > 0$ and $t < 0$) to the crossed t-channel, and construct integral representations of the crossed-channel partial-wave amplitudes $\tilde{a}_l(t)$. One finds then that the analytic properties of $\tilde{a}_l(t)$, as functions of the angular momentum variable l, have powerful direct implications on the high-energy behavior of the original amplitude $A(s,t)$. For example, assuming that the only singularities of $\tilde{a}_l(t)$ in the angular momentum plane are isolated poles, one finds that the large-s asymptotic behavior of the amplitude is given by a simple power law,

$$\tilde{a}_l(t) \sim \frac{1}{l-\alpha(t)} \qquad \longrightarrow \qquad A(s,t) \;\xrightarrow{\frac{|t|}{s}\to 0}\; f(t)\, s^{\alpha(t)}\,, \tag{1.2}$$

where $\alpha(t)$ is the Regge trajectory.

This general result follows just from physically motivated postulates about the S matrix, without reference to an underlying lagrangian field theory, and without resorting to a perturbative expansion. When one considers a specific field theory model, and uses the tools of perturbation theory, one may verify that the structure schematically described by eq. (1.2) does indeed emerge. In a gauge theory, the high-energy limit is dominated at tree level by the t-channel exchange of massless gauge bosons, and it is possible to show, at least at Leading Logarithmic (LL) accuracy [5], that virtual corrections dress the t-channel gauge boson propagator according to

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}\,, \tag{1.3}$$

where the Regge trajectory $\alpha(t)$ is now expressed as a perturbative expansion. A typical example of this structure is the resummed expression for the four-gluon amplitude in QCD, which can be written as

$$M^{gg\to gg}_{a_1a_2a_3a_4}(s,t) = 2g_s^2 \frac{s}{t}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]\left(\frac{s}{-t}\right)^{\alpha(t)}\left[(T_b)_{a_2a_4}C_{\lambda_2\lambda_4}(k_2,k_4)\right], \tag{1.4}$$

where we label momenta so that $s = (k_1+k_2)^2$ and $t = (k_1-k_3)^2$. The factorized form of eq. (1.4), which has been shown to hold at NLL accuracy [6] for the real part of the amplitude, is consistent with eq. (1.2), and thus with the assumptions that the only singularities in the complex angular momentum plane should be isolated poles. The functions $C_{\lambda_i\lambda_j}(k_i,k_j)$ are the gluon impact factors, and are universal at least to the stated logarithmic accuracy: it is expected that one may use the same impact factor in other high-energy processes involving gluons as well as other energetic particles, for example quarks. In perturbative QCD, of course, virtual corrections to the amplitude are infrared divergent, and these divergences will appear in both the impact factors and the Regge trajectory. To organize these divergences, it is useful employ dimensional regularization (with $d = 4-2\varepsilon$ and $\varepsilon < 0$), and express the trajectory as a perturbative series in powers of the d-dimensional running coupling. One writes then

$$\alpha(t) = \frac{\alpha_s(-t,\varepsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\varepsilon)}{4\pi}\right)^2\alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right), \tag{1.5}$$

where the d-dimensional running coupling [7] is given by

$$\alpha_s(-t,\varepsilon) = \left(\frac{\mu^2}{-t}\right)^{\varepsilon}\alpha_s(\mu^2) + \mathcal{O}\left(\alpha_s^2\right). \tag{1.6}$$

The first two perturbative coefficients $\alpha^{(1)}$ and $\alpha^{(2)}$ are known [8, 9, 10, 11, 12], and can be written as

$$\alpha^{(1)} = C_A\frac{\widehat{\gamma}_K^{(1)}}{\varepsilon}, \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\varepsilon^2} + \widehat{\gamma}_K^{(2)}\frac{2}{\varepsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]. \tag{1.7}$$

where $b_0 = (11C_A - 2n_f)/3$, and $\widehat{\gamma}_K^{(i)}$ are the perturbative coefficients of the cusp anomalous dimension [13], with the overall Casimir eigenvalue C_A scaled out.

One must naturally wonder to what extent the factorization in eq. (1.4) can be trusted to higher logarithmic accuracy, a question which is intimately connected to the underlying assumption that the only singularities in angular momentum should be isolated poles. This question was studied in the early days of Regge theory, both on general grounds and in specific models (see, for example, [2]). In general, one may expect to find cuts as well as poles in the complex angular momentum plane. It can be shown that, in a perturbative expansion, these 'Regge' cuts require the t-channel exchange of at least two particles, and further require a non-planar diagrammatic structure. The simplest class of diagrams with all the required features, which appears starting at three loops, is the 'Mandelstam double-cross', portrayed in Fig. 1. As we shall see, our approach to the high-energy limit will lead us to prove that the simple factorized form of the amplitude given in eq. (1.4) must break down at NNLL, and the first energy logarithm not correctly predicted by a naive Regge-pole-based factorization arises at three loops, from non-planar diagrams, in agreement with the expectation that Regge cuts should play a role at precisely that level of accuracy.

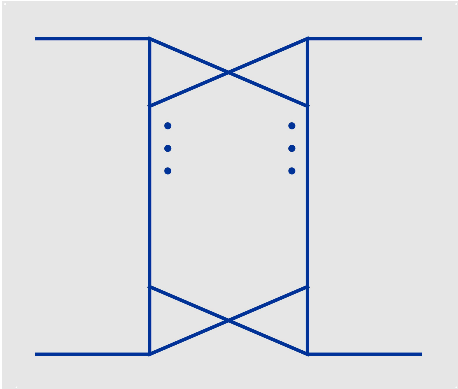

Figure 1: The Mandelstam double-cross diagram.

In the following, we will briefly describe our knowledge of the general structure of infrared poles in multi-particle fixed-angle massless gauge theory amplitudes, and we will discuss how this knowledge can be applied to the high-energy limit, summarizing the results of [14]. We will show that infrared divergences naturally 'Reggeize', for general t-channel color exchanges, and we will prove that the simplest form of Reggeization breaks down at NNLL accuracy. Interestingly, the general results of Regge theory also have implications on the structure of infrared poles, providing non-trivial constraints on possible corrections to the simplest ansatz for the soft anomalous dimension, the dipole formula [15, 16].

2. The infrared structure of gauge amplitudes

Let us consider a scattering amplitude in a massless gauge theory, involving n colored particles in arbitrary representations of the gauge group. Such an amplitude can be seen as a vector in the vector space spanned by all possible color tensors connecting the various representations involved. One writes

$$M_{a_1 \ldots a_n} \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \sum_J M_J \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) (c_J)_{a_1 \ldots a_n} \, , \tag{2.1}$$

where the color tensors c_J form a basis in the relevant vector space, p_i are the particle momenta, and a_i their color indices.

Such on-shell matrix elements are plagued by soft and collinear divergences order by order in perturbation theory. Studies performed over the space of several decades (see [17] for a review), and summarized in this context in Refs. [15, 18], have shown that infrared singularities factorize. Denoting by M the vector with components M_J, one can write [15, 16] the matrix equation

$$M \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = Z \left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right) \, H \left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon \right) , \tag{2.2}$$

where H is a vector of matching coefficients, which are finite as $\epsilon \to 0$, while all infrared singularities are collected in the matrix factor Z, and μ_f is a factorization scale. A key property of the infrared factor Z is that it obeys a (matrix) renormalization group equation with a finite anomalous dimension Γ, which reads

$$\mu \frac{d}{d\mu} Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = -Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) . \tag{2.3}$$

This equation is easily, if formally, solved in exponential form, as

$$Z\left(\frac{p_l}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = P \exp\left[\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right)\right] , \tag{2.4}$$

where we have used as a boundary condition the fact that the d-dimensional running coupling vanishes in the infrared for $d > 4$, so that one can set $Z(\mu = 0) = 1$. The anomalous dimension matrix Γ is the centerpiece dictating the infrared structure of multi-particle amplitudes. The key result of Refs. [15, 16] is that Γ itself, in the case of massless particles, obeys a set of exact evolution equations which strongly constrain its kinematic dependence, tying it to the color structure of the process. The simplest all-order solution to this set of equations is the dipole ansatz for the matrix Γ, which reads

$$\Gamma_{\rm dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = \frac{1}{4} \widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right) \sum_{(i,j)} \ln\left(\frac{-s_{ij}}{\lambda^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j - \sum_{i=1}^{L} \gamma_i\left(\alpha_s(\lambda^2)\right) . \tag{2.5}$$

Equation (2.5) must be understood as a color operator acting on the hard coefficients H by means of the gluon insertion operators $\mathbf{T}_i$, which act exclusively on the color indices of the i-th particle, modifying the color structure according to the Feynman rules for single gluon emission [19]. The solution is derived under the assumption that the cusp anomalous dimension corresponding to Wilson lines in color representation R, $\gamma_K^{(R)}(\alpha_s)$, is proportional, to all orders, to the quadratic Casimir eigenvalue C_R; one may then define the universal function $\widehat{\gamma}_K(\alpha_s)$ according to $\gamma_K^{(R)}(\alpha_s) = C_R \widehat{\gamma}_K(\alpha_s)$. The functions $\gamma_i(\alpha_s)$, on the other hand, carry no color structure, and are simply given by the anomalous dimensions of the fields corresponding to the the each hard external particle. Equation (2.5) has several remarkable features, which cannot be reviewed in detail here. Let us simply note that $\Gamma_{\rm dip}$ contains only two-particle correlations, which mirrors the structure of one-loop corrections; it thus preserves the simplicity of single-gluon exchange. This implies vast cancellations in higher-order diagrammatic calculations, and means in particular that the color structure $\Gamma_{\rm dip}$ is fixed at one loop, so that the path-ordering symbol in eq. (2.4) becomes superfluous. One also sees how the structure of double, soft-collinear poles is generated, even as one is simply solving a renormalization group equation, which would ordinarily generate only single poles: indeed, integrating the d-dimensional running coupling, accompanied by an extra logarithm of the scale as in eq. (2.5), generates precisely such poles.

Corrections to the dipole ansatz can arise from precisely two sources. First, it is possible that the cusp anomalous dimension may cease to be proportional to the quadratic Casimir eigenvalue, starting at some loop order. This can in principle happen starting at four loops, where quartic Casimirs can arise, however arguments have been given in [16, 21] indicating that this is in fact not the case, so that the first correction of this kind should be confined to even higher orders. The only other possible corrections to eq. (2.5), which can arise starting at three loops and for at least four external particles, must take the form of functions of conformal invariant cross-ratios of external momenta, $\rho_{ijkl} \equiv (p_i \cdot p_j p_k \cdot p_l)/(p_i \cdot p_k p_j \cdot p_j)$. One writes then in general

$$\Gamma\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\lambda^2)\right) . \tag{2.6}$$

The function Δ has been studied in detail in [20, 21]. A number of constraints can be imposed on Δ, but while they severely limit the available functional forms, they are not sufficient to prove that it vanishes, even at the three loop level. Whether such quadrupole corrections do indeed arise, starting at three loops, is a question that will probably need to be answered by a direct calculation.

3. Infrared constraints on high-energy logarithms

Studying the high-energy limit of the dipole formula is clearly of interest: indeed, the perturbative Regge trajectory is infrared divergent, and one may expect to be able to make general predictions, at least for divergent terms, to all orders in perturbation theory. In order to proceed, one must first note that the high-energy limit is formally outside the domain of applicability of the factorization theorem expressed by eq. (2.2), which was derived for fixed-angle amplitudes, *i.e.* assuming that all kinematic invariants are of the same order, and all large with respect to the strong interaction scale Λ. This objection does not affect our arguments: we can start with a fixed-angle configuration, where the factorized expression applies, and continuously deform the kinematics towards the high-energy limit, while staying in the perturbative regime; what happens is that energy logarithms, $\ln(s/|t|)$, become large, and they fail to properly factorize, but no new infrared poles are generated. We are thus in a position to make predictions, at least for the infrared divergent part of the various ingredients entering the Regge-factorized form of the amplitude.

Let us consider, for simplicity, the four-point amplitude (analogous considerations apply to the multiparticle amplitudes in multi-Regge kinematics, as shown explicitly in [14]). The key property of the infrared factor Z in the Regge limit $s \gg |t|$ is that, to leading power in $|t|/s$, but to all logarithmic accuracies, it factorizes into the product of two factors, one of which carries the s dependence and the non-trivial color information, while the second one depends only on t and is proportional to the identity in color space. We write this result as

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \varepsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) . \tag{3.1}$$

The color-singlet factor $Z_{\mathbf{1}}$ contains collinear (as well as soft) divergences to be associated with impact factors in a Regge-factorized expression. The central piece of eq. (3.1) is the s-dependent matrix $\widetilde{Z}$, which can be written in a very general and elegant way as

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \varepsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \varepsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + {\rm i}\pi\,\mathbf{T}_s^2\right]\right\} . \tag{3.2}$$

Here we have introduced notations for the color insertion operators corresponding to the color representations exchanged in the t and s channel [22], defining

$$\mathbf{T}_s \equiv \mathbf{T}_1 + \mathbf{T}_2 \quad , \quad \mathbf{T}_t \equiv \mathbf{T}_1 + \mathbf{T}_3 \, , \tag{3.3}$$

for the scattering process $1+2 \longrightarrow 3+4$. Furthermore, we have introduced the notation

$$K\Big(\alpha_s(\mu^2),\varepsilon\Big) \equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2,\varepsilon)\right). \tag{3.4}$$

It is straightforward to show that eqs. (3.1) and (3.2) imply LL Reggeization of infrared poles in t-channel exchanges for the scattering of generic color representations. Indeed, at LL accuracy, one may neglect the s-channel phase in the exponent of eq. (3.2), which starts contributing at NLL. One can then write the full amplitude as

$$M\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\varepsilon\right)\bigg|_{LL} = \exp\left\{K\Big(\alpha_s(\mu^2),\varepsilon\Big)\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2\right\} Z_{\mathbf{1}}\, H\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\varepsilon\right). \tag{3.5}$$

Whenever the Born amplitude (which gives the leading-order contribution to H) is dominated by t-channel exchanges at leading power in $|t|/s$, each such exchange is an eigenstate of the t-channel color operator $\mathbf{T}_t^2$, with an eigenvalue given by the appropriate Casimir operator. For gluon-gluon scattering, for example, the hard scattering is dominated in the high-energy limit by gluon exchange, and one may write

$$\mathbf{T}_t^2\, H^{gg\to gg} = C_A\, H_t^{gg\to gg} + \mathscr{O}(|t|/s). \tag{3.6}$$

This automatically leads to Reggeization, yielding

$$M^{gg\to gg} = \left(\frac{s}{-t}\right)^{C_A\, K\left(\alpha_s(\mu^2),\varepsilon\right)} Z_{\mathbf{1}}\, H_t^{gg\to gg}, \tag{3.7}$$

and thus providing an all-order expression for the Regge trajectory, given by eq. (3.4), reproducing early results [12].

Since the factorization in eq. (3.2) is valid at leading power in $|t|/s$, we can in principle extend our predictions to all subleading energy logarithms. One way to do it is to use the Baker-Campbell-Haussdorff formula to express $\widetilde{Z}$ as a product of exponentials. One finds

$$\begin{aligned}\widetilde{Z}\left(\frac{s}{t},\alpha_s(\mu^2),\varepsilon\right) &= \left(\frac{s}{-t}\right)^{K\,\mathbf{T}_t^2} \exp\left\{\mathrm{i}\,\pi K\,\mathbf{T}_s^2\right\}\exp\left\{-\mathrm{i}\,\frac{\pi}{2}K^2\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right\} \\ &\times \exp\left\{\frac{K^3}{6}\left(-2\pi^2\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right] + \mathrm{i}\pi\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]\right)\right\}\exp\left\{\mathscr{O}\left(K^4\right)\right\}.\end{aligned} \tag{3.8}$$

. Each exponential factor in eq. (3.8) carries increasing powers of the function $K(\alpha_s,\varepsilon)\sim\alpha_s/\varepsilon$. One observes that, as expected, only the first factor contributes at LL. At NLL accuracy, one finds a series of commutators, which in general are not diagonal in the t-channel basis, and will thus break the naive form of Regge factorization expressed for example by eq. (1.4). All these commutators however appear as phases, so that they will contribute only to the imaginary part of the amplitude: we see, on the other hand, that infrared poles for the real part of the amplitude Reggeize at NLL accuracy for arbitrary color exchanges in the t channel, extending the results of [6]. At NNLL accuracy, we predict that the simple form of Regge factorization, which is based on the assumption that only Regge poles arise in the angular momentum complex plane, must break down also for the real part of the amplitude. The first color operator effecting this breakdown arises at $\mathcal{O}(\alpha_s^3)$, and it is given by

$$E\left(\frac{s}{t},\alpha_s,\varepsilon\right) \equiv -\frac{\pi^2}{3} K^3(\alpha_s,\varepsilon) \ln\left(\frac{s}{-t}\right) \left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]. \tag{3.9}$$

The most important thing to note concerning eq. (3.9) is that Regge-breaking terms start arising at three loops, at NNLL level, and they involve non-planar contributions to the amplitude, as implied by the presence of color commutators; furthermore, they arise from the squaring of the s-channel phase in eq. (3.2). All these features closely match what is expected from the presence of Regge cuts in the angular momentum plane, as discussed in Sec. 1. A second important point is that possible corrections to the dipole formula arising at three loops (the function Δ in eq. (2.6)) cannot rescue naive Regge factorization, since they contribute terms of order α_s^3/ε to the amplitude, whereas eq. (3.9) contains terms of order α_s^3/ε^3. We also note that a trace of this Regge-breaking mechanism must arise already at two loops in terms with no energy logarithms, which are generated for example from the expansion of the second factor in eq. (3.8). Terms of precisely this form, proportional to $\alpha_s^2\pi^2/\varepsilon^2$, where indeed detected in [10] by direct calculation. Finally, we note that double color commutators terms such as those generated by eq. (3.9) have recently been shown to play a role [24] in the breakdown of strict collinear factorization described in [25]: in that context, these terms arise from the contributions of Glauber gluons which survive in finite hadronic cross sections which are not fully inclusive, in the form of super-leading logarithms [26].

4. High-energy constraints on infrared poles

We close by briefly illustrating how the known structure of the Regge limit can be used to impose further constraints on quadrupole corrections to the dipole formula, which may arise starting at the three loop order. Considering again for simplicity the four-point amplitude, and following Ref. [20], we note that the three relevant conformal cross ratio can be written as

$$\rho_{1234} = \left(\frac{s}{-t}\right)^2 \mathrm{e}^{-2\mathrm{i}\pi}, \qquad \rho_{1342} = \left(\frac{-t}{s+t}\right)^2, \qquad \rho_{1423} = \left(\frac{s+t}{s}\right)^2 \mathrm{e}^{2\mathrm{i}\pi}, \tag{4.1}$$

where the appropriate phases have been kept, and the constraint $\rho_{1234}\rho_{1342}\rho_{1423} = 1$ is verified. Ref. [20], after examining the constraints imposed on the function Δ in eq. (2.6) by soft-collinear factorization, Bose symmetry and transcendentality, found a small set of functions that would still give allowed contributions. The functions considered in [20] have simple dependences on logarithms or polylogarithms of the conformal cross ratios in eq. (4.1), and one can study their high-energy limit. Defining $L \equiv \ln(s/|t|)$, and $L_{ijkl} \equiv \ln\rho_{ijkl}$, one readily sees that, to leading power in $s/|t|$, the logarithms L_{ijkl} become

$$L_{1234} = 2(L - \mathrm{i}\pi), \qquad L_{1342} = -2L, \qquad L_{1423} = 2\mathrm{i}\pi. \tag{4.2}$$

One can now examine what happens to the non-trivial examples of Δ that were discussed in [20].

The simplest case, involving only logarithms, is the function

$$\Delta^{(212)}(\rho_{ijkl},\alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f^{ade}f^{cbe} L_{1234}^2 \left(L_{1423}L_{1342}^2 + L_{1423}^2 L_{1342}\right) + f^{cae}f^{dbe} L_{1423}^2 \left(L_{1234}L_{1342}^2 + L_{1234}^2 L_{1342}\right) + f^{bae}f^{cde} L_{1342}^2 \left(L_{1423}L_{1234}^2 + L_{1423}^2 L_{1234}\right)\Big] . \tag{4.3}$$

It is now immediate to verify that, in the high-energy limit

$$\Delta^{(212)}(\rho_{ijkl},\alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \, 32\,\mathrm{i}\,\pi \Big[\left(-L^4 - \mathrm{i}\pi L^3 - \pi^2 L^2 - \mathrm{i}\pi^3 L\right) f^{ade}f^{cbe} + \left(2\mathrm{i}\pi L^3 - 3\pi^2 L^2 - \mathrm{i}\pi^3 L\right) f^{cae}f^{dbe}\Big] + \mathcal{O}\left(|t/s|\right) . \tag{4.4}$$

The known structure of the Regge limit forbids the appearance of such a function, since it would generate super-leading high-energy logarithms of the form $\alpha_s^n L^{n+1}$ at three loops and beyond. Studying the high-energy limit is sufficient to rule out, individually, all explicit examples of Δ that were discussed in [20]. It is clear however that it may still be possible to consider linear combinations of the same functions, constructed precisely in order to cancel the logarithms that happen to be constrained by Regge theory arguments. Indeed, a recent study [21] has identified sets of functions that explicitly satisfy all known constraints, including those arising from the high-energy limit. The question of the existence of non-vanishing corrections to the dipole formula at three loops and beyond remains therefore open.

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