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When epsilon-expansion of hypergeometric functions is expressible in terms of multiple polylogarithms: the two-variables examples

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In this talk, we discuss the algorithm for the construction of analytical coefficients of higher order epsilon expansion of some Horn type hypergeometric functions of two variables around rational values of parameters.

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One-loop Feynman Diagrams and Hypergeometric functions.
Within dimensional regularization [1], the algorithm for analytical evaluation of one-loop multilegs Feynman Diagrams has been described a long time ago [2, 3]. It includes two steps: reduction of the amplitude to a set of master-integrals [2] with following analytical evaluation of master-integrals via Feynman parameter representation [3]. Due to the appearance of $1/\varepsilon$ terms coming from IR and/or UV singularities, the NNLO calculation would demand the knowledge of higher terms in the ε-expansion of the one-loop Feynman Diagrams (see also [4]). The above mentioned technology is suitable for the evaluation of the finite part of master-integrals [5, 6]. However, the direct application of this technique for analytical evaluation of the higher-order coefficients in power of ε gives rise to complicated results [7, 8, 9] even in a simple kinematic. A perspective approach to the construction of analytical coefficients of the ε-expansion of one-loop Feynman Diagrams is to explore the hypergeometric representation [10, 11]. Based on the approach developed in [11], the all-order ε-expansion of one-loop self-energy diagrams has been constructed in $d = 4-2\varepsilon$. More examples of hypergeometric representation for one-loop diagrams based on the technique of [10] are given in [13, 14]. However, still by now a systematic way to construct the analytical coefficients of the ε-expansion for Horn-type hypergeometric functions around rational values of parameters does not exist.
Definition of the hypergeometric function.
We remind the definition of *Horn-type Hypergeometric Functions*:
it is a formal (Laurent) power series in r variables of the following form,

$$H(\vec{J};\vec{z}) \equiv H(\vec{\gamma};\vec{\sigma};\vec{z}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\Pi_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\Pi_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \,, \tag{1}$$

with $\mu_{ab}, \nu_{ab} \in \mathbb{Q}$, $\gamma_j, \sigma_k \in \mathbb{C}$ and $\vec{J} \equiv \{\vec{\gamma}_j, \vec{\sigma}\}$.
The problem under consideration:
In arbitrary d-dimensional space time, $d = 4-2\varepsilon$, where ε is the parameter of dimension regularization [1], the set of discrete parameters $\vec{J} \equiv \{\vec{\gamma}_j, \vec{\sigma}\}$ of hypergeometric function, $H(\vec{\gamma};\vec{\sigma};\vec{z})$ is a linear combination of rational and ε-dependent coefficients: $J_k = A_{0,k} + a_k \varepsilon$, where $A_{0,k}$ and a_k are arbitrary rational numbers. The Laurent expansion of the hypergeometric function around the integer value $d = 4$, is called "construction of the analytical coefficients of ε-expansion" of the function:

$$H(\vec{A}_0 + \vec{a}\varepsilon;\vec{z}) = H(\vec{A}_0;\vec{z}) + \sum_{j=1}^{\infty} \varepsilon^j h_j(\vec{z}) \,, \tag{2}$$

where symbolically,

$$h_j(\vec{z}) = \frac{\partial}{\partial \vec{A}} H(\vec{A};\vec{z}) \Big|_{\vec{A}=\vec{A}_0} \,. \tag{3}$$

The goal is to write the *coefficient functions* h_j in terms of known functions, suitable for numerical evaluation [15], or to describe all analytical properties of h_j, treating them as a new class of functions.
Existing algorithms:
The first systematic algorithms for the construction of higher order coefficients of the ε-expansion

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of multivariable hypergeometric functions around integer values of parameters were suggested in [16]. In Ref. [17], the special set of rational values of parameters, the so called "zero-balance case" was analyzed. However, the partial results of [17] beyond the zero-balance case are in contradiction with partial results of Ref. [18].

Our method:

In a series of papers [19, 20, 21] it was shown that for the hypergeometric function of one variable, ${}_pF_{p-1}$, the analytical coefficients of the ε-expansion can be constructed via an *explicit solution of differential equations* for coefficients functions $h_j(z)$. Using Eq. (3) it is easy to show that the coefficients $h_j(\vec{z})$ satisfy the following linear system of (partial) differential equations (PDE):

$$\sum_{\vec{L}} P_{\vec{L}} \frac{\partial^{\vec{L}}}{\partial \vec{z}} H(\vec{A};\vec{z}) = 0 \Rightarrow \frac{\partial}{\partial \vec{A}} \left[\sum_{\vec{L}} P_{\vec{L}} \frac{\partial^{\vec{L}}}{\partial \vec{z}} H(\vec{A};\vec{z}) = 0 \right] \Bigg|_{\vec{A}=\vec{A}_0} = 0$$
$$\Rightarrow \left[\sum_{\vec{L}} P_{\vec{L}} \right] \Bigg|_{\vec{A}=\vec{A}_0} \frac{\partial^{\vec{L}}}{\partial \vec{z}} h(\vec{z}) = - \left[\frac{\partial}{\partial \vec{A}} \sum_{\vec{L}} P_{\vec{L}} \right] \Bigg|_{\vec{A}=\vec{A}_0} \frac{\partial^{\vec{L}}}{\partial \vec{z}} H(\vec{A};\vec{z}) \Bigg|_{\vec{A}=\vec{A}_0} . \tag{4}$$

When are non-homogeneous PDE solvable in terms of multiple polylogarithms?

We are interested in the question under which conditions the functions $h_j(z)$, solutions of Eq. (4), are expressible in terms of *multiple polylogarithms* [22, 23, 24, 25], or *generalized iterated integrals*, defined as:

$$G(z;R_k,R_{k-1},\cdots,R_1) = \int_0^z \frac{dt}{R_k(t)} I(t;R_{k-1},\cdots,R_1) = \int_0^z \frac{dt_k}{R_k(t)} \int_0^{t_k} \frac{dt_{k-1}}{R_{k-1}(t)} \cdots \int_0^{t_2} \frac{dt_1}{R_1(t_1)} , \tag{5}$$

where $R_k(t)$ are some rational functions. *Multiple polylogarithms* correspond to $R_k(t) = t - a_k$. When is the system of PDE with non-zero non-homogeneous part solvable in terms of (generalized) multiple polylogarithms? **Our algorithm** includes the following steps:

- *Factorization*: the differential operator(s) after ε-expansion are factorisable into product of differential operators of the first order;
- *Linear parametrization* to all orders in ε;
- The non-homogeneous part belongs to the class of functions of the special type (see below).

Example I

Let us consider the differential equation related to the hypergeometric function ${}_pF_{p-1}$ [20, 21]:

$$\sum_{k=0}^{p} P_k(z;\varepsilon) \left(\frac{d}{dz} \right)^k H(z;\varepsilon) = F(z;\varepsilon) , \tag{6}$$

where $P_k(z;\varepsilon)$ and $F(z;\varepsilon)$ are rational functions or iterated integrals over a rational 1-form:

$$P_k(z;\varepsilon) = \frac{\Pi_j(z-\alpha_j-\beta_j\varepsilon)}{\Pi_r(z-A_r-B_r\varepsilon)} , \quad F(z;\varepsilon) = \int^z \frac{dt}{t-\sigma} \frac{\Pi_j(t-\mu_j-\nu_j\varepsilon)}{\Pi_r(t-M_r-N_r\varepsilon)} . \tag{7}$$

We are looking for a solution of Eq. (6) of the following form: $H(z;\varepsilon) = \sum_{j=0}^{\infty} h_j(z)\varepsilon^j$.

Factorization. Factorization of differential operators after ε-expansion means the following:

$$\sum_{k=0}^{p} P_k(z;\varepsilon) \left(\frac{d}{dz} \right)^k = \sum_{r=0} \Pi_{k=1}^{l_r \leq p} \left[R_{k,r}(z) \frac{d}{dz} + Q_{k,r}(z) \right] \varepsilon^r ,$$

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where $R_{k,r}(z)$ and $Q_{k,r}(z)$ are some rational functions.
Linear parametrization. Let us consider as illustration the following differential equation

$$\left[R_1(z)\frac{d}{dz}+Q_1(z)\right]\left[R_2(z)\frac{d}{dz}+Q_2(z)\right]h(z)=F(z)\ . \tag{8}$$

Its iterated solution is:

$$f(z)=\int^z\frac{dt_3}{R_2(t_3)}\left[\exp^{-\int_0^{t_3}\frac{Q_2(t_4)}{R_2(t_4)}dt_4}\right]\int^{t_3}\frac{dt_1}{R_1(t_1)}\left[\exp^{-\int_0^{t_1}\frac{Q_1(t_2)}{R_1(t_2)}dt_2}\right]F(t_1)\ . \tag{9}$$

In accordance with Eq. (5), this iterated integral can be written as multiple polylogarithm, if there is a new variable $\xi:\xi=\Psi(t)$, converting this expression into ratio of polynomials [20]:

$$\int^z\frac{Q_i(t)}{R_i(t)}dt=\ln\frac{M_i(\xi)}{N_i(\xi)}\ ,\qquad \frac{dt}{R_2(t)}\bigg|_{t=t(\xi)}\frac{N_i(\xi)}{M_i(\xi)}=dx\frac{K_i(x)}{L_i(x)}\ , \tag{10}$$

where M_i,N_i,K_i,L_i are polynomial functions. The existence of such a parametrization we called *linear parametrization*.

When does such parametrization exist? To answer to this question, the non-homogeneous part of differential equation Eq. (6) should satisfy the system of linear PDE with *Factorization* and *Linear parametrization* in each order of ε:

$$F(z;\varepsilon)\ =\ \sum_{j=0}f_j(z)\varepsilon^j\ ,\quad \Pi_{i=1}^r\left[P_i(\xi)\frac{d}{d\xi}+S_i(\xi)\right]f_j(\xi)=T_j(\xi)\ , \tag{11}$$

where P_i,S_i,T are rational functions.

Multivariable generalization

Generalization of this technique for the Horn-type hypergeometric functions is straightforward:

1. Convert the system of linear PDE with polynomial coefficients into Pfaff form:

$$\sum_{J,k}P_{\vec{J};k}(\vec{a};\vec{z})\frac{\partial}{\partial z_k}F(\vec{a};\vec{z})=0\ \Rightarrow\ \left\{d_k\omega_i(\vec{z})=\Omega_{ij}^k(\vec{z})\omega_j(\vec{z})dz_k\ ,\quad d_r\left[d_k\omega_i(\vec{z})\right]=0\right\}\ .$$

2. Find the values of parameters when the last system of linear PDE can be converted into triangular form and when *Factorization* is valid.

3. Find a linear parametrization: validity of Eq.(10) for each variable .

Simplification of the procedure of Factorization.

To simplify the procedure of *Factorization* of differential operators, the following trick is very useful. Any Horn type hypergeometric function, defined by Eq. (1), satisfies the system of linear PDE with polynomial coefficients:

$$P_{\vec{L}}(\vec{z})\frac{\partial^{\vec{L}}}{\partial\vec{z}}H(\vec{J};\vec{z})=0\ , \tag{12}$$

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where $\frac{\partial^{\vec{L}}}{\partial \vec{z}} = \frac{\partial^{l_1+\cdots+l_k}}{\partial z_1^{l_1}\cdots\partial z_k^{l_k}}$ and $P_{\vec{L}}(\vec{z})$ are polynomial. Moreover, there are linear differential operators that change the value of each parameter J_a by ± 1:

$$R_{a,\vec{K}}(\vec{z})\frac{\partial^{\vec{K}}}{\partial \vec{z}}H(J_1,\cdots,J_{a-1},J_a,J_{a+1},\cdots,J_r;\vec{z}) = H(J_1,\cdots,J_{a-1},J_a\pm 1,J_{a+1},\cdots,J_r;\vec{z}) \,. \tag{13}$$

In accordance with [26], the differential operators inverse to the operators defined by Eq. (13) can be constructed. Applying direct/inverse differential operators to the hypergeometric function the values of parameters can be changed by an arbitrary integer numbers:

$$Q_0(\vec{z})H(\vec{J}+\vec{m};\vec{z}) = \sum_{j=0}^{r} Q_{\vec{J}}(\vec{z})\theta_{\vec{J}}H(\vec{J};\vec{z}) \,, \tag{14}$$

where $\vec{m}$ is a set of integers and $Q_0(\vec{z})$ and $Q_{\vec{J}}$ are polynomials and r is the holonomic rank of system (12). More details are given in [27].

Example II: F_3 hypergeometric function

In [28, 21] we applied our algorithm for the construction of the ε-expansion of F_1 and F_3 Appell hypergeometric functions. The results of the ε expansion for F_1- [28] and F_3- functions [21] around integer values of parameters are in agreement with results of [29]. Let us consider the ε-expansion of the Appell hypergeometric function F_3 around rational values of parameters. The ε-expansion around this set of parameters does not follow from the algorithms described in [16, 17].

Let us consider the Appell hypergeometric function F_3:

$$\begin{aligned} F_3\left(\frac{p_1}{q}+a_1\varepsilon,\frac{p_2}{q}+a_2\varepsilon,\frac{r_1}{q}+b_1\varepsilon,\frac{r_2}{q}+b_2\varepsilon,1-\frac{p}{q}+c\varepsilon;x,y\right) \\ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{p_1}{q}+a_1\varepsilon\right)_m\left(\frac{p_2}{q}+a_2\varepsilon\right)_n\left(\frac{r_1}{q}+b_1\varepsilon\right)_m\left(\frac{r_2}{q}+b_2\varepsilon\right)_n}{\left(1-\frac{p}{q}+c\varepsilon\right)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \,, \end{aligned} \tag{15}$$

where $\{p_i,r_j,p,q\}$ are integers. Applying our technology step-by-step, we find that the system of linear PDE for the coefficient functions is factorisable and has a triangular form only when $p_jr_j = 0$ for $j=1,2$. After that, the original system of linear PDE with polynomial coefficients is transformed into a linear system of PDE with algebraic coefficients. To convert this system into a class of linear PDE, the linear parametrization should exist simultaneously for the each element of the singular locus of F_3:

$$\{x\}\cup\{1-x\}\cup\{y\}\cup\{1-y\}\cup\{xy-x-y\} \,, \tag{16}$$

as well as for the auxiliary functions H_j, $j=1,2,3$ defined as

$$\begin{aligned} H_1(x) &= (-1)^{\frac{s_1}{q}}\left[\frac{x^p}{(x-1)^{s_1+p}}\right]^{\frac{1}{q}}, \quad H_2(x) = (-1)^{\frac{s_2}{q}}\left[\frac{y^p}{(y-1)^{s_2+p}}\right]^{\frac{1}{q}}, \\ H_3(x,y) &= (-1)^{\frac{s_1+s_2}{q}}\left[\frac{x^{s_2+p}y^{s_1+p}}{(xy-x-y)^{s_1+s_2+p}}\right]^{\frac{1}{q}} \,, \end{aligned} \tag{17}$$

where $s_j = p_j + r_j$ and $j=1,2$. We find that the linear parametrization exists when:

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- The functions H_j are constant polynomial: $s_1 = s_2 = p = 0$. It corresponds to [16].
- One of three functions H_j are equal to 1: $s_1 = 0, s_2 = -p$, and $(1 \leftrightarrow 2)$.
- Two of three functions H_j coincide: $s_1 \neq 0, s_2 = p = 0$, and $(1 \leftrightarrow 2)$.

Unfortunately, for another physically interesting set of parameters [14], we failed to rewrite the iterated integral in terms of multiple polylogarithms. For example, for $s_1 = s_2 = 0$ and $p \neq 0$, the statement about the existence of a linear parametrization is equivalent to the existence of three different (rational) polynomial functions of two variables $P_1(x,y), P_2(x,y)$ and $P_3(x,y)$, such that

$$P_1^q + P_2^q + P_3^q = 1 \,, \tag{18}$$

where q is integer and $q \geq 2$. To our knowledge, this equation has a solution only in the class of elliptic functions. However, the finite part of the F_3-function with this set of parameters is expressible in terms of polylogarithms [30].

As result we got, that only for the two cases:

$$F_3\left(I_1 + \frac{p_1}{q} + a_1\varepsilon, I_2 + a_2\varepsilon, I_3 + b_1\varepsilon, I_4 + b_2\varepsilon, I_5 + \frac{p_1}{q} + c\varepsilon; x, y\right) \,, \tag{19}$$

$$F_3\left(I_1 + \frac{p_1}{q} + a_1\varepsilon, I_2 + a_2\varepsilon, I_3 + b_1\varepsilon, I_4 + b_2\varepsilon, I_5 + c\varepsilon; x, y\right) \,, \tag{20}$$

where I_j, p_1, q are integers, the analytical coefficients of the ε-expansion of F_3 hypergeometric function are explicitly expressible in terms of multiple polylogarithms [24].

Hypergeometric Functions vs. Feynman Diagram

These two building blocks, *Factorization* and *Linear parametrization*, are sufficient to rewrite an iterative solution of system of linear PDE in terms of multiple polylogarithms. It is a cornerstone of all modern multiloop analytical evaluations of master-integrals in QCD and our results are in full agreement with available QCD calculations [31, 32, 33].

Conclusion:

The algorithm described in [19, 20, 21] has been applied to the construction of the analytical coefficients of ε-expansion of Horn-type hypergeometric functions of two variables [34] as well as Mellin-Barnes integrals [35]. In particular, we analyzed the following linear system:

$$\begin{aligned} U_0 \theta_{11} \omega(\vec{z};\varepsilon) &= \left\{ U_1 \theta_{12} + P_1 \theta_1 + P_2 \theta_2 + P_0 \right\} \omega(\vec{z};\varepsilon) \,, \\ T_0 \theta_{22} \omega(\vec{z};\varepsilon) &= \left\{ T_1 \theta_{12} + R_1 \theta_1 + R_2 \theta_2 + R_3 \right\} \omega(\vec{z};\varepsilon) \,, \end{aligned} \tag{21}$$

where $\vec{z} = (z_1, z_2)$ are independent variables, $\theta_j = z_j \partial_{z_j}$, $j = 1, 2$, and $\theta_{i_1 \cdots i_k} = \theta_{i_i} \cdots \theta_{i_k}$. The functions $G_0 \equiv \{U_0, T_0, U_1, T_1\}$ are polynomial of variables z_1 and z_2:

$$G_0 = \sum_{i,j=0} \sigma_{i,j} z_1^i z_2^j \,, \tag{22}$$

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all other function $E_0 \equiv \{P_i, R_j\}$ are polynomial of variables z_1, z_2 and ε:

$$E_0 = \sum_{i,j,k=0} \gamma_{i,j;k} z_1^i z_2^j \varepsilon^k \, , \tag{23}$$

and $\sigma_{i,j}$ and $\gamma_{i,j;k}$ are rational. The analytical structure of the coefficients $h_j^{(r)}(\vec{z})$ defined via Laurent expansion around $\varepsilon = 0$ of the solution $\omega^{(r)}(\vec{z};\varepsilon)$ of this system have been analyzed

$$\omega^{(r)}(\vec{z};\varepsilon) = \sum_j h_j^{(r)}(\vec{z}) \varepsilon^j \, . \tag{24}$$

for the physically interesting set of parameters [14]. In particular, the hypergeometric functions considered in [16, 17] correspond to a system of linear PDE (21) with polynomial coefficients and with singularity locus

$$L := \{z_1\} \cup \{U_0\} \cup \{z_2\} \cup \{T_0\} \cup \{U_0 T_0 - U_1 T_1\} \, , \tag{25}$$

where

$$U_i(\vec{z};\vec{a}) \; = \; a_{0,i} + a_{1,i} z_1 + a_{2,i} z_2 \, , \quad T_j(\vec{z};\vec{a}) = b_{1,j} z_1 + b_{2,j} z_2 + b_{3,j} z_1 z_2 \, , \tag{26}$$

$i, j = 1, 2$ and $a_{k,j}, b_{k,i} \in \{0, \pm 1\}$. See also [36, 37].

The ε-expansion around rational values of parameters with one unbalanced rational parameter, corresponds to a system of linear PDE with algebraic or elliptic coefficients. Imposing only *Factorization* conditions gives rise to iterative integrals with algebraic functions, that in general, are not expressible in terms of *multiple polylogarithms*. Only when additional *Linear parametrization* conditions are valid, we are able to rewrite the results of the integration in terms of 2-dimensional polylogarithms [24]. The *Linear parametrization* should exist simultaneously for the each element of singular locus, Eq. (25), of the differential system Eq. (21) and for algebraic functions defined as q-roots of ratios of elements of L: $(L_i/(1-L_i))^{\frac{1}{q}}$ and/or $((L_i L_j)/(L_i + L_j - L_i L_j))^{\frac{1}{q}}$ (see Eq. (17)). It is in agreement with the one-variable case analysed in [20].

We got, see also [20], that even when the finite part of a hypergeometric function is expressible in terms of multiple polylogarithms (existence of Liouvillian solution of a linear system of PDE in $d = 4$) it does not follow that higher order terms of the ε-expansion are expressible in terms of multiple polylogarithms, too.

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