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One–loop correction to the soft–gluon current with massive fermions

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In this talk, we present the calculation of the one–loop correction to the soft–gluon current in the case of massive fermions, which describes the singular behaviour of massive one–loop amplitudes when one of the external gluons becomes soft. This current is process independent and can be used in the context of numerical evaluations of massive observables to next–to–next–to leading order (NNLO) at hadron colliders.

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1. Introduction

For the calculation and numerical evaluation of QCD observables at hadron colliders, one has to deal with the occurring divergences, ultra–violet (UV) and infra–red (IR) (soft and collinear). The former can be removed by renormalization. The latter cancel in observables, however, they have to be dealt with at intermediate steps of the calculation, in particular in numerical integrations where they have to be regularized and treated appropriately. In this talk, a method for handling these singularities to next–to–next–to leading order (NNLO) in the case of massive fermions is described, which was derived in [1].

At next–to leading order (NLO), several subtraction schemes in the massless and massive case have been proposed for this issue [2, 3, 4, 5, 6, 7]. The idea is to construct an approximation to the corresponding real emission partonic amplitude, which is simple enough to analytically extract the IR singularities and calculate this quantity. The remaining difference of the full amplitude and its approximation is then IR finite and can therefore be integrated numerically in a straightforward manner.

To construct a subtraction scheme at NNLO, one needs to know, among other ingredients, the limiting behavior of one–loop amplitudes when one of the external on–shell gluons becomes soft. The massless case at NNLO has been addressed in Refs. [8, 9, 10]. In [1], we generalized the process–independent approach of Catani and Grazzini, [10], to the case of massive fermions which will be explained in the following. Details about our calculation and all mathematical expressions can be found in [1].

The calculation of the massive one–loop soft–gluon current was performed as one contribution to the project of the calculation of the full NNLO top–quark pair production for which first results are given in [11, 12], making use of the soft one–loop current calculated here. Different other groups have approached the topic of NNLO heavy quark production, either at the level of the full (two–loop) amplitude or subtraction terms. More details concerning the various ingredients for the above–mentioned calculation as well as the work of other groups, can be found, e.g., in [11, 12, 13, 14] and references therein.

2. Factorization of the amplitude in the soft limit

Let $M_a(n+1;q)$ be the amplitude for the production of $n+1$ on–shell partons, where at least one final–state parton is a gluon. Here, $a = 1,\dots,N_c^2$ is its color index and q its momentum, with $q^2 = 0$. In general we do not make the distinction between initial and final state partons, unless stated otherwise. We are considering the limit when the external gluon becomes soft, meaning that $q \to 0$, or, more precisely, its momentum scales as $q \to \lambda q$, $\lambda \to 0$. In this limit, the amplitude satisfies the following factorization property:

$$M_a(n+1;q) = J_a(q)M(n) + \mathcal{O}(\lambda)\,. \tag{2.1}$$

The n–parton amplitude $M(n)$ denotes the remaining amplitude after removing the external gluon with momentum q from $M_a(n+1;q)$, while $J_a(q)$ is the soft–gluon (eikonal) current, the main subject to be calculated to one–loop order here. Note that this current is process–independent and therefore, calculated once and for all, can be used in different contexts. Each factor in Eq. (2.1) has a loop expansion in powers of the strong coupling constant α_S indicated by a superscript, $^{(n)}$, $n \in \mathbb{N}$, and depends on the dimensional regularization parameter ε, in $d = 4-2\varepsilon$ dimensions:

$$\begin{aligned} M_a(n+1;q) &= M_a^{(0)}(n+1;q) + M_a^{(1)}(n+1;q) + \dots\,, \\ M(n) &= M^{(0)}(n) + M^{(1)}(n) + \dots\,, \\ J_a(q) &= g_S \mu^{\varepsilon} \left(J_a^{(0)}(q) + J_a^{(1)}(q) + \dots \right)\,. \end{aligned} \tag{2.2}$$

The dots indicate terms at higher orders in α_S. Since the leading–order amplitude $M^{(0)}(n)$ contains a process–dependent power of the strong coupling constant, the powers are not put explicitly in this notation. The index $^{(0)}$ thus always names the leading order, $^{(1)}$ the next–to leading order in α_S, and so on, such that $J_a^{(0)}$ denotes the leading order result for the soft current given explicitly in Eq. (2.3), $J_a^{(1)}$ stands for its next–to leading order in α_S, etc.

Since the same considerations apply for UV unrenormalized as well as renormalized amplitudes, the concentration in the following will be on bare amplitudes and the reader is referred to [1] for questions concerning UV renormalization.

Except for the soft external gluon with momentum q, external momenta of the remaining external on–shell partons are denoted by p_i. The massive case is then given by $p_i^2 = m_i^2 > 0$, the massless by $p_i^2 = 0$. In both cases, the tree–level soft–gluon current reads:

$$J_a^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q} \equiv \sum_{i=1}^{n} T_i^a e_i^\mu\,, \tag{2.3}$$

with $J_a^{(n)}(q) \equiv \varepsilon^\mu(q) J_a^{\mu(n)}(q)$. Throughout the following, the conventions of Ref. [10] for the signs of color generators are applied. We will also follow the general strategy developed in Ref. [10] for the calculation of the one–loop soft–gluon current in Eq. (2.4). The approach consists in the evaluation of all appearing one–loop diagrams connecting (on–shell) external legs and attaching a real gluon to either the external legs or the gluon propagator of the virtual gluon in the loop. The calculation is performed in the eikonal approximation. The diagrams are split into $1P$ and $2P$ contributions, where the $1P$ contributions are defined as the ones that depend on a single external hard momentum p_i, while the $2P$ contributions involve two external hard momenta p_i and p_j. In Ref. [10] it is shown that the calculation of the one–loop soft–gluon current is reduced to the calculation of the sum of two (non–abelian) $2P$–contribution diagrams (cf. Fig. 4 in Ref. [10]). Looking at these diagrams more closely, one realizes that at the integrand level the eikonal approximation is identical for the massive and the massless case and hence we can simply use the sum of the expressions given in Eqs. (46), (47) of Ref. [10]. However, for the massive case, which we are considering here, the subsequent reduction differs from the one done there, because it produces terms that explicitly depend on the masses $m_{i,j}^2$, which can be non–zero for the cases at hand. Applying partial fractioning and omitting scaleless integrals, the following expression for the the one–loop UV–unrenormalized soft–gluon current $J_a^{\mu(1)}(q)$ is derived:

$$J_a^{\mu(1)}(q) = i f_{abc} \sum_{i \neq j = 1}^{n} T_i^b T_j^c \left(e_i^\mu - e_j^\mu \right) g_{ij}^{(1)}(\varepsilon, q, p_i, p_j)\,. \tag{2.4}$$

The function $g_{ij}^{(1)}(\varepsilon, q, p_i, p_j)$ in Eq. (2.4) reads:

$$
\begin{aligned}
g_{ij}^{(1)} &= a_S^b \mu^{2\varepsilon} \frac{p_i \cdot p_j}{m_i^2 (p_j\cdot q)^2 - 2(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q) + m_j^2 (p_i\cdot q)^2} \\
&\times \Bigg\{ (p_i\cdot q)(p_j\cdot q)\left[(p_j\cdot q)M_1 + (p_i\cdot q)\hat{M}_1\right] \\
&+\frac{1}{2}(p_j\cdot q)\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2 (p_j\cdot q)\right] M_2 + \frac{1}{2}(p_i\cdot q)\left[(p_i\cdot p_j)(p_j\cdot q) - m_j^2(p_i\cdot q)\right]\hat{M}_2 \\
&+ \left[(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q) - m_i^2(p_j\cdot q)^2 - m_j^2 (p_i\cdot q)^2\right] \frac{(p_i\cdot q)(p_j\cdot q)}{p_i\cdot p_j} M_3 \Bigg\}, \qquad (2.5)
\end{aligned}
$$

using $a_S^b = \alpha_s^b S_\varepsilon/(2\pi)$ for the bare coupling with $S_\varepsilon = (4\pi)^\varepsilon \exp(-\varepsilon\gamma_E)$. The functions $M_{1,2,3}$ denote the following integrals:

$$
\begin{aligned}
M_1 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][-p_j\cdot k]}, \\
M_2 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][p_i\cdot k + p_i\cdot q][-p_j\cdot k]}, \qquad (2.6) \\
M_3 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][p_i\cdot k + p_i\cdot q][-p_j\cdot k]},
\end{aligned}
$$

with $\hat{M}_k \equiv M_k(p_i \leftrightarrow p_j),\ k = 1,2,3$. Each propagator has an implicit $+i\delta$ imaginary part. The momenta p_i, p_j can in general be massive or massless and the momentum q, corresponding to the soft gluon, is assumed outgoing and massless. The normalization factor is $\Phi = 8\pi^2(4\pi)^{-\varepsilon} e^{\varepsilon\gamma_E}$. Noting that $\hat{M}_3 = M_3$, we see that $g_{ij}^{(1)}$ is symmetric, $g_{ij}^{(1)} = g_{ji}^{(1)}$. The formula for $g_{ij}^{(1)}$ in Eq. (2.5) is generic for different kinematical regions, which we consider next.

3. The one–loop current in the kinematic regions

For all following cases, we define $p_i^2 = m_i^2 > 0$ and p_i as well as the momentum q of the soft gluon in the final state. We consider three kinematical configurations:

$$
1.)\ p_j^2 = 0,\ p_j \text{ incoming}, \quad 2.)\ p_j^2 = 0,\ p_j \text{ outgoing}, \quad 3.)\ p_j^2 = m_j^2 > 0,\ p_j \text{ outgoing}. \qquad (3.1)
$$

We will refer to the *Case 1* as being the "space–like" (SL) and the *Cases 2* and 3 as the "time–like" (TL) cases, depending on the sign of the scalar products $\sigma_k \equiv (p_k\cdot q)/|p_k\cdot q| = \pm 1$, with $\sigma_i = -\sigma_j = 1$ for the space–like and $\sigma_i = \sigma_j = 1$ for the time–like kinematics. For all three cases, the major bottleneck is the calculation of the integrals, which become increasingly complicated. Details for the calculation of these integrals and discussion about their results can be found in the Appendix A of [1]. The general structure of these integrals is the following: the integral M_1 can straightforwardly be calculated into a short expression, while the integral M_2 results in a hypergeometric function ${}_2F_1$. The most complicated integral is M_3. For the space–like kinematics, M_3^{SL} can be expressed in terms of two Appell hypergeometric functions F_1, which can be expanded in ε in terms of multiple polylogarithms $\mathrm{Li}_{m_k,\ldots,m_1}(t_k,\ldots,t_1)$ with the help of the library *Nestedsums* [15]. In the case of one non–zero mass, as needed for *Case 1*, the Appell functions in $M_3^{(SL)}$ collapse into two hypergeometric functions ${}_2F_1$. The evaluation of M_3 is hardest in the "time–like" kinematics of *Case 3*. For this case, we did not calculate a closed analytic form, but expanded the integral in ε up to the desired order and determined each term in this expansion.

Expanding the integrals M_i to the desired order in ε, one can evaluate the kinematical configurations for the function $g_{ij}^{(1)}$ directly. The generic result for the unrenormalized one–loop soft current can be written in the following form, treating the real and imaginary parts separately:

$$g_{ij}^{(1)}(Case\ x) = R_{ij}^{[Cx]} + i\pi I_{ij}^{[Cx]} \equiv a_S^b \left(\frac{2(p_i \cdot p_j)\mu^2}{2(p_i \cdot q)2(p_j \cdot q)} \right)^{\varepsilon} \sum_{n=-2}^{r} \varepsilon^n \left(R_{ij}^{(n)[Cx]} + i\pi I_{ij}^{(n)[Cx]} \right), \quad (3.2)$$

where r is the order in ε needed to produce finite contributions after phase–space integration. The purpose of the overall d–dimensional prefactor in Eq. (3.2) is to exactly extract the leading power scaling behavior of the one–loop soft–gluon current in the limit $q \to 0$. The remainder is given as an expansion in ε, which has a well–defined limit $q \to 0$. All results can be found in [1] either analytically or attached in electronic form.

Looking at the cases explicitly, one makes the following observations: *Case* 1 is available in fully analytic form and can be expanded up to order ε^2, as needed here. *Case* 2 can be obtained by performing an analytic continuation of the result of *Case* 1. Since in these cases $p_j^2 = 0$, the continuation amounts to the exchange of $p_j \to -p_j$ (cf. Appendix D of [1]). One easily observes that the result of *Case* 1 remains unchanged by such a transformation and hence *Case* 2 is identical to *Case* 1. *Case 3* is calculated up to and including terms of $\mathscr{O}(\varepsilon)$ which is sufficient, since the $\mathscr{O}(\varepsilon^2)$ term contributes only if multiplied by a term $\sim 1/\varepsilon^2$ originating from phase–space integration. Such a leading pole $\sim 1/\varepsilon^2$ stems from the emission of soft and collinear radiation. Since both partons i and j are massive, however, collinear singularities are regularized and the leading pole is of order $1/\varepsilon$.

For illustration, Eq. (3.4) shows the first orders up to $O(\varepsilon)$ of *Case 1*:

$$g_{ij}^{(1)}(Case\ 1) = R_{ij}^{[C1]} + i\pi I_{ij}^{[C1]} \equiv a_S^b \left(\frac{2(p_i \cdot p_j)\mu^2}{2(p_i \cdot q)2(p_j \cdot q)} \right)^{\varepsilon} \sum_{n=-2}^{2} \varepsilon^n \left(R_{ij}^{(n)[C1]} + i\pi I_{ij}^{(n)[C1]} \right), \quad (3.3)$$

with the real and imaginary parts given by:

$$\begin{aligned}
I_{ij}^{(-2)[C1]} &= 0, \qquad (3.4)\\
I_{ij}^{(-1)[C1]} &= -\frac{1}{2},\\
R_S I_{ij}^{(0)[C1]} &= 2m_i^2(p_j \cdot q)\ln\left(\frac{\alpha_i}{2}\right),\\
R_S I_{ij}^{(1)[C1]} &= 4\left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2(p_j \cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right) + m_i^2(p_j \cdot q)\ln^2\left(\frac{\alpha_i}{2}\right)\\
&\quad + \pi^2 \frac{-2(p_i \cdot p_j)(p_i \cdot q) + m_i^2(p_j \cdot q)}{2},\\
R_{ij}^{(-2)[C1]} &= -\frac{1}{2},\\
R_{ij}^{(-1)[C1]} &= 0,
\end{aligned}$$

$$
\begin{aligned}
R_S R_{ij}^{(0)[C1]} &= m_i^2(p_j\cdot q)\ln^2\left(\frac{\alpha_i}{2}\right) - \pi^2\frac{5(2(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q))}{6}\,,\\
R_S R_{ij}^{(1)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_3\left(\frac{\alpha_i}{2}\right) - \zeta_3\frac{4\left[7(p_i\cdot p_j)(p_i\cdot q) - 5m_i^2(p_j\cdot q)\right]}{3}\\
&\quad +2\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\ln\left(1-\frac{\alpha_i}{2}\right)\ln^2\left(\frac{\alpha_i}{2}\right)\\
&\quad +\ln\left(\frac{\alpha_i}{2}\right)\left(\pi^2\frac{-2(p_i\cdot p_j)(p_i\cdot q) - 5m_i^2(p_j\cdot q)}{3}\right.\\
&\quad \left. +4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)\right)\,,
\end{aligned}
$$

where we expressed the result through the variables R_S and α_i defined as:

$$
R_S = 4\left[m_i^2(p_j\cdot q) - 2(p_i\cdot p_j)(p_i\cdot q)\right]\,,\ \alpha_i = \frac{m_i^2(p_j\cdot q)}{(p_i\cdot q)(p_i\cdot p_j)}\,,\ \alpha_j = \frac{m_j^2(p_i\cdot q)}{(p_j\cdot q)(p_i\cdot p_j)}\,. \tag{3.5}
$$

We have performed various consistency checks on the results for the one–loop soft–gluon current (see the Appendices of [1] for more information). As expected, the result for *Case* 3 agrees in the limit $m_j \to 0$ with the result for the soft current in *Case* 2. This agreement is also a non–trivial check on the analytic continuation used to derive the result in *Case* 2 from that in *Case* 1. We have also numerically checked the result for the most difficult integral M_3 in the "time–like" kinematics *Case* 3. Additionally, the massless limit $m_i = 0$, $m_j = 0$ of the one–loop unrenormalized soft current reproduces the massless results of Ref. [10].

4. Squared matrix elements

The knowledge of the one–loop soft–gluon current makes it possible to construct an approximation to the squared one–loop matrix element for any process in the limit of the soft gluon as defined before, which is correct up to power–suppressed terms, as indicated in Eq. (2.1). The one–loop current (2.4) is calculated as an expansion in ε sufficient for the derivation of the terms $\mathcal{O}(\varepsilon^0)$ in any observable at NNLO. The interference term between the Born and one–loop amplitude in this limit is:

$$
\begin{aligned}
&\langle M_a^{(0)}(n+1;q)|M_a^{(1)}(n+1;q)\rangle + c.c. = -4\pi\alpha_S\mu^{2\varepsilon}\Bigg\{\\
&2C_A\sum_{i\neq j=1}^{n}(e_{ij}-e_{ii})R_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle - 4\pi\sum_{i\neq j\neq k=1}^{n} e_{ik}I_{ij}\langle M^{(0)}(n)|f^{abc}T_i^aT_j^bT_k^c|M^{(0)}(n)\rangle\\
&+\left(\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(1)}(n)\rangle + c.c.\right) + \left(\sum_{i=1}^{n}\mathscr{C}_i e_{ii}\langle M^{(0)}(n)|M^{(1)}(n)\rangle + c.c.\right)\Bigg\} + \mathcal{O}(\lambda)\,,
\end{aligned} \tag{4.1}
$$

where we have split $g_{ij}^{(1)} \equiv R_{ij} + i\pi I_{ij}$ into its real and imaginary parts as defined, e.g., in Eqs. (3.2), (3.3) and introduced $e_{ij} \equiv e_i\cdot e_j$ and $\mathscr{C}_i \equiv T_i\cdot T_i$.

5. Conclusion and outlook

We have studied the behaviour of one–loop QCD amplitudes with an arbitrary number of external massive fermions, in the limit of one external gluon becoming soft. This amplitude factorizes as in the massless case into a product of an amplitude, where the soft gluon has been removed and a process–independent soft–gluon current. This statement is correct up to power–suppressed terms. We have calculated this current up to and including one loop in three kinematic regions and performed a number of checks on the result.

This result can now be used for the evaluation of any cross section with massive fermions at next–to–next–to leading order within a subtraction approach. An immediate application for it is the calculation of the $t\bar{t}$ cross–section at NNLO, for which the one–loop soft–gluon current calculated here has been used in [11, 12].

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