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Evaluating Five-Loop Konishi in $\mathcal{N}=4$ SYM

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The strategy of the recent evaluation of the five-loop correction to the anomalous dimension of the Konishi operator in $\mathcal{N}=4$ supersymmetric Yang–Mills theory is outlined, with the emphasis on the methods of evaluating multiloop Feynman integrals used within this project.

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1. Introduction

The Konishi operator is defined as

$$\mathscr{K} = \mathrm{tr}\left(\Phi^I \Phi^I\right) \tag{1.1}$$

with Φ^I (with $I = 1, \ldots, 6$) in the adjoint representation of $SU(N_c)$. It is the simplest unprotected gauge invariant Wilson operator in $\mathscr{N} = 4$ SYM, whose scaling dimension receives anomalous contribution to all loops:

$$\Delta_{\mathscr{K}} = 2 + \gamma_{\mathscr{K}}(a) = 2 + \sum_{\ell=1}^{\infty} a^\ell \gamma_{\mathscr{K}}^{(\ell)}$$

with $a = g^2 N_c/(4\pi^2)$ and

$$\gamma_{\mathscr{K}}(a) = 3a - 3a^2 + \frac{21}{4}a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4 + \left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}\zeta_3{}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2).$$

The five-loop correction recently evaluated in Ref. [1] has perfect agreement with calculations based on integrability in AdS/CFT [2, 3, 4, 5].

The evaluation of Ref. [1] was based on the operator-product expansion (OPE) of two stress-tensor multiplet operators

$$\mathscr{O}(x_1,y_1)\mathscr{O}(x_2,y_2) = c_{\mathscr{I}} \frac{(Y_1 \cdot Y_2)^2}{x_{12}^4} \mathscr{I} + c_{\mathscr{K}}(a) \frac{(Y_1 \cdot Y_2)^2}{(x_{12}^2)^{1-\gamma_{\mathscr{K}}/2}} \mathscr{K}(x_2) + c_{\mathscr{O}} \frac{(Y_1 \cdot Y_2)}{x_{12}^2} \mathscr{O}^{IJ}_{\mathbf{20'}}(x_2) + \ldots \tag{1.2}$$

where $x_2 \to x_1$,

$$\mathscr{O}(x,y) \equiv Y_I Y_J \, \mathscr{O}^{IJ}_{\mathbf{20'}}(x) = Y_I Y_J \, \mathrm{tr}\left(\Phi^I(x)\Phi^J(x)\right), \tag{1.3}$$

$$\mathscr{O}^{IJ}_{\mathbf{20'}} = \mathrm{tr}\left(\Phi^I \Phi^J\right) - \tfrac{1}{6}\delta^{IJ}\,\mathrm{tr}\left(\Phi^K \Phi^K\right),$$

and Y_I are auxiliary $SO(6)$ harmonic variables defined as (complex) null vectors, $Y^2 \equiv Y_I Y_I = 0$.

To obtain the Konishi anomalous dimension, the four-point correlation function of the operators (1.3) in the double coincidence limit was evaluated [1]. The integrand of the four-point correlation function was taken from the results of Refs. [6, 7] where it was constructed up to six loops. As is typical for renormalization-group calculations, the evaluation of the Konishi anomalous dimension was reduced in Ref. [1] to the evaluation of the pole part of a linear combination of Feynman integrals.

Quite recently, the six- and seven-loop corrections to the Konishi anomalous dimension were evaluated using integrability in AdS/CFT [8, 9]. What about extending the results of Ref. [1] to higher loops? It is clear that the results of Refs. [6, 7] can be extended to the seven-loop level. So, the feasibility of higher-loop quantum-field theoretical calculations depends on whether the tools for Feynman integrals used in Ref. [1] can be applied. In the next three section, these tools are briefly characterized and discussed, from this point of view.

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2. IRR

The pole part of the linear combination of five-loop Feynman integrals contributing to the Konishi anomalous dimension was reduced [1] to a linear combination of four-loop integrals by means of the coordinate-space version of the method of infrared rearrangement (IRR) [10]. The IRR is a standard important tool in renormalization-group calculations because it provides the possibility to reduce the number of loops by one in Feynman integrals necessary for the evaluation. It was rediscovered several times – see, in particular, [11].

To describe the method, let us consider as an example the following four-loop integral in Euclidean $D-$dimensional space-time (with $D=4-2\varepsilon$)

$$I(x_{13}) = \frac{e^{4\gamma\varepsilon}}{\pi^{2D}} \int \frac{(x_{13}^2)^4 \, d^D x_5 \dots d^D x_8}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{35}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}\,, \tag{2.1}$$

where $x_{ij} = x_i - x_j$. The integral (2.1) has a simple pole in ε

$$I(x_{13}) = (x_{13}^2)^{-4\varepsilon} \left[\frac{C}{\varepsilon} + O(\varepsilon^0) \right] \tag{2.2}$$

which comes from integration over the region where $x_5, \dots, x_8$ are all close to x_1 and from the symmetrical region where $x_5, \dots, x_8$ are all close to x_3. Since the integration variables are true coordinates in Euclidean space, the pole $1/\varepsilon$ has to be interpreted as an UV divergence.

In general, the UV divergences in coordinate space come from regions where the integrand considered as a generalized function of x_i (tempered distribution, i.e. linear functional on a space of test functions) is ill-defined. In our example, the product of x^2-factors in the denominator of (2.1) turns out to be unintegrable in a vicinity of the two external points, x_1 and x_3. In the first case, we consider the product

$$F(x_1, x_5, \dots, x_8) = \frac{1}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2} \tag{2.3}$$

as a tempered distribution. Its divergent part is described by an UV counterterm

$$\Delta(x_1, x_5, \dots, x_8) = \frac{C}{2\varepsilon} \delta(x_1 - x_5) \dots \delta(x_1 - x_8)\,, \tag{2.4}$$

with the constant C determined below. Similar counterterm $\Delta(x_3, x_5, \dots, x_8)$ describes singular behaviour of the integrand (2.1) in the vicinity of x_3. Thus, the pole part of (2.1) is just twice the factor $C/(2\varepsilon)$ in (2.4)

$$I = \int d^D x_5 \dots d^D x_8 \left[\Delta(x_1, x_5, \dots, x_8) + \Delta(x_3, x_5, \dots, x_8) \right] + O(\varepsilon^0) = \frac{C}{\varepsilon} + O(\varepsilon^0)\,, \tag{2.5}$$

leading to (2.2).

To evaluate the constant C in (2.4) we apply the infrared rearrangement (IRR) method originally proposed by Vladimirov in Ref. [10] in momentum space. It makes use of the fact that, for an infrared finite but logarithmically UV-divergent Feynman integral without subdivergences, the contribution of the counterterm is just a constant. The idea of IRR is to set the external momenta to zero and then, in order to avoid the appearance of IR divergences, to introduce an external momentum (or a mass) in such a way that the calculation becomes simpler.

Let us apply the IRR method to (2.3) in coordinate space and treat the coordinates x_1, x_5 as external and x_6, x_7, x_8 as internal points. Notice that setting an external momentum to zero corresponds to integrating over the corresponding coordinate. Then, the constant C in (2.4) can be obtained by integrating both sides of (2.3) with respect to internal points

$$F(x_1,x_5) = \int \frac{\mathrm{d}^D x_5 \mathrm{d}^D x_6 \mathrm{d}^D x_7}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2} = \frac{C}{2\varepsilon}\delta(x_1 - x_5) + O(\varepsilon^0). \tag{2.6}$$

The integral on the left-hand side depends on the two external points and is of propagator type. We can check it has no IR divergences, i.e. divergences at large values of coordinates, and has the following form by dimensional arguments

$$F(x_1,x_5) = f(\varepsilon)\frac{1}{(x_{15}^2)^{2+3\varepsilon}}. \tag{2.7}$$

Here the only source of the simple pole in ε is hidden in the second factor (which is considered as a distribution) so that $f(\varepsilon)$ is analytic in a vicinity of the point $\varepsilon = 0$. The simplest way to reveal the $1/\varepsilon$ pole of the distribution $1/(x_{15}^2)^{2+3\varepsilon}$ is to take its D-dimensional Fourier transform with a help of the identity

$$\mathscr{F}\left[\frac{1}{(x^2)^\lambda}\right] = \frac{1}{\pi^{D/2}}\int \mathrm{d}^D x\, \mathrm{e}^{ipx}\frac{1}{(x^2)^\lambda} = \frac{4^{D/2-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(D/2-\lambda)}{(p^2)^{D/2-\lambda}}. \tag{2.8}$$

In particular, for $\lambda = 2+3\varepsilon$ we find from (2.7) (for $x_5 = 0$)

$$\mathscr{F}[F(x_1,0)] = f(\varepsilon)\frac{4^{-4\varepsilon}\Gamma(-4\varepsilon)}{\Gamma(2+3\varepsilon)}\frac{1}{(p^2)^{-4\varepsilon}} = -\frac{f(0)}{4\varepsilon} + O(\varepsilon^0). \tag{2.9}$$

At the same time, replacing $F(x_1,0)$ by its expression (2.6) we obtain the left-hand side of this relation as $C/(2\varepsilon)+O(\varepsilon^0)$ leading to

$$C = -\frac{1}{2}f(0) = -\frac{1}{2}F(x_1,0)\Big|_{x_1^2=1, D=4}. \tag{2.10}$$

It is easy to see that the integral $F(x_1,x_5)$, Eq. (2.6), corresponds to a planar graph After going to the dual momenta, one finds that it coincides with a well-known three-loop *V in O* graph. This gives

$$C = -10\,\zeta(5). \tag{2.11}$$

Similarly to this example, the method of IRR in coordinate space was applied in Ref. [1] to a linear combination of five-loop integrals. As a result, the problem was reduced to four-loop integrals. Clearly, this step is feasible also at least in six loops.

3. IBP

After applying IRR it was necessary to evaluate around seventeen thousands of integrals

$$G(a_1,\ldots,a_{14}) = \int\ldots\int \frac{\mathrm{d}^D x_6 \mathrm{d}^D x_7 \mathrm{d}^D x_8 \mathrm{d}^D x_9}{(x_{16}^2)^{a_1}(x_{17}^2)^{a_2}(x_{18}^2)^{a_3}(x_{19}^2)^{a_4}(x_6^2)^{a_5}(x_7^2)^{a_6}(x_8^2)^{a_7}} \times \frac{1}{(x_9^2)^{a_8}(x_{67}^2)^{a_9}(x_{68}^2)^{a_{10}}(x_{69}^2)^{a_{11}}(x_{78}^2)^{a_{12}}(x_{79}^2)^{a_{13}}(x_{89}^2)^{a_{14}}}, \tag{3.1}$$

with various integer (positive and negative) indices $a_1,\ldots,a_{14}$. To do this, the standard tool called integration by parts (IBP) [12] was used with the help of the C++ version of the code FIRE [13]. As a result, every integral was reduced to a linear combination, with rational coefficients in d, of twenty two master integrals.

This IBP reduction was not at the level of world records. The complexity is more or less equal to the complexity of the IBP reduction of the corresponding four-loop momentum-space integrals. So, such a reduction is feasible for the Baikov's algorithm [14] as well for some other public and private codes of IBP reduction.

I think, an IBP reduction of six-loop massless propagator integrals is not feasible within existing computer codes. However, at least the C++ version of the code FIRE can work with five-loop massless propagator integrals which depend on twenty indices and are necessary for the evaluation of the six-loop correction to the Konishi anomalous dimension. So, this step of the evaluation seems to be feasible here.

4. Evaluating master integrals

Among the master integrals appeared in the calculation of Ref. [1] only two master integrals were associated with non-planar graphs. Moreover, they belong to the same sector, i.e. a subset in the set of the indices $(a_1,\ldots,a_{14})$ where certain indices are positive and other indices are non-positive. In fact, in the family of the twenty eight master integrals for momentum-space massless propagator integrals (see Ref. [15]), there is at most one master integral in any sector, so that the coordinate-space family of the master integrals[1] is more complicated.

The evaluation of the twenty planar master integrals was simple: we introduced the dual momenta $k_i = x_i - x_{i+1}$, represented the same integrals as four-loop propagator master (momentum) integrals of Ref. [15] and took results from that paper in terms of ε expansions up to transcendentality weight seven. For the two non-planar integrals, we applied the method of gluing of Refs. [12, 15]. I do not believe that it will feasible to evaluate this way master integrals at the next loop level. However, we now have a much more general method of Ref. [17] based on dimensional recurrence relations. I think, it is indeed feasible for five-loop massless propagator integrals, both in momentum and coordinate space. One of its useful features is that going to higher orders of the ε expansions can be done easily. For example, the twenty eight master integrals for momentum-space massless propagators were evaluated up to transcendentality weight twelve [16]. A new important feature of this method which appeared quite recently and was described in Ref. [18] is that it can now work also in situations with two and more master integrals in a given sector when one is forced to solve matrix difference equations.

To summarize, I believe that the quantum-field theoretic evaluation of the six-loop anomalous dimension of the Konishi operator is feasible. It would be interesting to do this and check whether the agreement with the results based on integrability in AdS/CFT holds also at the six-loop level.

[1]We only considered master integrals necessary for our calculation where we met twenty two master integrals and did not analyze the whole family.

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