Efficient contraction of 1-loop N-point tensor integrals

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A new approach for the reduction of tensor integrals is described. The standard decomposition à la Davydychev is applied. Integrals with higher indices are then expressed in terms of scalar higher-dimensional integrals with generic indices. The approach allows to perform contractions with external momenta in a particularly efficient manner. This is due to the possibility to perform analytically the resulting sums over the indices of products of signed minors and scalar products of chords. Advantages of this approach for the calculation of cross sections are described. We are preparing the numerical package OLEC (One Loop External Contractions) with interfaces in Mathematica for algebraic and in C++ for numerical calculations. First numerical results are discussed.

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1. Introduction

Starting from the one-loop level, calculations of the squared amplitudes in Quantum Field Theory bring us to consider n-point tensor integrals of rank R, (n,R)-integrals

$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}}, \tag{1.1}$$

where $d = 4 - 2\varepsilon$ and denominators c_j have indices ν_j and chords q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon. \tag{1.2}$$

There are many strategies how to solve such tensor integrals, see e.g. [1 − 4] and references therein..

Our procedure and notations are described in [5, 6] and in references quoted therein. There are two main steps which are applied to (1.1). First, higher dimensional scalar integrals are used. This is Davydychev's decomposition [7], e.g. for a $R = 2$ tensor with a general number of external legs n

$$I_n^{\mu\nu} = \int^d k^\mu k^\nu \prod_{r=1}^n c_r^{-1} = \sum_{i,j=1}^n q_i^\mu q_j^\nu \nu_{ij} I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_n^{[d+]}. \tag{1.3}$$

We use the notation

$$[d+]^l = d + 2l. \tag{1.4}$$

Second, recursive relations as derived in [8, 9] bring the expression into a simpler form, e.g. explicitly for pentagons

$$\nu_{ij} I_{5,ij}^{[d+]^2} = -\frac{\binom{0}{j}_5}{()_5} I_{5,i}^{[d+]} + \sum_{s=1,s\neq i}^5 \frac{\binom{s}{j}_5}{()_5} I_{4,i}^{[d+],s} + \frac{\binom{i}{j}_5}{()_5} I_5^{[d+]}. \tag{1.5}$$

As may be seen, there is a decrease in both dimension and indices. Finally, recursively, we go down to basic scalar integrals in higher dimension - but at the price of appearance of inverse Gram determinants $()_5$ in (1.5). It is known that inverse Gram determinants can cause trouble in realistic physical applications, because there are kinematical regions where they can be small or even vanish.

Here, the main aspect of the new approach comes into action: After insertion of (1.5) into (1.3), the chords q_i, q_j are contracted with external momenta (expressed as simple differences of chords q_a, q_b). One observes the appearance of "auxiliary vectors"

$$Q_s^\nu = \sum_{j=1}^5 q_j^\nu \frac{\binom{s}{i}_5}{()_5}, \quad s = 0, \dots, 5. \tag{1.6}$$

Contractions of the vectors with a chord q_a immediately eliminate the unwanted inverse of $()_5$. At first glance this is really surprising and we want to stress this property in particular. If we would arrange cancellations of inverse Gram determinants $()_5$ in (1.3) in a standard way, much more effort is needed [10, 11, 5].

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To be more precise, let us give a manifest example, taken from subsection 2.1:

$$(q_a \cdot Q_s) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{s}{j}_n}{()_n} \;\; = \;\; \frac{1}{2} (\delta_{as} - \delta_{ns}), \quad s = 1, \ldots, 5.. \tag{1.7}$$

The Q vectors, when contracted with a chord in (1.7), give just Kronecker δ-functions. The inverse Gram determinant $()_5$, present in intermediate steps, disappears automatically.

Let us account now the main advantages of the new approach where chords are contracted with Q vectors (signed minors):

1. Cancellation of inverse Gram determinants $()_5$.
   Explained as above - occasionally also cancellation by subtraction.

2. Elimination of the metric tensor $g_{\mu\nu}$.
   The integral (1.3) in its original form includes the $g_{\mu\nu}$ metric tensor.
   The identity (valid in $d = 4$ dimensions)

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^{5} \frac{\binom{i}{j}_5}{()_5} q_i^\mu q_j^\nu \tag{1.8}$$

   introduces the inverse of $()_5$ again. So, as argued before, it should cancel out. In fact, e.g. for pentagons $(n = 5)$,

$$I_5^{\mu\nu} = \sum_{i,j=1}^{4} q_i^\mu q_j^\nu v_{ij} I_{5,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_5^{[d+]}, \tag{1.9}$$

   and the last term in (1.5) cancels against the $g^{\mu\nu}$ term in (1.9). After the $g^{\mu\nu}$ cancellation and the contractions with chords, the remaining inverse of $()_5$, which was produced in intermediate steps, disappears as well.

3. Presence of Kroneckerδ-functions and simple kinematical factors Y.
   This is probably the main feature of the approach. Due to relations like (1.7) and

$$(q_a \cdot Q_0) \equiv \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{0}{j}_n}{()_n} = -\frac{1}{2} (Y_{an} - Y_{nn}), \tag{1.10}$$

   where $Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2$ are simple kinematical functions, the expressions for contracted tensor integrals presented in the next section take a simple form. Take as an example the (2.33) from the next section. For given indices a, b and t, most of the terms vanish.

The features are independent of multiplicity (i.e. the number of external legs) and rank of the tensor integrals under consideration.

2. The contractions

We consider now 5-point functions and eliminate the inverse 5-point Gram determinants $()_5$. Inverse 4-point Gram determinants $\binom{s}{s}_5$ will be treated separately. The explicit formulae cover tensor integrals until rank $R = 3$.

2.1 The vector integral: I_5^μ

The vector integral may be written in terms of the scalar 5-point function E and 4-point scalar functions I_4^s, where s indicates the scratched line of the 5-point topology:

$$I_5^\mu = E Q_0^\mu - \sum_{s=1}^5 I_4^s Q_s^\mu, \qquad E = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{s}{0}_5 I_4^s. \tag{2.1}$$

For $q_n = 0$, $a = 1,\ldots,n-1$, $s = 1,\ldots n$, the multiplication with chords introduces (1.7) and (1.10), and the final expression for a contraction of the vector integral I_5^μ with an arbitrary chord $q_{a,\mu}$ becomes

$$(q_a \cdot I_5) = E\,(q_a \cdot Q_0) - \sum_{s=1}^5 I_4^s\,(q_a \cdot Q_s)$$

$$= -\frac{1}{2} E\,(Y_{a5} - Y_{55}) - \frac{1}{2} \sum_{s=1}^5 I_4^s\,(\delta_{as} - \delta_{5s}). \tag{2.2}$$

Here we have applied (1.7) and (1.10) for $n = 5$. In fact no inverse Gram determinants have appeared.

2.2 The tensor integrals of rank $R = 2$: $I_5^{\mu\nu}$

From equation (3.5) of [5] one gets

$$I_5^{\mu\nu} = I_5^\mu Q_0^\nu - \sum_{s=1}^5 I_4^{\mu,s} Q_s^\nu, \tag{2.3}$$

with

$$I_4^{\mu,s} = -\sum_{i=1}^4 q_i^\mu I_{4,i}^{[d+],s} = Q_0^{s,\mu} I_4^s - \sum_{t=1}^5 Q_t^{s,\mu} I_3^{st}. \tag{2.4}$$

There are two kinds of contractions to be considered: with the metric tensor or with two chords. In principle, one may also meet in chiral theories contractions of the form

$$q^{a\alpha} q^{b\beta} \, \varepsilon_{\alpha\beta\mu\nu} \, I_5^{\mu\nu}. \tag{2.5}$$

Proper organization of the matrix element evaluation will exclude the appearance of such terms since cross sections are either helicity and spin summed and thus pure scalar quantities or written in terms of helicity states. In the latter case, the scalar factors with the loop functions are also free of γ_5, and the ε tensor might influence the matrix element basis. Nevertheless, we add one rank $R = 2$ example in order to demonstrate that this kind of sums may also be treated.

2.2.1 Contractions with chords: $q_{a\mu} q_{b\nu} I_5^{\mu\nu}$

By contraction, (2.3) rewrites into

$$q_{a\mu} q_{b\nu} I_5^{\mu\nu} = (q_a \cdot I_5)(q_b \cdot Q_0) - \sum_{s=1}^5 (q_a \cdot I_4^s)(q_b \cdot Q_s). \tag{2.6}$$

Here no inverse $()_5$ occurs anymore. Inserting the sums $\Sigma_a^{2,s}$ and $\Sigma_a^{1,st}$ from [6], $(q_a \cdot I_4^s)$ can be written as

$$(q_a \cdot I_4^s) = \frac{1}{\binom{s}{s}_5}\left[\Sigma_a^{2,s}I_4^s - \Sigma_a^{1,st}I_3^{st}\right] = -\frac{1}{2}\left\{(Y_{a5} - Y_{55})I_4^s + \sum_{t=1}^{5}(\delta_{at} - \delta_{5t})I_3^{st} + (\delta_{as} - \delta_{5s})R^s\right\},\quad (2.7)$$

where we introduced the abbreviation

$$R^s \equiv \frac{1}{\binom{s}{s}_5}\left[\binom{s}{0}_5 I_4^s - \sum_{t=1}^{5}\binom{s}{t}_5 I_3^{st}\right] = \frac{1}{\binom{0s}{0s}_5}\left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^{5}\binom{0s}{0t}_5 I_3^{st}\right]. \quad (2.8)$$

The 2nd representation of R^s in (2.8) can be used if the four-point Gram determinant $\binom{s}{s}_5$ is small or even vanishes.

2.2.2 Contractions with the metric tensor: $g_{\mu\nu}I_5^{\mu\nu}$

The second scalar which can be constructed from the tensor of rank 2 is $g_{\mu\nu}I_5^{\mu\nu}$. Due to (2.3) we have to evaluate the following scalar products:

$$(Q_0 \cdot Q_0) = \frac{1}{2}\left[\frac{\binom{0}{0}_5}{()_5} + Y_{55}\right],$$

$$(Q_0 \cdot Q_s) = \frac{1}{2}\left[\frac{\binom{s}{0}_5}{()_5} - \delta_{s5}\right],$$

$$(Q_0^s \cdot Q_s) = -\frac{1}{2}\delta_{s5},$$

$$(Q_t^s \cdot Q_s) = 0. \quad (2.9)$$

In this case the terms with $\frac{1}{()_5}$ cancel and, not surprisingly, the result finally is

$$g_{\mu\nu}I_5^{\mu\nu} = \frac{Y_{55}}{2}E + I_4^5. \quad (2.10)$$

2.2.3 Contractions with the antisymmetric tensor: $q^{a\mu}q^{b\nu}\varepsilon_{\mu\nu\alpha\beta}I_5^{\alpha\beta}$

A reduction of pseudoscalar contractions leads to expressions of the type

$$P[C] = q_a^\rho q_b^\lambda \, \varepsilon_{\mu\nu\rho\lambda} \sum_{i,j=1}^{4} q_i^\mu q_j^\nu C_{ij}. \quad (2.11)$$

This may be evaluated as follows:

$$P[C] = q_a^\rho q_b^\lambda \, \varepsilon_{\mu\nu\rho\lambda} \, g^\mu_{\ \mu'} \, g^\nu_{\ \nu'} \sum_{i,j=1}^{4} q_i^{\mu'} q_j^{\nu'} C_{ij}$$

$$= \frac{4}{()_5^2} q_a^\rho q_b^\lambda \, \varepsilon_{\mu\nu\rho\lambda} \sum_{k,l=1}^{4}\binom{k}{l}_5 q_k^\mu \, q_{l,\mu'} \sum_{m,n=1}^{4}\binom{m}{n}_5 q_m^\nu \, q_{n,\nu'} \sum_{i,j=1}^{4} q_i^{\mu'} q_j^{\nu'} C_{ij}$$

$$= \frac{4}{()_5^2} \sum_{\substack{k,m=1 \\ a+b+k+m=10}}^{4}\left[q_a^\rho q_b^\lambda \, \varepsilon_{\mu\nu\rho\lambda} q_k^\mu q_m^\nu\right] \sum_{l,n=1}^{4}\binom{k}{l}_5\binom{m}{n}_5 \sum_{i,j=1}^{4}(q_l \cdot q_i)(q_n \cdot q_j)C_{ij}$$

$$= \frac{4}{()_5^2}\det[q_1,q_2,q_3,q_4]\sum_{\substack{k,m=1 \\ a+b+k+m=10}}^{4} S(a,b,k,m)\sum_{l,n=1}^{4}\binom{k}{l}_5\binom{m}{n}_5 \Sigma_{ln}[C], \quad (2.12)$$

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where $S(a,b,k,m)$ gives the sign of permutations needed to place the indices in increasing order. Further we introduced the abbreviation

$$\Sigma_{ln}[C] = \sum_{i,j=1}^{4} (q_l \cdot q_i)(q_n \cdot q_j)C_{ij}, \tag{2.13}$$

the calculation of which has to be performed for specific cases.

Let us choose for demonstrational purposes $C_{ij} = \binom{is}{js}_5$:

$$\Sigma_{ln}[C] = \frac{1}{2}(q_l \cdot q_n)\binom{s}{s}_5 - \frac{1}{4}()_5(\delta_{ls} - \delta_{5s})(\delta_{ns} - \delta_{5s}) \tag{2.14}$$

$$= \begin{cases} \frac{1}{2}(q_l \cdot q_n)\binom{s}{s}_5 - \frac{1}{4}()_5\delta_{ls}\delta_{ns} & s \neq 5, \\ \frac{1}{2}(q_l \cdot q_n)\binom{s}{s}_5 - \frac{1}{4}()_5 & s = 5. \end{cases}$$

Next we have to evaluate the sum $\sum_{l,n=1}^{4}\binom{k}{l}_5\binom{m}{n}_5 \cdot \Sigma_{ln}[C]$. Taking the first term in (2.14), to begin with, we have

$$\frac{1}{2}\sum_{l,n=1}^{4}(q_l \cdot q_n)\binom{k}{l}_5\binom{m}{n}_5 = \frac{1}{2}\sum_{l=1}^{4}\binom{k}{l}_5\sum_{n=1}^{4}(q_l \cdot q_n)\binom{m}{n}_5$$

$$= \frac{1}{4}\sum_{l=1}^{4}\binom{k}{l}_5\delta_{ml}()_5$$

$$= \frac{()_5}{4}\binom{k}{m}_5, \tag{2.15}$$

and for $s \neq 5$ we thus have

$$\sum_{l,n=1}^{4}\binom{k}{l}_5\binom{m}{n}_5 \cdot \Sigma_{ln}[C] = \frac{()_5}{4}\left[\binom{k}{m}_5\binom{s}{s}_5 - \binom{k}{s}_5\binom{m}{s}_5\right]$$

$$= \frac{()_5^2}{4}\binom{ks}{ms}_5. \tag{2.16}$$

Similarly we have for $s = 5$

$$\sum_{l,n=1}^{4}\binom{k}{l}_5\binom{m}{n}_5 \cdot \Sigma_{ln}[C] = \frac{()_5}{4}\left[\binom{k}{m}_5\binom{5}{5}_5 - \sum_{l,n=1}^{4}\binom{k}{l}_5\binom{m}{n}_5\right]. \tag{2.17}$$

Due to

$$\sum_{l=1}^{5}\binom{k}{l}_5 = 0, \quad \text{i.e.} \sum_{l=1}^{4}\binom{k}{l}_5 = -\binom{k}{5}_5, \tag{2.18}$$

we get for $s = 5$ the same result as in (2.16).

Thus the total result reads for the specific choice of C_{ij}:

$$P[\binom{is}{js}_5] = \det[q_1, q_2, q_3, q_4] \sum_{\substack{k,m=1 \\ a+b+k+m=10}}^{4} S(a,b,k,m)\binom{ks}{ms}_5. \tag{2.19}$$

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In the summation of (2.19) all indices a, b, k, m must be different, i.e. the sum contains only 2 terms, with k and m exchanged. Since these terms come with opposite sign and (2.16) is symmetric in k and m, the final result for the present example vanishes, i.e.

$$P[\binom{is}{js}_5] = q_a^\rho q_b^\lambda \, \varepsilon_{\mu\nu\rho\lambda} \sum_{i,j=1}^4 q_i^\mu q_j^\nu \binom{is}{js}_5$$
$$= 0. \tag{2.20}$$

2.3 The tensors of rank $R = 3$: $I_5^{\mu\nu\lambda}$

From eq.(3.5) of [5] one gets

$$I_5^{\mu\nu\lambda} = I_5^{\mu\nu} \cdot Q_0^\lambda - \sum_{s=1}^5 I_4^{\mu\nu,s} \cdot Q_s^\lambda, \tag{2.21}$$

where according to (3.12) in [5] it is

$$I_4^{\mu\nu,s} = I_4^{\mu,s} Q_0^{s,\nu} - \sum_{t=1}^5 I_3^{\mu,st} Q_t^{s,\nu} - \frac{()_5}{\binom{s}{s}_5} Q_s^\mu Q_s^\nu I_4^{[d+],s},$$

$$I_3^{\mu,st} = -\sum_{i=1}^4 q_i^\mu I_{3,i}^{[d+],st} = Q_0^{st,\mu} I_3^{st} - \sum_{u=1}^5 Q_u^{st,\mu} I_2^{stu}. \tag{2.22}$$

2.3.1 Contractions with chords: $q_{a\mu} q_{b\nu} q_{c\lambda} I_5^{\mu\nu\lambda}$

The contraction of (2.21) with $q_{c\lambda}$ is trivial due to (1.7) and (1.10). Further, the term $q_{a\mu} q_{b\nu} I_5^{\mu\nu}$ is known from subsection 2.3.1. The only new object to be investigated is therefore

$$q_{a\mu} q_{b\nu} I_4^{\mu\nu,s} = (q_a \cdot I_4^s)(q_b \cdot Q_0^s) - \sum_{t=1}^5 (q_a \cdot I_3^{st})(q_b \cdot Q_t^s) - \frac{()_5}{\binom{s}{s}_5}(q_a \cdot Q_s)(q_b \cdot Q_s) I_4^{[d+],s}. \tag{2.23}$$

New sums may be expressed by those derived in [6]:

$$(q_b \cdot Q_0^s) = \frac{1}{\binom{s}{s}_5} \Sigma_b^{2,s}, \tag{2.24}$$

$$(q_b \cdot Q_t^s) = \frac{1}{\binom{s}{s}_5} \Sigma_b^{1,st}, \tag{2.25}$$

and

$$(q_a \cdot I_3^{st}) = \frac{1}{\binom{st}{st}_5} \left[\Sigma_a^{3,st} I_3^{st} - \sum_{u=1}^5 \Sigma_a^{1,stu} I_2^{stu} \right]$$
$$= -\frac{1}{2} \left\{ (Y_{a5} - Y_{55}) I_3^{st} + \sum_{u=1}^5 (\delta_{au} - \delta_{5u}) I_2^{stu} + (\delta_{as} - \delta_{5s}) R^{st} + (\delta_{at} - \delta_{5t}) R^{ts} \right\}, \tag{2.26}$$

where we introduced as a further abbreviation

$$R^{st} \equiv \frac{1}{\binom{st}{st}_5} \left[\binom{st}{0t}_5 I_3^{st} - \sum_{u=1}^5 \binom{st}{ut}_5 I_2^{stu} \right]$$
$$= \frac{1}{\binom{0st}{0st}_5} \left[\binom{st}{0t}_5 (d-2) I_3^{[d+],st} - \sum_{u=1}^5 \binom{0st}{0ut}_5 I_2^{stu} \right]. \tag{2.27}$$

From

$$\Sigma_b^{2,s} = -\frac{1}{2}\left\{ \binom{s}{s}_5 (Y_{b5} - Y_{55}) + \binom{s}{0}_5 (\delta_{bs} - \delta_{5s}) \right\}, \tag{2.28}$$

$$\Sigma_b^{1,st} = \frac{1}{2}\left\{ \binom{s}{s}_5 (\delta_{bt} - \delta_{5t}) - \binom{s}{t}_5 (\delta_{bs} - \delta_{5s}) \right\}, \tag{2.29}$$

multiplying (2.7) and (2.26), respectively, we see that the $(\delta_{bs} - \delta_{5s})$-terms in (2.29) combine with the first terms in (2.7) and (2.26) to yield the contribution

$$\frac{1}{4}(Y_{a5} - Y_{55})(\delta_{bs} - \delta_{5s})R^s. \tag{2.30}$$

Collecting all contributions, the following relation is useful in order to to eliminate some $\frac{1}{\binom{s}{s}_5}$-terms:

$$R^{st} + \frac{\binom{s}{t}_5}{\binom{s}{s}_5}R^{ts} = \frac{\binom{s}{0}_5}{\binom{s}{s}_5}I_3^{st} - \sum_{u=1}^5 \frac{\binom{s}{u}_5}{\binom{s}{s}_5}I_2^{stu}. \tag{2.31}$$

The result for (2.23) is explicitly symmetric in the indices a, b:

$$q_{a\mu}q_{b\nu}I_4^{\mu\nu,s} = q_{a\mu}q_{b\nu}\bar{I}_4^{\mu\nu,s} - \frac{1}{4}(\delta_{as} - \delta_{5s})(\delta_{bs} - \delta_{5s})\frac{()_5}{\binom{s}{s}_5}I_4^{[d+],s}, \tag{2.32}$$

and the first term on the right hand side is

$$\begin{aligned}
q_{a\mu}q_{b\nu}\bar{I}_4^{\mu\nu,s} &= \frac{1}{4}(Y_{a5} - Y_{55})(Y_{b5} - Y_{55})\cdot I_4^s \\
&+ \frac{1}{4}(\delta_{as} - \delta_{5s})(\delta_{bs} - \delta_{5s})\frac{1}{\binom{s}{s}_5}\left[\binom{s}{0}_5 R^s - \sum_{t=1}^5 \binom{s}{t}_5 R^{st} \right] \\
&+ \frac{1}{4}\left[(\delta_{as} - \delta_{5s})(Y_{b5} - Y_{55}) + (Y_{a5} - Y_{55})(\delta_{bs} - \delta_{5s}) \right]R^s \\
&+ \frac{1}{4}\sum_{t=1}^5 \left[(\delta_{as} - \delta_{5s})(\delta_{bt} - \delta_{5t}) + (\delta_{bs} - \delta_{5s})(\delta_{at} - \delta_{5t}) \right]R^{st} \\
&+ \frac{1}{4}\sum_{t=1}^5 \left[(\delta_{at} - \delta_{5t})(Y_{b5} - Y_{55}) + (Y_{a5} - Y_{55})(\delta_{bt} - \delta_{5t}) \right]I_3^{st} \\
&+ \frac{1}{4}\sum_{t=1}^5 (\delta_{at} - \delta_{5t})(\delta_{bt} - \delta_{5t})R^{ts} + \frac{1}{4}\sum_{t,u=1}^5 (\delta_{bt} - \delta_{5t})(\delta_{au} - \delta_{5u})I_2^{stu}. \quad (2.33)
\end{aligned}$$

It is worth mentioning that the second term in (2.32) may be written as

$$()_5 I_4^{[d+],s} = \binom{0}{0}_5 I_4^s - \sum_{t=1}^5 \binom{t}{0}_5 I_3^{st} - \binom{s}{0}_5 R^s. \tag{2.34}$$

The R^s, R^{st}, $\sum_{t=1}^5 \binom{s}{t}_5 R^{st}$ and the scalar integrals in (2.33) are independent of the indices a, b, c and can be considered as buildung blocks. The summation over s, t, u is simplified by the explicit appearance of the Kronecker δ-functions, i.e. all contractions are just given in terms of different combinations of the buildung blocks.

There still occur contributions with factor $\frac{1}{\binom{s}{s}_5}$ in (2.32) and in (2.33). At first we observe that this contribution is indeed quite "localized": in general one will be able to avoid the small $\binom{s}{s}_5$ to appear on line $s = 5$ by choosing the integration momenta properly. Let say $\binom{s_0}{s_0}_5$ is small. Then the critical contribution will only occur if $a = b = c = s_0$, i.e. only in a single contraction.

Nevertheless, if one wants to avoid the $\frac{1}{\binom{s}{s}_5}$ completely, one can express the corresponding contribution for the sum of these terms in (2.32) and (2.33) in terms of higher dimensional integrals, making again use of (2.8). Let us call the sum J_4^s:

$$J_4^s \equiv \frac{1}{\binom{s}{s}_5}\left\{ -()_5 I_4^{[d+],s} + \binom{s}{0}_5 R^s - \sum_{t=1}^{5}\binom{s}{t}_5 R^{st}\right\} \tag{2.35}$$

$$= \frac{-1}{\binom{0s}{0s}_5}\left\{ ()_5 (d-2)(d-1) I_4^{[d+]^2,s} - \binom{0}{0}_5 I_4^{[d+],s} + \sum_{t=1}^{5}\binom{t}{0}_5 (d-2) I_3^{[d+],st} + \sum_{t=1}^{5}\binom{0s}{0t}_5 R^{st}\right\}.$$

The higher dimensional integrals can then be evaluated as described in [5].

Let us call the second term on the right hand side of (2.21) $C_{5,abc}$. It becomes, when contracted with momenta q_a, q_b and q_c:

$$C_{5,abc} = -q_{a\mu}q_{b\nu}q_{c\lambda}\sum_{s=1}^{5} I_4^{\mu\nu,s}\cdot Q_s^{\lambda}$$

$$= -\sum_{s=1}^{5} q_{a\mu}q_{b\nu} I_4^{\mu\nu,s}\cdot\frac{1}{2}\left(\delta_{cs}-\delta_{5s}\right). \tag{2.36}$$

The last term again contains (2.32). Introducing $Y_a = Y_{a5} - Y_{55}$ and making use of (2.35), we may collect terms and get

$$C_{5,abc} = \frac{1}{8}\ \left\{ J_4^5 - \delta_{ab}\delta_{ac}J_4^a + Y_aY_b\left(I_4^5 - I_4^c\right)\ + Y_a\left(I_3^{b5} + I_3^{c5} - I_3^{bc}\right) + Y_b\left(I_3^{a5} + I_3^{c5} - I_3^{ac}\right)\right.$$

$$+ I_2^{ab5} + I_2^{ac5} + I_2^{bc5} - I_2^{abc}\ - Y_a\left(R^5 + \delta_{bc}R^b\right) - Y_b\left(R^5 + \delta_{ac}R^a\right)$$

$$\left. - R^{5a} - R^{5b} - R^{5c}\ + \delta_{ab}\left(R^{a5} - R^{ac}\right) + \delta_{ac}\left(R^{a5} - R^{ab}\right)\ + \delta_{bc}\left(R^{b5} - R^{ba}\right)\right\}. \tag{2.37}$$

2.3.2 Contractions with the metric tensor: $g_{\mu\nu}q_{a\lambda}I_5^{\mu\nu\lambda}$

In order to calculate $g_{\mu\nu}I_5^{\mu\nu\lambda}$, we need $g_{\mu\nu}I_4^{\mu\nu}$ and thus the further scalar products, see (2.22):

$$(Q_0^s\cdot Q_0^s) = \frac{1}{2\binom{s}{s}_5}\left[\binom{0s}{0s}_5 + 2\binom{s}{0}_5\delta_{s5}\right] + \frac{1}{2}Y_{55},$$

$$(Q_s\cdot Q_s) = \frac{1}{2}\frac{\binom{s}{s}_5}{()_5},$$

$$(Q_t^s\cdot Q_0^s) = \frac{1}{2\binom{s}{s}_5}\left[\binom{ts}{0s}_5 - \binom{s}{s}_5\delta_{t5} + \binom{s}{t}_5\delta_{s5}\right],$$

$$(Q_t^s\cdot Q_0^{st}) = \frac{1}{2\binom{s}{s}_5}\left[\qquad - \binom{s}{s}_5\delta_{t5} + \binom{s}{t}_5\delta_{s5}\right],$$

$$(Q_t^s\cdot Q_u^{st}) = 0, \tag{2.38}$$

which yields

$$g_{\mu\nu}I_4^{\mu\nu,s} = \frac{Y_{55}}{2}I_4^s + I_3^{s5} + \delta_{s5}R^s \tag{2.39}$$

and finally

$$g_{\mu\nu}q_{a\lambda}I_5^{\mu\nu\lambda} = -\frac{Y_{55}}{4}\left[(Y_{a5}-Y_{55})E + I_4^a - I_4^5\right] - \frac{1}{2}\left[(Y_{a5}-Y_{55})I_4^5 + I_3^{a5} - R^5\right]. \tag{2.40}$$

It is remarkable that (2.39) is trivial again for $s \neq 5$. For $s = 5$, however, the standard cancellation of propagators does not work and for this case (2.39) is indeed a useful result.

3. The OLEC package – first numerical results

The relations introduced in the previous sections have been implemented in Mathematica. A corresponding file OLECv0.9.m can be found at [12]. Certainly, computer algebra programs like Mathematica are not the optimal and efficient tools for ultimate numerical applications, they are slow. However, there are some reasons to start with Mathematica. First, standard tensor reductions for five and six point functions discussed in [11] have been also implemented in the Mathematica package hexagon.m [13, 14]. The new method of direct contractions of tensors as introduced in [6, 5] and presented here can be compared with hexagon.m. Second, if basic scalar integrals are known analytically, then the complete reductions are automatically given also in an analytic form. This approach could be used for instance to investigate infrared structure of amplitudes [15 – 17, 11]. Of course, for numerical applications one will prefer packages written in Fortran or C++. For efficient reductions of five-point tensor integrals, see V. Yundin's C++ package PJFry [18 – 20]).

For the *contracted* 5-point functions with which we are dealing here, the following notation is used: CEx(chords), where "C" stands for "contracted" and "E" stands traditionally for pentagons. The argument "chords" represents a list of indices, and their length depends on the rank "x" of the considered object. In this way, $CE1(a)$ is equivalent to (2.2), $CE2(a,b)$ is equivalent to (2.6) and $CE3(a,b,c)$ is equivalent to appropriate relations given in subsection 2.3.1. For the objects contracted with the metric tensor $g_{\mu\nu}$ we have two objects up to rank 3: $CE2g$ and $CE3g(a)$. They correspond to (2.10) and (2.40), respectively.

Now, we can compare these objects with corresponding objects which come from standard reductions. Below we will use the notation of LoopTools/FF [21, 22] for coefficients of corresponding tensorial objects $E0i$:

$$CE1(a) \equiv q_a^{\mu_1}\left\{\sum_{i=1}^4 q_{\mu_1}^i E0i[eei,invs]\right\}, \tag{3.1}$$

$$CE2(a,b) \equiv q_a^{\mu_1}q_b^{\mu_2}\left\{\sum_{\{i,j\}=1}^4 q_{\mu_1}^i q_{\mu_2}^j E0i[eeij,invs] + g_{\mu_1\mu_2}E0i[ee00,invs]\right\}, \tag{3.2}$$

$$CE3(a,b,c) \equiv q_a^{\mu_1}q_b^{\mu_2}q_c^{\mu_3}\left\{\sum_{\{i,j,k\}=1}^4 q_{\mu_1}^i q_{\mu_2}^j q_{\mu_3}^j E0i[eeijk,invs]\right.$$
$$\left. + \sum_{i=1}^4 \left(g_{\mu_1\mu_2}q_{\mu_3}^i + g_{\mu_1\mu_3}q_{\mu_2}^i + g_{\mu_2\mu_3}q_{\mu_1}^i\right)E0i[ee00i,invs]\right\}. \tag{3.3}$$

For contractions with the metric tensor:

$$CE2g \equiv E0i[ee001, invs], \tag{3.4}$$

$$CE3g(a) \equiv g^{\mu_1\mu_2}q_a^{\mu_3}\left\{ \sum_{\{i,j,k\}=1}^{4} q_{\mu_1}^i q_{\mu_2}^j q_{\mu_3}^j E0i[eeijk, invs] \right.$$

$$\left. + \sum_{i=1}^{4}\left(g_{\mu_1\mu_2}q_{\mu_3}^i + g_{\mu_1\mu_3}q_{\mu_2}^i + g_{\mu_2\mu_3}q_{\mu_1}^i\right)E0i[ee00i, invs]\right\}. \tag{3.5}$$

Numerical results are shown in the file `Contracts_examples.nb`. They are compared for various chord indices using the Looptools/FF and OneLoop [23] packages.

Below we tabularize an example of results extracted from `Contracts_examples.nb` for kinematical points defined by:
$p_1^2 = p_2^2 = 0$; $p_3^2 = p_5^2 = 49/256$; $p_4^2 = 9/100$;
$s_{12} = 4$; $s_{23} = -1/5$; $s_{34} = 1/5$; $s_{45} = 3/10$; $s_{15} = -1/2$;
$m_1^2 = m_2^2 = m_3^2 = 49/256$; $m_4^2 = m_5^2 = 81/1600$.

```
In:
Do[Print[i, ") LoopTools: ", Rank1LoopTools[i], ", Hexagon: ",
  Rank1hexagon[i], ", Contracts: ", CE1[i]], {i, 1, 5}]
Out:
1) LoopTools: -19.1597+12.6772 I, Hexagon: -19.1597+12.6772 I,
Contracts: -19.1597+12.6772 I
2) LoopTools: -80.4038+91.9442 I, Hexagon: -80.4038+91.9442 I,
Contracts: -80.4038+91.9442 I
3) LoopTools: -19.1525+12.4364 I, Hexagon: -19.1525+12.4364 I,
Contracts: -19.1525+12.4364 I
4) LoopTools: -17.496+10.67060 I, Hexagon: -17.496+10.67060 I,
Contracts: -17.496+10.67060 I
5) LoopTools: 0, Hexagon: 0. +0. I, Contracts: 0
```

In addition, comparisons concerning the speed of calculation have been made. As expected, the Mathematica package OLEC.m v.0.9 is at least one order of magnitude faster than hexagon.m but still slower than the Fortran package LoopTools/FF.

Thus, a second package OLEC v.0.9 written in C++ has been prepared for numerical studies. It should be assumed as the skeleton for the first version of the ultimate package. It has to be further optimized and additional refinements for specific kinematical points are needed, similar to those which were implemented for the standard reductions within the PJFRY package.

The OLEC package can be downloaded from [12]. It is linked with both the LoopTools/FF and the OneLoop libraries. Our first package, where the contractions replace the explicit tensor reductions is at the moment comparable in speed to numerical results of 5-point tensor reductions implemented in LoopTools/FF. This has been tested for specific rank 2 tensors in (3.2) and tensors of rank 3 in (3.1). Details can be found in the module `example_2` of the Makefile in [12] where

$CE2(1,2)$ and $CE3(1,2,3)$ can be calculated and compared with corresponding LoopTools objects using different flags, for users convenience. The result is not too surprizing. Evidently, for higher tensor ranks the advantage of contractions compared to rank reductions, introducing multiple sums, should become more pronounced.

4. Conclusions and outlook

A new method of evaluation of contracted tensor integrals has been presented for five-point one loop integrals. So far individual objects like $CE1$, $CE2$ and $CE3$ have been resolved and implemented into first codes. What remains is to apply it to the calculation of whole amplitudes for some physical processes.

In the near future the first version 1.0 of the complete code in C++ and Fortran 90 is planned to be released. Here we have shown the backbone of the code with first numerical estimations. In addition, higher than rank 3 tensor structures for 5-point functions and tensorial integrals with more than five external legs are in preparation.

Acknowledgments

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