

New developments in PJFry

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We report on recent progress in numerical evaluation of one loop tensor integrals. A public C++ package PJFry [1, 2] implementing algorithms from Ref. [3] and its extension to hexagons up to rank 6 are presented.

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1. Introduction

The stable numerical evaluation of tensor integrals is one of the central ingredients of one loop Feynman diagram calculations. It can also be efficiently applied in modern techniques like tensorial reconstruction at integrand level [4] or the Open Loops method [5].

The classic Passarino-Veltman reduction scheme allows to express tensor integrals in terms of a basis of $(4-2\epsilon)$ -dimensional scalar 1-, 2-, 3- and 4-point integrals with kinematic coefficients [6–8]. While it works well for processes with up to 4 external states, for 5 and more legs the numerical stability of the reduction coefficients is spoiled by the appearance of inverse Gram determinants. A number of methods have been proposed to avoid inverse Gram determinants and to improve the numerical stability [9–16, 3].

In this contribution we introduce the open-source tensor reduction package PJFry which avoids Gram determinant instability problems by using dimensional recurrence-based algorithms developed in [1–3, 14–22].

2. The reduction algorithm

We define dimensionally regulated n -point 1-loop tensor integral of rank R as

$$I_n^{\mu_1 \dots \mu_R} = (2\pi\mu)^{2\epsilon} \int \frac{d^d k}{i\pi^{d/2}} \frac{k^{\mu_1} \dots k^{\mu_R}}{((k-q_1)^2 - m_1^2 + i\epsilon) \dots ((k-q_n)^2 - m_n^2 + i\epsilon)} \quad (2.1)$$

where the chords q_i are defined by the external momenta, see Fig. 2:

$$q_1 = p_1, \quad q_2 = p_1 + p_2, \quad q_3 = p_1 + p_2 + p_3, \quad \dots, \quad q_n = \sum_{i=1}^n p_i \quad (2.2)$$

Any one loop tensor integral can be rewritten in terms of scalar tensor form-factors separating the Lorentz structure into products of external momenta and metric tensors:

$$I_n^{\mu_1 \dots \mu_R} = \sum_{i_1, \dots, i_R}^n q_{i_1}^{[\mu_1} \dots q_{i_R}^{\mu_R]} F_{i_1 \dots i_R}^{(n)} + \sum_{i_3, \dots, i_R}^n g^{[\mu_1 \mu_2} q_{i_3}^{\mu_3} \dots q_{i_R}^{\mu_R]} F_{00 i_3 \dots i_R}^{(n)} + \dots \quad (2.3)$$

where square brackets denote non-equivalent symmetrization, which gives the set of all non-equivalent permutations.

Following Ref. [18] tensor form-factors can be mapped to scalar integrals in higher dimension and shifted powers of denominators:

$$F_i^{(n)} = -I_{n,i}^{[2]}, \quad F_{ij}^{(n)} = n_{ij} I_{n,ij}^{[4]}, \quad F_{00}^{(n)} = -\frac{1}{2} I_n^{[2]}, \quad F_{ijk}^{(n)} = -n_{ijk} I_{n,ijk}^{[6]}, \quad F_{00k}^{(n)} = \frac{1}{2} I_{n,k}^{[4]}. \quad (2.4)$$

where $I_{n,i_1 i_2 \dots}^{[2l], s_1 s_2 \dots}$ is a generalized scalar loop integral in shifted dimension:

$$I_{n,i_1 i_2 \dots}^{[2l], s_1 s_2 \dots} = (2\pi\mu)^{2\epsilon} \int \frac{d^{d+2l} k}{i\pi^{d/2+l}} \prod_{r=1}^n \frac{1}{((k-q_r)^2 - m_r^2 + i\epsilon)^{1+\delta_{r i_1} + \delta_{r i_2} + \dots - \delta_{r s_1} - \delta_{r s_2} - \dots}} \quad (2.5)$$

The symbols $n_{i_1 i_2 \dots}$ in (2.4) are shorthand notations for combinatorial factors introduced in [16].

Now one can use dimensional recurrence relations derived in Ref. [19, 20] to express tensor form-factors in terms of (4-2 ϵ)-dimensional scalar integrals:

$$({}_n(d+2l - \sum_{i=1}^n v_i + 1)I_n^{[2(l+1)]}) = \binom{0}{0}_n I_n^{[2l]} - \sum_{s=1}^n \binom{0}{s}_n s^- I_n^{[2l]} \quad (2.6)$$

$$({}_n v_j \mathbf{j}^+ I_n^{[2(l+1)]}) = - \binom{j}{0}_n I_n^{[2l]} + \sum_{s=1}^n \binom{j}{s}_n s^- I_n^{[2l]} \quad (2.7)$$

where s^- is s^{th} denominator power lowering operator, \mathbf{j}^+ is j^{th} denominator power raising operator and v_j is the power of the j^{th} denominator.

However the direct application of eqs. (2.6) and (2.7) introduces inverse Gram determinants $({}_n)$ which will spoil numerical accuracy in large regions of the physical phase space.

The problem of inverse 5-point Gram determinants has been addressed, among others, in [3] where expressions for tensor pentagons have been derived using signed minor algebraic relations.

The basis shown in Fig. 1 can be further reduced using an additional recurrence relation:

$$\binom{0}{0}_n v_j \mathbf{j}^+ I_n^{[2(l+1)]} = - \binom{j}{0}_n (d+2l - \sum_{i=1}^n v_i + 1) I_n^{[2(l+1)]} + \sum_{s=1}^n \binom{0s}{0j}_n s^- I_n^{[2l]}. \quad (2.8)$$

With the help of the above relation one can reduce powers of all denominators to 1 without introduction of inverse Gram determinants, however it does not reduce the dimension. Thus we cannot recurse all the way down to standard (4-2 ϵ)-dimensional scalar integrals and we have to extend our integral basis:

$$I_4^{[8],s}, \quad I_4^{[6],s}, \quad I_4^{[4],s}, \quad I_4^{[2],s}, \quad I_3^{[8],st}, \quad I_3^{[6],st}, \quad I_3^{[4],st}, \quad I_3^{[2],st}. \quad (2.9)$$

For the numerical evaluation of these additional integrals (2.9) we employ a series expansion in the small Gram region [3]. This method treats all mass-kinematic combinations uniformly with dimensional recurrence, therefore works equally well for both massive and massless cases.

Our expansion formula is easily derived from (2.6), which can be written in the following form:

$$X(d+2l-n+1)I_n^{[2(l+1)]} = I_n^{[2l]} - Z_n^{[2l]}, \quad X = \frac{({}_n)}{({}_n)}, \quad Z_n^{[2l]} = \sum_{s=1}^n \frac{\binom{s}{0}_n}{\binom{0}{0}_n} I_n^{[2l],s}. \quad (2.10)$$

solving the recurrence we get

$$I_n^{[2l]} = \sum_{m=0}^M a_m^{(l)} X^m Z_n^{[2(l+m)]} + [a_M^{(l)} X^M I_n^{[2(l+M)]}], \quad a_m^{(l)} = 2^m \binom{d+2l-n+1}{2}_m \quad (2.11)$$

where $(a)_m = \Gamma(m+a)/\Gamma(a)$ is the Pochhammer symbol.

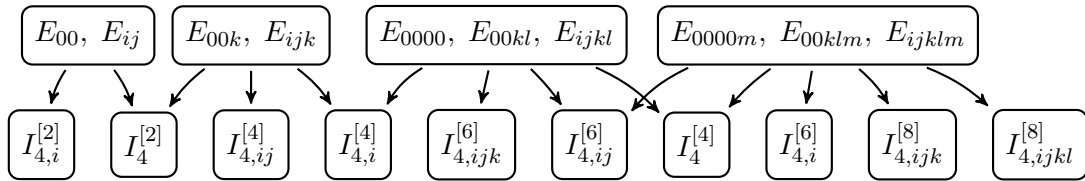


Figure 1: Basis integrals for the tensor reduction of pentagons

3. The PJFry reduction library

The PJFry package [1, 2] is an open source library for the numerical evaluation of one loop tensor integrals. It is licensed under the GNU Lesser General Public License. The latest version can be obtained from the program webpage at <https://github.com/Vayu/PJFry/>.

The program needs an external library for the evaluation of $(4-2\epsilon)$ -dimensional scalar integrals. Currently QCDLoop [23] and OneLoop [24] are supported. The tensor reduction formulae of the PJFry package rely essentially on Ref. [3], where the dimensional recurrence algorithms developed in Refs. [19, 20] were explored. Due to the recursive nature of the numerical algorithms in PJFry, one can greatly benefit from reusing building blocks throughout the calculation.

The main features of the current public version are:

- Reduction of up to rank 5 pentagon tensor integrals
- Any combination of real internal or external masses
- Leading Gram determinants are avoided by the reduction procedure
- Subleading small Gram determinants are treated with asymptotic expansions
- A cache system gives lower point tensor integrals at no extra cost
- Interfaces for C, FORTRAN, C++, Mathematica and GoSam [25].

The different algorithms used in the program are schematically illustrated for tensor pentagon form factors of ranks 2 and 3 in Fig. 3. The blue lines correspond to the reduction formulae of [3]. In the reduction of 4-point functions teal lines represent direct downward recursions for large Gram determinant (2.6), while the solid red paths represent an alternative scheme with Cayley determinants (2.8). Relations which are shared by both schemes are drawn with two-colored dashed lines. The small Gram series expansion procedure (2.11) is depicted for boxes by dotted red lines. Similarly for 3-point functions we use green lines for (2.6), solid magenta lines for (2.8) of $I_{3,i}^{[2]}$ and dotted magenta lines for the series expansions of scalar $I_3^{[2]}$ in the small Gram case.

Due to the 4-dimensionality of space-time the problem of inverse Gram determinants does not appear beyond the 5-point case. Therefore after establishing stable numerical methods for the evaluation of pentagons we can use them to construct higher point functions without additional effort.

In PJFry we implemented a well known expression [14, 15, 20–22, 26] for tensor hexagons up to rank 6 written in terms of pentagons up to rank 5:

$$I_6^{\mu_1 \dots \mu_{R-1} \mu_R} = - \sum_{s=1}^6 I_5^{\mu_1 \dots \mu_{R-1} s} \sum_{i=1}^6 q_i^{\mu_R} \frac{\binom{0s}{0i}_6}{\binom{0}{0}_6} \quad (3.1)$$

Similar representations exist also for 7 and 8-point functions [27, 28]. For higher point functions one might consider using optimized expressions based on contracted tensors [29].

In addition to six point functions the upcoming version of PJFry supports complex internal masses and extended precision via the qd library [30]. Both options are available only if using OneLoop [24].

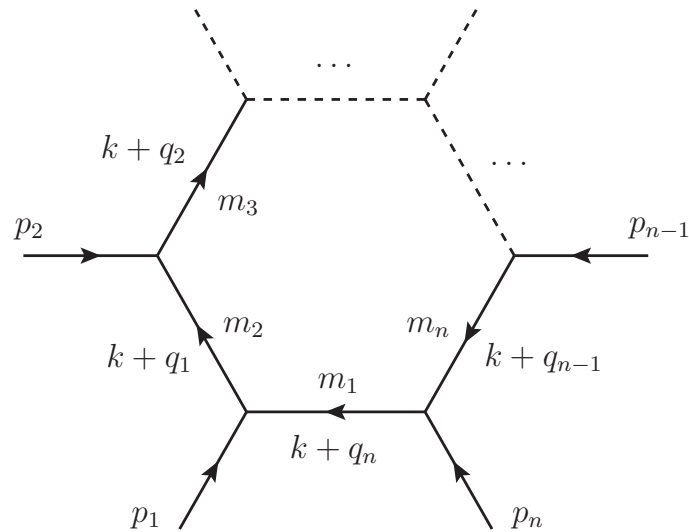


Figure 2: Momenta labeling

The installation is done via a `configure` script. See the `INSTALL` file for a detailed description of available options.

The Mathematica interface follows the conventions of LoopTools [31] (see Fig. 2) and can be called from a Mathematica session:

```
In:= Install["PJFry"]
Out:= PJFry MathLink
Out:= Type Names["PJFry`*"] to show exported names
```

The list of exported functions can be obtained by typing

```
In:= Names["PJFry`*"]
Out:= {"A0v0", "B0v0", "B0v1", "B0v2", "C0v0", "C0v1", "C0v2",
      "C0v3", "ClearCache", "D0v0", "D0v1", "D0v2", "D0v3",
      "D0v4", "E0v0", "E0v1", "E0v2", "E0v3", "E0v4", "E0v5",
      "F0v0", "F0v1", "F0v2", "F0v3", "F0v4", "F0v5", "F0v6",
      "GetMu2", "SetMu2"}
```

All functions have a short help message which can be accessed by prepending the function name in a call with a question mark (e.g. `?F0v1`).

As an example we demonstrate a hexagon tensor form-factor evaluation for one selected phase-space point shown in Table 1.

The scalar hexagon F_0 :

```
In:= k6 = Sequence @@ {ps1, ps2, ps3, ps4, ps5, ps6, s12, s23, s34,
      s45, s56, s16, s234, s345, s456, m1, m2, m3, m4, m5, m6};
In:= Table[F0v0[k6, ep], {ep, 0, 2}]
Out:= {-30.3579 - 205.8213 I, -48.17500 - 42.76719 I, -10.58591}
```

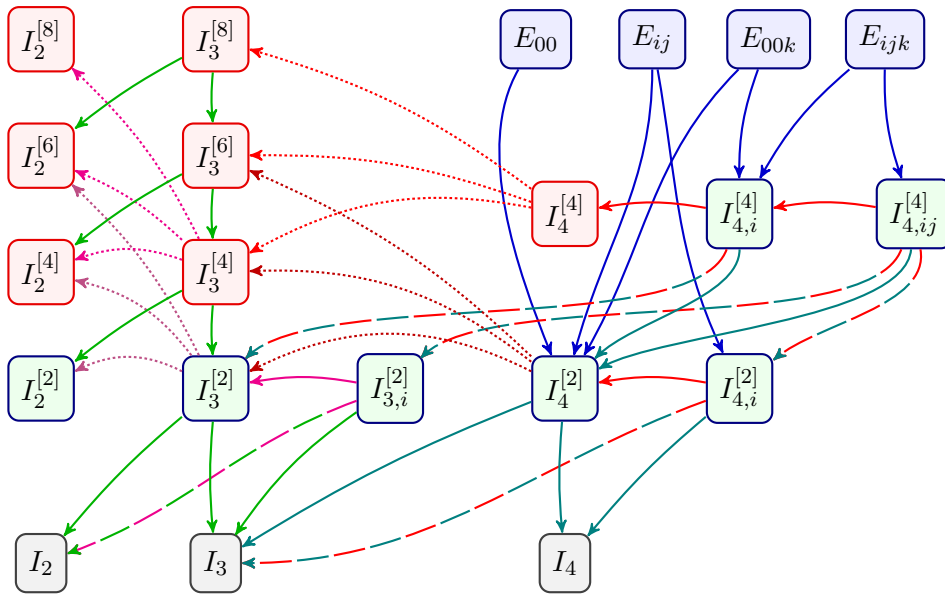


Figure 3: Rank 2 and 3 pentagon form-factors calculation flowchart

p_1	2.1774554	0	0	2.1774554
p_2	2.1774554	0	0	-2.1774554
p_3	-2.0369414560538	-0.4757951224	0.4212682252	0.8409718065
p_4	-2.0907236589150	0.5521596147	-0.466920343	-0.9001008672
p_5	-0.068463307640486	0.0530631952	0.02969826743	-0.03145687079
p_6	-0.15878243799973	-0.1294276875	0.01595385037	0.09058593149
$m_1 = 0, m_2 = 0, m_3 = 0, m_4 = 3.038049, m_5 = 0, m_6 = 0$				

Table 1: An example phase-space point.

The vector hexagon components F^μ :

```
In:= Transpose[Table[
  q1 F0v1[1, k6, ep] + q2 F0v1[2, k6, ep] + q3 F0v1[3, k6, ep]
+ q4 F0v1[4, k6, ep] + q5 F0v1[5, k6, ep], {ep, 0, 2}]]
Out:= {{4.76115 + 38.31618 I, 8.400862 + 7.559091 I, 1.748247},
  {5.93303 + 20.79025 I, 5.326493 + 4.122928 I, 1.080011},
  {0.162103 - 6.809284 I, -1.119560 - 1.201034 I, -0.2433933},
  {-4.02995 - 15.03100 I, -3.642480 - 3.047085 I, -0.7563652}}
```

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