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Length and distance on a quantum space

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This contribution is an introduction to the metric aspect of noncommutative geometry, with emphasize on the Moyal plane. Starting by questioning "how to define a standard meter in a space whose coordinates no longer commute ?", we list several recent results regarding Connes's *spectral distance* calculated between eigenstates of the quantum harmonic oscillator [8], as well as between coherent states [28]. We also question the difference (which remains hidden in the commutative case) between the spelral distance and the notion of *quantum length* inherited from the length operator defined in various models of noncommutative space-time (DFR and θ-Minkowski). We recall that a standard procedure in noncommutative geometry, consisting in doubling the spectral triple, allows to fruitfully confront the spectral distance with the quantum length. Finally we refine the idea of discrete vs. continuous geodesics in the Moyal plane, introduced in [27].

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1. Introduction

Till 1960 the standard meter was the distance between two marks on an iridium-platinum bar in the *Bureau des poids et mesures* in Paris. In 1960, the meter has been defined as 1 650 763.73 wave lengths from an orange radiation of ^{86}Kr. Since 1983 [37] 1 meter is the length of the path travelled by light in vacuum during $\frac{1}{299\,792\,458}$ of a second, the latest being the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom. The definition of the meter thus relies on both special relativity - the speed of light is constant by definition - and quantum properties of nature - transitions between energy levels correspond to radiations with a fixed wavelength. Nowadays metrology mainly consists in measuring the proper time interval[1]

$$\Delta\tau = \int d\tau = \frac{1}{c}\int ds \quad \text{with} \quad ds = \sqrt{g_{\mu\nu}(x)dx^\mu dx^\mu} \tag{1.1}$$

between two suitably chosen events, like the emission and the reception of a light signal at one extremity of the object one is measuring (with a mirror reflecting the signal at the other extremity).

The choice of the worldline along which the integral (1.1) is performed becomes crucial as soon as the gravitational field is non-negligible with respect to the uncertainties of the clock used to measure $\Delta\tau$. For instance, measuring by laser telemetry the height of a tower of 400m thanks to a clock localized at the bottom or at the top of the tower yields two distinct values $h_{\text{top}} = \frac{1}{2}c\Delta\tau_{\text{top}}$, $h_{\text{bottom}} = \frac{1}{2}c\Delta\tau_{\text{bottom}}$ which differ in relative value by an amount bigger than the uncertainty measurement of the best atomic clock [22]. Say differently, in presence of a sufficiently non-uniform gravitational field, the length of an object is a frame dependent notion. To quote [22], the 1983 definition of the meter "fixes the unit of proper length in the tangent three-dimensional space orthogonal to the world line of the caesium atom providing the second (where the theory of special relativity applies)" and it is valid in "regions of space small enough that the non-uniformity of the gravitational field has negligible effects with respect to the uncertainties of measurement". But if the precision of atomic-clocks were improved by - say - two orders of magnitude, one could not unambiguously measure the height of a 1m tower. This would make the very notion of "meter" problematic: two observers located at the extremities A, B of the (vertical) iridium-platinum bar would disagree on the numerical value of the meter: is $1\text{m} = \frac{1}{2}c\Delta\tau_A$ or $1\text{m} = \frac{1}{2}c\Delta\tau_B$?

In principle one could escape the problem by considering as a unit of length a sufficiently small submultiple of the meter, so that the non-uniformity of the gravitational field becomes negligible. Not arbitrarily small however, since at the Planck scale the classical picture of spacetime as a smooth manifold is expected to loose any operational meaning, due to the impossibility of simultaneously measuring with arbitrary accuracy the four spacetime coordinates x_μ. This comes as a consequence of the *principle of gravitational stability against localization* [17, 18], which can be stated as follows: *The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localise an event in spacetime should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.* In other term, to prevent the formation of black-hole during an arbitrarily accurate localization process, one postulates a non-zero minimal uncertainty in the *simultaneous* measurement of *all* coordinates of

[1] All along the paper we use Einstein sum on repeated indices in alternate up/down position

space-time. Typically, in a flat four dimensional space-time $\mathscr{M}$ and assuming suitable symmetry for the measuring-probe, one has [19]

$$\Delta x_0(\Delta x_1+\Delta x_2+\Delta x_3)\geq\lambda_P^2,\quad \Delta x_1\Delta x_2+\Delta x_2\Delta x_3+\Delta x_3\Delta x_1\geq\lambda_P^2 \tag{1.2}$$

where λ_P is the Planck length. A way to implement these uncertainty relations is to view the coordinates in a chart U of $\mathscr{M}$ no more as functions $x\in U\subset\mathscr{M}\mapsto x_\mu\in\mathbb{R}$, but as quantum operators q_μ satisfying non trivial commutation relations,

$$[q_\mu,q_\nu]=i\lambda_P^2Q_{\mu\nu}, \tag{1.3}$$

where the $Q_{\mu\nu}$'s are operators whose properties depend on the model. Generalization to curve space-time have been investigated in [38, 20].

Of course the accuracy of today's atomic-clocks is far too small compare to the inhomogeneity of the gravitational field on Earth to make an effective measurement of λ_P frame dependant. However the 1983 definition of the unit of length relies in an essential manner on the differential nature of the line element (1.1) and is therefore incompatible with a quantum structure of spacetime as the one postulated in (1.3). To be more specific, assuming that space-time at small scale is accurately described by quantum coordinate operators q_μ, how does one extract some metric information from it ? None of the objects entering formula (1.1) makes sense in a quantum context. By this we mean:

$$\overset{?}{\int_x^{y\to ?}}\sqrt{\underset{?}{g_{\mu\nu}(x)}\underset{?}{dx^\mu dx^\nu}}$$

- what would be a "quantum metric tensor" $g_{\mu\nu}$, and what would be its evaluation at some "quantum point" ?
- in case such an object is well defined (and there do exist proposals for a quantized version of the metric tensor, see e.g. [24]), how does one contract it with some "quantum differentials" dq^μ so that to obtain a "quantum line element" ?
- between which "quantum points" and along which "quantum geodesics" should one integrate this line element ? And with respect to which theory of "quantum integration" ?

In this contribution, we review two proposals to answer these questions in the Riemannian context. This does not allow to work out what could be the equivalent of the standard meter in a quantum *space-time*, but it allows to give an answer to all the question marks above for a quantum *space*. These two proposals rely on an algebraic definition of the length/distance, that makes sense in a noncommutative framework and gives back the usual geodesic distance (i.e. the length of the shortest path) when applied to the commutative coordinates. The first proposal consists in defining a quantum *length operator* L, whose eigenvectors are interpreted as "eigenstates of length" and the minimum l_P of its spectrum is the minimal value that may come out from a length measurement. As soon as l_P is non-zero, one inherits a natural notion of *minimal length*. The second proposal is Connes' *spectral distance* formula in noncommutative geometry.

Let us begin with the first proposal. As stressed above, the quantum coordinates satisfying (1.3) are well defined in the flat case, so a natural candidate as a length operator is

$$L = \sqrt{\sum_{\mu} (dq_\mu)^2} \quad \text{where} \quad dq_\mu \doteq q_\mu \otimes 1 - 1 \otimes q_\mu, \tag{1.4}$$

for it mimics the formula of the Euclidean distance (details in section 2). Explicit computations of

$$l_P = \min\left(\mathrm{Sp}(L)\right) \tag{1.5}$$

have been made in [19, 4] for the model of Doplicher, Fredenhagen and Roberts (DFR) in which the commutators $Q_{\mu\nu}$'s are central operators with selfadjoint closure, covariant under the action of the Poincaré group; as well as in [1] for the canonical noncommutative space θ-Minkowski where the $Q_{\mu\nu}$'s are constant. These results are recalled in section 5.1.

Both models are in fact viewed as quantum deformations of Minkowski spacetime, in that they carry an action of either the usual Poincaré group (DFR), or a quantum group deformation of it (θ-Minkowski). However, regarding the computation of the *quantum length* (defined below as a mean value of L), these actions do not play any role (see the appendix). Alternatively, one could wonder why we do not consider instead the Lorentzian operator

$$(dq_0)^2 - \sum_{\mu} (dq_\mu)^2. \tag{1.6}$$

There are two difficulties: on the one hand the spectrum of this operator is not bounded below from zero and its physical interpretation as a quantum observable is not transparent (see [4]); on the other hand the comparison with the spectral distance formula would not be easy, since the latter makes sense only in the Riemannian case (although some generalization to the Minkovskian case have been investigated in [31, 21]).

The second proposal is thus Connes spectral distance in noncommutative geometry [11]. Before introducing it, let us make a short digression: in order to be interpreted as a quantum-length operator with *real* spectrum, the operator L discussed above cannot be seen as an element of $\mathscr{A}_F \otimes \mathscr{A}_F$, where $\mathscr{A}_F$ denotes the free algebra generated by relation (1.3) and the identity $\mathbb{I}$. Indeed, as nicely explained in [32], any element in $\mathscr{A}_F$ which is not a multiple of $\mathbb{I}$ is a polynomial p in the q_μ's, so that for any complex number λ one has $(p-\lambda)^{-1} \notin \mathscr{A}_F$. This means that the spectrum of the element p is the whole complex plane, which makes the interpretation of the q_μ's as physical observables difficult (a physical observable is expected to have real spectrum). The point is thus to determine an Hilbert space $\mathscr{H}$ on which the q_μ's act as (unbounded) selfadjoint operators. Their spectrum,

$$\mathrm{Sp}(q_\mu) \doteq \left\{\lambda \in \mathbb{C}, (q_\mu - \lambda\mathbb{I}) \text{ has no inverse in } \mathscr{B}(\mathscr{H})\right\} \tag{1.7}$$

would then be real, making the q_μ's acceptable physical observables. Such an Hilbert space is obtained by viewing the q_μ's as operators affiliated to a suitable noncommutative algebra $\mathscr{A}$. In the models of quantum spaces studied in this paper, $\mathscr{A}$ turns out to be the algebra $\mathbb{K}$ of compact operators (details are given in section 3).

This digression illustrates how a suitably chosen noncommutative $*$-algebra $\mathscr{A}$, viewed as the algebra of bounded continuous functions on the quantum space, can be a more tractable (and chart independent) way to describe a quantum space than the algebra of coordinates. Such an idea is at the heart of Connes' approach to noncommutative geometry [12] in which all the geometric information is encoded within a *spectral triple*, that is an operator D - acting on some Hilbert space $\mathscr{H}$ that carries a representation of $\mathscr{A}$ - which generalizes the Dirac operator $\not{\partial} = -i\gamma^\mu \partial_\mu$ of quantum field theory. A distance on the state space $\mathscr{S}(\mathscr{A})$ of $\mathscr{A}$ (that is the set of positive, normalized, linear applications from $\mathscr{A}$ to $\mathbb{C}$, see sections 2.1) is defined by

$$d_D(\varphi_1, \varphi_2) \doteq \sup_{a \in \mathscr{A}} \{ |\varphi_1(a) - \varphi_2(a)|, \|[D,a]\| \leq 1 \}. \tag{1.8}$$

This formula generalizes the Riemannian distance to the noncommutative framework (proposition 2.1) and relies only on the spectral properties of $\mathscr{A}$ and D. From a mathematical point of view, one can check without difficulty that (1.8) does define a (possibly infinite) distance on $\mathscr{S}(\mathscr{A})$, that is a function from $\mathscr{S}(\mathscr{A}) \times \mathscr{S}(\mathscr{A}) \to \mathbb{R}^+ \cup \{\infty\}$ which is symmetric in the exchange of its arguments, vanishes on the diagonal ($d_D(\varphi, \tilde{\varphi}) = 0$ iff $\tilde{\varphi} = \varphi$) and satisfies the triangle inequality.

In the following, we review several recent results on the spectral distance and the length operator, mainly from [8, 27, 28, 15]. We begin by recalling in section 2 how to retrieve, in the commutative case $Q_{\mu\nu} = 0$, the Euclidean distance from either the length operator L or Connes' formula (1.8). Then we go to the noncommutative case and, under the assumption that the $Q_{\mu\nu}$'s are non-zero central operators, we single out in section 3 the Moyal algebra as a suitable algebra to describe the quantum space (1.3). In section 4 we make precise our notions of "quantum points" as pure states of the C^*-closure of the Moyal algebra, which turns out to be the algebra of compact operators $\mathbb{K}$. In section 5 we discuss the quantum length d_L of a two-"quantum point" state $\tilde{\omega} \otimes \omega$, defined as

$$d_L(\tilde{\omega}, \omega) \doteq (\tilde{\omega} \otimes \omega)(L). \tag{1.9}$$

We also recall the results on the spectral distance d_D for various classes of states of the Moyal algebra, including the eigenstates of the Hamiltonian of the quantum harmonic oscillator [8] and the coherent states [28]. Section 6 presents the strategy developed in [27] in order to compare the quantum length with the spectral distance, despite an obvious discrepancy (the latter vanishes between a state and itself, the former does not as soon as $l_P \neq 0$). This idea is to compare the quantum length with the spectral distance on a double Moyal plane, that is the product of the Moyal plane by $\mathbb{C}^2$. By a Pythagoras theorem for the product of spectral triples, we show that this comparison is equivalent to compare the spectral distance on a single Moyal plane with a new quantity d'_L build from the length operator, called the *modified quantum length*. The comparison of the spectral distance d_D with the modified quantum length d'_L is the object of section 7. This is mainly the analysis developed in [27], with further clarifications on the notion of discrete geodesics (between eigenstates of the harmonic oscillator) vs. continuous geodesics (between coherent states) in the Moyal plane.

2. Commutative case

In this section, we recall how to retrieve the Euclidean distance from the commutative coordinates x_μ, either as the mean value of the length operator L on the state $\delta_x \otimes \delta_y$, or as the spectral distance $d_{\mathcal{D}}(\delta_x, \delta_y)$. We begin with the algebraic characterization of the notion of "points", and close with a discussion on how the geodesic curves emerge in this picture.

2.1 Points as characters

By Gelfand theorem, the points of a locally compact topological space $\mathscr{X}$ are retrieved as the characters of the commutative algebra $C_0(\mathscr{X})$ of continuous functions vanishing at infinity. Recall that a character is an algebra morphism between $C_0(\mathscr{X})$ and $\mathbb{C}$. Gelfand theorem simply means that rather than viewing a point $x \in \mathscr{X}$ as being acted upon by a function $f \in C_0(\mathscr{X})$ in order to give a number $f(x)$, a point can be equivalently viewed as the object that acts on a function f to yields a number $\delta_x(f)$. This point of view is more compatible with quantum mechanics: classical physics assumes that space is the object that comes first, and functions acts on space to give number; in quantum mechanics there is no a priori given space, and observables come first.

Let us make some mathematical comments, in order to prepare the generalization to the non-commutative setting carried out in section 4. $C_0(\mathscr{X})$ is a C^*-algebra, whose elements have norm

$$\|f\| \doteq \sup_{x \in \mathscr{X}} |f(x)|. \tag{2.1}$$

Characters are linear functionals on $C_0(\mathscr{X})$ which are positive - i.e. $\delta_x(\bar{f}f) = \bar{f}(x)f(x) \in \mathbb{R}^+$ - and normalized - i.e. $\|\delta_x\| = 1$ - for the norm

$$\|\delta_x\| \doteq \sup_{f \in C_0(\mathscr{X})} \frac{|\delta_x(f)|}{\|f\|}. \tag{2.2}$$

Normalized positive linear functional on a C^*-algebra are called *states*. This is a generalization of the notion of states in quantum mechanics. Moreover, a general result of operator algebra guarantees that the space of state $\mathscr{S}(\mathscr{A})$ of any C^*-algebra $\mathscr{A}$ is convex, so that there exist extremum elements, that is states that cannot be written as a convex combination $\lambda\varphi_1 + (1-\lambda)\varphi_2$ of two other states φ_1, φ_2. Such states are called *pure*. We denote $\mathscr{P}(\mathscr{A})$ the space of pure states of $\mathscr{A}$. In the commutative case, characters are precisely the pure states of $C_0(\mathscr{X})$: Gelfand theorem then reads

$$\mathscr{P}(C_0(\mathscr{X})) \simeq \mathscr{X}. \tag{2.3}$$

2.2 Quantum length and spectral distance

Let us consider the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, with Cartesian coordinates $\{x_\mu\}_{\mu=1}^d$. Let q_μ denote the (unbounded, densely defined) selfadjoint coordinate operators whose action on $L^2(\mathbb{R}^d)$ reads

$$(q_\mu \psi)(x) \doteq x_\mu \psi(x). \tag{2.4}$$

The q_μ's do not belong to $C_0(\mathbb{R}^d)$ but are affiliated[2] to it in the sense of Woronowicz [40]. The space being classical is traced back in the vanishing of the commutator $[q_\mu, q_\nu]$.

Proposition 2.1. *On pure states of $C_0(\mathbb{R}^d)$, the spectral distance (1.8) associated to the spectral triple $(C_0^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d), \not{\partial})$ as well as the quantum length d_L introduced in (1.9) coincide with the Euclidean distance:*

$$d_{\not{\partial}}(\delta_x, \delta_y) = d_{Eucl}(x,y) = d_L(\delta_x, \delta_y) \quad \forall x, y \in \mathbb{R}^d. \tag{2.5}$$

The proof is standard and can be found e.g. in [27].

Notice that our definition of the length operator $L = \sqrt{\sum (dq_\mu)^2}$ heavily relies on the choice of the coordinate system: the dq_μ's are relevant only because the distance can be written as a function of the difference of the coordinates, that is on a flat space.

For the spectral distance, a more general result holds: viewing a state[3] φ of $C_0(\mathscr{M}) = \overline{C_0^\infty(\mathscr{M})}$ as a probability measure μ on $\mathscr{M}$,

$$\varphi(f) = \int_{\mathscr{M}} f d\mu, \tag{2.6}$$

then on a geodesically complete Riemannian manifold $\mathscr{M}$ the spectral distance $d_{\not{\partial}}$ between states coincides with the Wasserstein distance W of order 1 between probability distributions [35, 14]. The latter is defined in the theory of optimal transport as the supremum of $\varphi_1(f) - \varphi_2(f)$ on all 1-Lipschitz functions. The spectral distance being the same as the Wasserstein distance follows from noticing that for any $f \in C_0^\infty(\mathscr{M})$, then

$$\|[\not{\partial}, f]\| = \sup_{x \in \mathscr{M}} \|\text{grad } f\| \tag{2.7}$$

is precisely the Lipschitz norm of f. One then checks that any 1-Lipschitz function on $\mathscr{M}$ can be approximated by a sequence of smooth 1-Lipschitz functions vanishing at infinity. For instance, for pure states δ_x, δ_y, the supremum in the spectral distance formula is attained by the function

$$x \to d_{\text{geo}}(y,x), \tag{2.8}$$

that we approximate by the sequence

$$x \to \tilde{d}_{\text{geo}}(y,x)\, e^{-\frac{\tilde{d}_{\text{geo}}(y,x)}{n}} \tag{2.9}$$

of functions in $C_0^\infty(\mathscr{M})$, with $\tilde{d}_{\text{geo}}$ a smooth approximation of d_{geo}. Notice that $\mathscr{M}$ being complete is important to make $\tilde{d}_{\text{geo}}$ vanish at infinity.

[2] An element T is affiliated to a C^*-algebra $\mathscr{A}$ if bounded continuous functions of T belong to the multiplier algebra $M(\mathscr{A})$ of $\mathscr{A}$. In our context the unbounded operator q_μ's are affiliated to $C_0(\mathbb{R}^d)$, meaning that for any bounded continuous function f on $\mathbb{R}^d$, $f(q_\mu) \in M(C_0(\mathbb{R}^d)) = C_b(\mathbb{R}^d)$ where $C_b(\mathbb{R}^d)$ is the algebra of bounded continuous functions on $\mathbb{R}^d$.

[3] Strictly speaking, one should talk of "state" only for C^*-algebras. In case $\mathscr{A}$ is not C^*, we consider states of its C^*-closure $\bar{\mathscr{A}}$ with respect to the operator norm coming from the representation on $\mathscr{H}$. This is always possible, for from the axioms of spectral triples it follows that $\mathscr{A}$ is a pre-C^* algebra.

2.3 Geodesics

Although the length operator and the spectral distance are both an "algebraic version" of the usual distance formula, the way of addressing the problem is different:

- the definition (1.4) of the length operator supposes that the Euclidean distance

$$l(x_\mu) = \sqrt{\sum x_\mu^2} \tag{2.10}$$

is known a-priori;

- the spectral distance formula (1.8) can be seen as an equation whose solution is (a suitable approximation of) the distance function d_{geo}.

More generally, given any spectral triple $(\mathscr{A},\mathscr{H},D)$, we call *optimal element* between two states $\tilde{\varphi},\varphi$ an element of $\mathscr{A}$ that attains the supremum in (1.8), or the sequence of elements tending to this supremum in case the latter is not attained. It is far from unique: given two points x,y on a compact manifold, any function $d_{\text{geo}}(z,.)$ - with z a point on the geodesic between x and y not contained in the segment of curve between x and y - is an optimal element between δ_x and δ_y. On a non-compact complete manifold, an optimal element is the sequence (2.9). We then call $d_{\text{geo}}(z,.)$ an optimal element *up to regularization*.

On the Euclidean plane, the function l in (2.10) is an optimal element up to regularization between two points x and $y=\lambda x$, $\lambda\in\mathbb{R}^+$. Identifying $C(\mathbb{R}^d)$ with its representation on $L^2(\mathbb{R}^d)$, the optimal element up to regularization $l(q_\mu)$ and the length operator $L=l(dq_\mu)$ are the image of the same function l, under the functional calculus of either the coordinates q_μ, or their universal differential dq_μ. This formulation in terms of functional calculus may sounds artificially complicated in the commutative case, but it is helpful to understand the difference between the quantum length and the spectral distance in the Moyal plane, where precisely the picture is different: the function l_1 yielding the length operator no longer yields the optimal element (see section 7.1).

Notice also that, while a unique length operator L gives the distance between any $x,y\in\mathbb{R}^d$ as $(\delta_x\otimes\delta_y)(L)=|x-y|$, there is no function $\tilde{l}$ that would be an optimal element between any two points: $|\tilde{l}(x)-\tilde{l}(y)|$ cannot equal $|x-y|$ $\forall x,y\in\mathbb{R}^d$. Furthermore, if $y\neq\lambda x$ then none of the optimal element

$$l_z(x_\mu) \doteq \sqrt{\sum (x_\mu - z_\mu)^2} \tag{2.11}$$

where z is a fixed point such that $y-z=\lambda(x-z)$ yields a length operator, for

$$(\delta_x\otimes\delta_y)(l_z(dq_\mu)) = \sqrt{\sum (x_\mu - y_\mu - z_\mu)^2} \neq |x-y|. \tag{2.12}$$

3. The algebras of quantum spacetime

A "commutative" space $\mathscr{M}$ can be characterized algebraically as the space of pure states of the commutative algebra $C_0(\mathscr{M})$. Similarly, a quantum space may be defined as the set of pure states of a noncommutative algebra $\mathscr{A}$. To find out the correct algebra associated with the non-commuting coordinates operators q_μ's (1.3), one needs to specify the property of their commutators $Q_{\mu\nu}$. In the following, we make the assumption used in both the DFR model and θ-Minkowski, namely the $Q_{\mu\nu}$ are central:

$$[Q_{\mu\nu},q_\alpha]=0 \qquad \forall\alpha,\mu,\nu. \tag{3.1}$$

Doing so we exclude another model that gained interest in the recent time: κ-Minkowski. As a matter of fact, the metric aspect of κ-Minkowski space has been little studied, and is still an open problem, that we shall not address here.

3.1 The Moyal algebra

A pathway to determine a "natural" $\mathscr{B}(\mathscr{H})$ on which the coordinate operators q_μ's (1.3) act as unbounded operators comes from group theory. Assuming the $Q_{\mu\nu}$'s are central and the representation of the q_μ's is faithful and irreducible, by Schur lemma we write

$$Q_{\mu\nu} = \theta_{\mu\nu}\mathbb{I}, \tag{3.2}$$

where $\Theta \doteq \{\theta_{\mu\nu}\}$ is an antisymmetric matrix. We assume that Θ is non-degenerate, which forces the dimension $d = 2N$ to be even, and for $x, y \in \mathbb{R}^{2N}$ we denote

$$\sigma(x,y) \doteq x^\mu \theta_{\mu\nu} y^\nu \tag{3.3}$$

the symplectic form induced by Θ. Equation (1.3) then defines the Heisenberg Lie algebra of dimension $2N$ with central element[4]

$$c = i\lambda_P^2. \tag{3.4}$$

By exponentiation, one gets the Heisenberg group $H \doteq \mathbb{R}^{2N} \ltimes \mathbb{R}$ with group law

$$(x,\lambda).(x',\lambda') = (x+x', \lambda+\lambda'+\frac{1}{2}\sigma(x,x')) \quad \forall x,x' \in \mathbb{R}^{2N}, \lambda,\lambda' \in \mathbb{R}. \tag{3.5}$$

Now, to any locally compact group G one naturally associates the enveloping C^*-algebra $C^*(G)$, obtained by completing the Banach *-algebra $L^1(G)$ [5] with respect to the norm

$$\|f\|_* \doteq \sup_\pi \{\|\pi(f)\|\}, \tag{3.7}$$

where the supremum runs over all representations of $L^1(G)$. Therefore it is tempting to consider $C^*(H)$ as the natural C^*-algebra associated to the quantum space. However a careful examination of the representations of $L^1(H)$ indicates that $C^*(H)$ is too big:

- First of all in (3.7) it is reasonable to take into account irreducible representations only, since the q_μ's are known as soon as one has determined their irreducible action on $\mathscr{H}$.

[4] Recall that given a symplectic vector space (V,σ) of real dimension n, the Heisenberg algebra with central element c is the real central extension of the Lie algebra $\mathfrak{v}^n = \mathbb{R}^n$ of the additive group $\mathbb{R}^n$, characterized by the relations $[v,c] = 0, [v,v'] = \sigma(v,v')c$ for all $v,v' \in \mathbb{R}^n$. This is the Lie algebra of the Heisenberg group $H(V)$, namely the central extension $\mathbb{R} \ltimes V$ with group law similar to (3.5).

[5] i.e. the completion with respect to the the L^1-norm $\|f\|_1 \doteq \int_G |f(t)|dt$ - dt a Haar measure - of the algebra of compactly supported function on G, equipped with the convolution product

$$(f \star g)(t) \doteq \int_G f(s)g(t^{-1}s)ds \tag{3.6}$$

and involution $f^*(g) = \overline{f(g^{-1})}$.

- Then, recall that the (irreducible) non-degenerate representations of $L^1(H)$ are in 1-to-1 correspondence with the (irreducible) unitary representations $(\pi, \mathscr{H}_\pi)$ of H: to any $f \in L^1(H)$ corresponds the bounded operator $\pi(f)$ defined by the Bochner integral [16]

$$\pi(f) \doteq \int_{H_N} f(x,\lambda)\pi(x,\lambda)\, dx d\lambda \tag{3.8}$$

acting on $\mathscr{H}_\pi$ and any bounded non degenerate representation of $L^1(H)$ comes in this way [16].

- Irreducible unitary representations of H are of two kinds, depending on their central character χ[6]. Since $Z(H) \simeq \mathbb{R}$, any χ is of the form $\lambda \mapsto e^{it\lambda}$ for a fixed $t \in \mathbb{R}$. For $t = 0$, the constant function $\chi(\lambda) \doteq 1$ is the central character of an infinite number of nonequivalent 1-dimensional representations (see [16] for details). For $t \neq 0$ one gets the central character of the irreducible unitary representation on $L^2(\mathbb{R}^N)$ (unique up to equivalence, by von Neumann uniqueness theorem):

$$\pi(x,\lambda)\varphi(u) \doteq e^{it(x_2 u+\lambda)}\varphi(u+x_1) \qquad x = (x_1,x_2) \in \mathbb{R}^{2N}, \lambda \in \mathbb{R}. \tag{3.9}$$

Since $\pi(\exp c) = \pi_t(0,1) = e^{it}\mathbb{I}$ the value of t is fixed by (3.4) as $\doteq \lambda_P^2$.

Therefore, rather than (3.7) it is legitimate to consider the closure of $L^1(H)$ with respect to the single representation with central character λ_P^2. Let us denote this representation π_λ, with kernel J. As a Banach algebra, one gets

$$L^1(H)/J \simeq L^1(\mathbb{R}^{2N},\times) \tag{3.10}$$

where $\times$ is the *twisted convolution*

$$(f\times g)(x) \doteq \int_{\mathbb{R}^{2N}} f(x')g(x-x')\, e^{-\frac{i\lambda_P^2\sigma(x',x)}{2}}\, dz' \qquad \forall f,g \in L^1(\mathbb{R}^{2N}). \tag{3.11}$$

The norm closure of $\pi(L^1(\mathbb{R}^{2N})/J)$ is $C^*(L^1(\mathbb{R}^{2N},\times))$, which turns out to be isomorphic to the algebra of compact operators [19]

$$C^*(L^1(\mathbb{R}^{2N},\times)) \simeq \mathbb{K}. \tag{3.12}$$

To retrieve the algebra of compact operators from the traditional Moyal product $\star$, let us recall that the latter is obtained as the pull-back through the Fourier transform F of the twisted convolution (3.11),

$$f\star g \doteq F^{-1}\left[F[f]\times F[g]\right]. \tag{3.13}$$

To close an algebra, one may for instance restricts to Schwartz functions $f,g \in S(\mathbb{R}^{2N})$, for the twisted convolution, as the Fourier transform, maps Schwartz function into Schwartz functions. Writing

$$\theta \doteq \lambda_P^2, \tag{3.14}$$

standard Fourier theory yields the usual form of the Moyal product, that is

$$(f\star g)(x) = \left(\frac{1}{\pi\theta}\right)^{2N} \int_{\mathbb{R}^{2N}\times\mathbb{R}^{2N}} du\, dv\, f(x+u)g(x+v)e^{-\frac{2i}{\theta}u\Theta_0^{-1}v}, \tag{3.15}$$

[6] i.e. the homomorphism from the center $Z(H) = (0,\lambda)$ of H to S^1 defined by $\chi(a)\mathbb{I} \doteq \pi(a)$.

where

$$\Theta_0 = \begin{pmatrix} 0 & \mathbb{I}_N \\ -\mathbb{I}_N & 0 \end{pmatrix}. \tag{3.16}$$

Consequently, in the same way that $C_0(\mathscr{M})$ is the natural C^*-algebra associated to a manifold $\mathscr{M}$, the C^*-closure $\mathbb{K}$ of the Moyal algebra $\big(S(\mathbb{R}^{2N}),\star\big)$ [7] is the natural algebra associated to the quantum space (1.3) with central commutators. As well, in the same way that the commutative coordinates x_μ do not belong to $C_0(\mathbb{R}^d)$, the noncommutative coordinate operators q_μ do not belong to $\mathbb{K}$, but they are affiliated to it (see footnote p.7). This means that to find out the desired representation of the q_μ's on the Hilbert space $\mathscr{H}$ in equation (1.7), one needs to study the representations of $\mathbb{K}$. This is the object of the next paragraph.

Notice that the analysis developed in this section does not take into account the action of the Poincaré group (DFR model) or a deformed version of it (θ-Minkowski) on the quantum coordinates. We explain in appendix why this action does not play any role regarding the computation of distance and length.

3.2 The left regular and the Schrödinger representations

The natural *left regular* action $\mathscr{L}$ of the Moyal algebra $(S(\mathbb{R}^{2N}),\star)$ on $L^2(\mathbb{R}^{2N})$,

$$\mathscr{L}(f)\psi = f\star\psi \qquad \forall \psi \in L^2(\mathbb{R}^{2N}), \tag{3.17}$$

is not irreducible (see e.g. [8]). As recalled in the next section, to characterize the pure states of the Moyal algebra, it is convenient to have an irreducible representation. The latter is nothing but the usual Schrödinger representation π_S of quantum mechanics. Let us work it out explicitly, restricting ourselves to the Moyal plane $N=1$ in order to fix notations.

On $\mathscr{H}_S \doteq L^2(\mathbb{R})$, we denote the position and momentum operators,

$$\mathfrak{q} : (\mathfrak{q}\psi)(x) = x\psi(x), \quad \mathfrak{p} : (\mathfrak{p}\psi)(x) = -i\theta\partial_x\psi_{|x}, \quad \psi \in L^2(\mathbb{R}),\ x\in\mathbb{R}. \tag{3.18}$$

Let $|n\rangle$ denote the eigenfunctions of the Hamiltonian $\mathfrak{h} = \frac{1}{2}(\mathfrak{q}^2+\mathfrak{p}^2)$ of the quantum harmonic oscillator. They form an orthonormal basis of $L^2(\mathbb{R})$ and span an invariant dense domain $\mathscr{D}_S$ of analytic vectors for the operators $\mathfrak{q},\mathfrak{p}$. Let W denote the unitary operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R})\otimes L^2(\mathbb{R})$ defined as

$$W h_{mn} = |m\rangle\otimes|n\rangle \qquad m,n\in\mathbb{N} \tag{3.19}$$

where $\{h_{mn}, m,n\in\mathbb{N}\}$ is the basis of $L^2(\mathbb{R}^2)$ spanned by Wigner transition functions (see e.g. [7] for the explicit form of the h_{mn}'s). One has

$$W\mathscr{L}(x_1)W^* = \mathfrak{q}\otimes\mathbb{I}, \quad W\mathscr{L}(x_2)W^* = \mathfrak{p}\otimes\mathbb{I}. \tag{3.20}$$

[7] The C^*-completion of the Moyal algebra in the operator norm coming from the (non-irreducible) left regular representation introduced below is a multiple of the N-fold (C^*) tensor product of $\mathbb{K}$ with itself, which is isomorphic to $\mathbb{K}$.

As a consequence, for $f \in S(\mathbb{R}^2)$,

$$W\mathscr{L}(f)W^* = \pi_S(f)\otimes\mathbb{I} \tag{3.21}$$

where π_S is the Schrödinger representation (or the Weyl prescription), namely

$$\pi_S(f) \doteq \int \hat{f}(k_1,k_2)e^{\frac{i}{\theta}(\mathfrak{q}k_1+\mathfrak{p}k_2)}dk_1dk_2. \tag{3.22}$$

Proposition 3.1. $\overline{\pi_S(S(\mathbb{R}^2))} \simeq \mathscr{K}(L^2(\mathbb{R}))$ *is an irreducible representation of* $\mathbb{K}$.

Here $\mathscr{K}(\mathscr{H})$ denotes the set of compact operators on a Hilbert space $\mathscr{H}$. This last proposition is a standard result, whose prove is recalled for instance in [28].

3.3 Quantum coordinates

To summarize, the abstract quantum coordinates operators q_1, q_2, once viewed as operators affiliated to the algebra of compact operators $\mathbb{K}$, have two two natural representations:

- a reducible one $\mathscr{L}(x_1), \mathscr{L}(x_2)$ on $L^2(\mathbb{R}^2)$,

- an irreducible one, the Schrödinger representation, $\pi_S(x_1) = \mathfrak{q},\ \pi_S(x_2) = \mathfrak{p}$ on $L^2(\mathbb{R})$.

In complex coordinates,

$$z = \frac{x_1+ix_2}{\sqrt{2}}, \qquad \bar{z} = \frac{x_1+ix_2}{\sqrt{2}}, \tag{3.23}$$

one similarly has

$$W\mathscr{L}(\bar{z})W^* = \mathfrak{a}^*\otimes\mathbb{I}, \quad W\mathscr{L}(z)W^* = \mathfrak{a}\otimes\mathbb{I} \tag{3.24}$$

where

$$\mathfrak{a} \doteq \frac{1}{\sqrt{2}}(\mathfrak{q}+i\mathfrak{p}), \qquad \mathfrak{a}^* \doteq \frac{1}{\sqrt{2}}(\mathfrak{q}-i\mathfrak{p}), \tag{3.25}$$

are the creation/annihilation operators, satisfying

$$[\mathfrak{a},\mathfrak{a}^*] = \lambda_P^2\mathbb{I}. \tag{3.26}$$

So the abstract complex quantum coordinates

$$a \doteq \frac{1}{\sqrt{2}}(q_1+iq_2), \quad a^* = \frac{1}{\sqrt{2}}(q_1-iq_2) \tag{3.27}$$

have representation $\mathscr{L}(z)$, $\mathscr{L}(\bar{z})$ on $L^2(\mathbb{R}^2)$ and $\pi_S(z) = \mathfrak{a}$, $\pi_S(\bar{z}) = \mathfrak{a}^*$ on $L^2(\mathbb{R})$.

4. Quantum points

Having shown that the algebra of compact operators $\mathbb{K} = \overline{(S(\mathbb{R}),\star))}$ plays for a quantum space the role of the algebra $C_0(\mathscr{M})$ for a Riemannian manifold, and remembering that a point $x \in \mathscr{M}$ is nothing but the pure state $\delta_x \in \mathscr{P}(C_0(\mathscr{M}))$, we take as "quantum points" the pure states of $\mathbb{K}$.

One immediately gets that a usual point is not a quantum point. Indeed in the noncommutative case δ_x is not longer a pure state, not even a state, for

$$\delta_x(f^* \star f) = (f^* \star f)(x) \tag{4.1}$$

has no reason to be positive (unlike the commutative case where $\delta_x(f^*.f) = \bar{f}(x)f(x) = |f(x)|^2 \in \mathbb{R}^+$). Said differently, by going to the noncommutative framework one replaces the pointwise product by the non-local Moyal product. By this we mean that the evaluation of $f \star g$ at a point x involves the values of f and g not only at x, but on all $\mathbb{R}^2$. This is what forbids δ_x to be a positive linear form. In this precise mathematical sense, "points become fuzzy" in a quantum space.

To determine "how much fuzzy", one needs to work out explicitly the pure state space $\mathscr{P}(\mathbb{K})$.

4.1 Quantum points as pure states

Having single out the Moyal algebra, or equivalently its C^*-closure $\mathbb{K}$, as the relevant algebra to describe the quantum space, it becomes very easy to explain our claim in section 2.1, that a "state" defined as a positive normalized linear form on an algebra is the generalization of the "state" as it appears in quantum mechanics. The latter usually denotes the state vector (the ket) $|\psi\rangle$ in some Hilbert space $\mathscr{H}$, describing a quantum state of the physical system under studies. The state, in the sense of a positive application, is simply the corresponding mean value of observables, namely

$$\omega_\psi(a) \doteq \langle\psi|a|\psi\rangle. \tag{4.2}$$

So to any *state vector* ψ of quantum mechanics corresponds a *vector state* ω_ψ in the sense of C^*-algebra (linearity and positivity of ω_ψ is obvious, normalization comes from $\||\psi\rangle\| = 1$, required by the probabilistic interpretation of quantum mechanics). To a mixed state in quantum mechanics, characterized by a density matrix $\rho = \sum_j p_j|j\rangle\langle j|$ in $\mathscr{B}(\mathscr{H})$, corresponds the state

$$\omega_\rho(a) \doteq \mathrm{Tr}(\rho a) \tag{4.3}$$

in the sense of C^*-algebra. Furthermore, ω_ρ is non pure whereas ω_ψ - viewed as states of the C^*-algebra $\mathscr{B}(\mathscr{H})$ - is pure. Viewed as a state of a C^*-sub algebra of $\mathscr{B}(\mathscr{H})$, ω_ψ may be pure or not, depending whether $\mathscr{B}$ acts irreducibly on $\mathscr{H}$.

However it is not true that to any state of a C^*-algebra corresponds a state vector in $\mathscr{H}$ or a density matrix in $\mathscr{B}(\mathscr{H})$. More exactly, given a C^*-algebra $\mathscr{A}$ and a state φ, one can always build a representation π_φ of $\mathscr{A}$ on a Hilbert space $\mathscr{H}_\varphi$ so that that there exists a vector $|\varphi\rangle \in \mathscr{H}_\varphi$ such that $\varphi(a) = \langle\varphi|\pi_\varphi(a)|\varphi\rangle$ (this is the GNS construction). But most often one cannot build a unique representation π on some Hilbert space $\mathscr{H}$ such that any state φ comes either as a vector state or as a density matrix.[8] However this happens to be true for $\mathscr{A} = \mathbb{K}$.

Indeed, it is a classical result of operator algebra that any non-degenerate representation[9] of $\mathbb{K}$ is unitary equivalent to a multiple of the unique (up to unitary equivalence) irreducible representation π_S of $\mathbb{K}$ on $\mathscr{H}_S$ (the index S is for Schrödinger, see below). So limiting ourselves to irreducible

[8] See for instance the commutative case: the pure state δ_x is not a vector state in the representation of $C_0(\mathbb{R})$ on $\mathscr{H} = L^2(\mathbb{R})$, for there is no $\psi \in \mathscr{H}$ such that $f(x) = \langle\psi, f\psi\rangle$. Each pure state δ_x however is indeed a vector state in the 1-dimensional representation $\pi_x(f) \doteq f(x)$, that is $\delta_x(f) = \langle 1, f1\rangle$ where the scalar product on $\mathscr{H}_x = \mathbb{C}$ is simply the multiplication of complex numbers. But for $x \neq y$, the pure state δ_y is not a vector state in the π_x representation: there does not exist a real number $\psi \in \mathscr{H}_x$ such that $f(y) = \langle\psi, f\psi\rangle$ for any $f \in C_0(\mathbb{R})$.

[9] That is $\pi(a)\psi = 0\ \forall a \in \mathbb{K} \Longrightarrow a = 0$.

representations, we can identify the abstract C^*-algebra $\mathbb{K}$ with the algebra of compact operators on $\mathscr{H}_S$, that is $\pi_S(\mathbb{K}) = \mathscr{K}(\mathscr{H}_S)$ (see [6, IV.1.2]). Now, any state φ in $\mathscr{S}(\mathbb{K})$ extends to a *normal* state on $\mathscr{B}(\mathscr{H}_S)$, still denoted φ. "Normal" means that there exists a trace-class operator $s_\varphi \in \mathscr{B}(\mathscr{H}_S)$ such that

$$\varphi(a) = \mathrm{Tr}(s_\varphi a) \quad \forall a \in \mathscr{B}(\mathscr{H}_S). \tag{4.4}$$

If s_φ has rank one, $\mathrm{rank}(s_\varphi) = \{|\psi\rangle \in \mathscr{H}_S\}$, then

$$\varphi(a) = \mathrm{Tr}(s_\varphi a) = \langle \psi, a\psi \rangle \tag{4.5}$$

so that φ is pure. We summarize these remarks in the following proposition, using as well proposition 3.1.

Proposition 4.1. *Any pure state ω of the algebra $\mathbb{K}$ of quantum space-time is a vector state in the irreducible representation π_S on $\mathscr{H}_S$, that is there exists a state vector $\psi \in L^2(\mathbb{R})$ such that*

$$\omega(f) = \langle \psi, \pi_S(f)\psi \rangle \quad \forall f \in S(\mathbb{R}^2). \tag{4.6}$$

4.2 Generalized coherent states

Among the vector states (4.6), we will pay attention to two particular classes. The first one are those states given by a vector $\psi \in L^2(\mathbb{R})$ whose decomposition on the basis $\{|n\rangle, n \in \mathbb{N}\}$ has only one non-zero component, namely the eigenstates of the quantum harmonic oscillator

$$\omega_m(f) \doteq \langle m|\pi_S(f)|m\rangle. \tag{4.7}$$

Another class of interesting states are the coherent states. Recall [10] that a coherent (or semi-classical) state of the quantum harmonic oscillator is a quantum state that reproduces the behaviour of a classical harmonic oscillator. The movement of such an oscillator with given mass m and angular velocity ω is fully characterised by one complex number $\kappa = |\kappa| e^{i\Xi}$ giving the amplitude of oscillation $|\kappa|$ and the phase Ξ. The same is true for a quantum coherent state (see e.g. [28]).

Definition 4.2. *A coherent state of the Moyal algebra $\mathscr{A}$ is a linear form*

$$\omega^c_\kappa(f) \doteq \langle \kappa, \pi_S(f)\kappa \rangle \quad \forall f \in \mathscr{A} \tag{4.8}$$

where $|\kappa\rangle \in L^2(\mathbb{R})$, $\|\kappa\|_{L^2(\mathbb{R})} = 1$, is a solution of

$$\mathfrak{a}|\kappa\rangle = \lambda_P \kappa |\kappa\rangle \quad \kappa \in \mathbb{C}. \tag{4.9}$$

The development of a coherent states on the basis of eigenstates is

$$|\kappa\rangle = \sum_{m\in\mathbb{N}} c^\kappa_m \varphi_m, \quad c^\kappa_m = e^{-\frac{|\kappa|^2}{2}} \frac{\kappa^m}{\sqrt{m!}}. \tag{4.10}$$

The exists another characterization of a coherent state, in terms of translation.

Given $\kappa \in \mathbb{R}^2 \simeq \mathbb{C}$, we denote $\alpha_\kappa f$ the translated of $f \in S(\mathbb{R}^2)$, that is $(\alpha_\kappa f)(z) = f(z+\kappa)$, and $\alpha_\kappa \varphi$ the κ-translated of a state φ, that is.

$$(\alpha_\kappa \varphi)(f) \doteq \varphi(\alpha_\kappa f). \tag{4.11}$$

Proposition 4.3. [see e.g. [28]] *The coherent state ω^c_κ is the translated of the ground state of the quantum harmonic oscillator, with translation $\sqrt{2}\lambda_P\kappa$. That is to say*

$$\omega^c_\kappa(f) = \alpha_{\sqrt{2}\lambda_P\kappa}\omega_0(f). \tag{4.12}$$

The coherent states are particularly important for the DFR model since they are the states of optimal localization, that is those which minimize the uncertainty (1.2) in the measurement of the coordinates [19].

In the following, we will consider a larger classes of states.

Definition 4.4. *We call a* generalized coherent state *any element in $\mathscr{P}(\mathbb{K})$ obtained by translation of an eigenstate of the Hamiltonian of the quantum harmonic oscillator. The set of all generalized coherent states is*

$$\mathscr{C} \doteq \bigcup_{m\in\mathbb{N}} \mathscr{C}(\omega_m) \quad \text{where} \quad \mathscr{C}(\omega_m) \doteq \left\{\alpha_\kappa\,\omega_m,\ \kappa\in\mathbb{R}^2\right\}. \tag{4.13}$$

5. Quantum length and spectral distance in the Moyal plane

In this section, we list the results on the quantum length and the spectral distance between generalized coherent states obtained in [27] for the former, in [8] and [28] for the latter.

5.1 Quantum length

It is easier to compute the square root of a mean value, that the mean value of a square root. So, rather than the quantum length d_L introduced in (1.9) we will consider the square-root of the quantum square-length

$$d_{L^2}(\omega,\tilde{\omega}) \doteq (\omega\otimes\tilde{\omega})(L^2). \tag{5.1}$$

Thanks to the similarities between the length operator and the Hamiltonian of the quantum harmonic oscillator, d_{L^2} is not difficult to calculate.

Proposition 5.1. [27, 4] *The quantum square-length on the set $\mathscr{C}$ of generalized coherent states introduced in definition 4.4 is*

$$d_{L^2}(\alpha_\kappa\omega_m,\alpha_{\tilde{\kappa}}\omega_n) = 2E_m + 2E_n + |\kappa-\tilde{\kappa}|^2 \tag{5.2}$$

for any $m,n\in\mathbb{N},\ \kappa,\tilde{\kappa}\in\mathbb{R}^2$, with

$$E_m = \lambda_P^2(m+\frac{1}{2}). \tag{5.3}$$

the n^{th} eigenvalue of the Hamiltonian H of the quantum harmonic oscillator. Hence the quantum square length is invariant by translation. Moreover one has

$$d_L(\alpha_\kappa\omega_m,\alpha_{\tilde{\kappa}}\omega_n) \leq \sqrt{d_{L^2}(\alpha_\kappa\omega_m,\alpha_{\tilde{\kappa}}\omega_n)} \tag{5.4}$$

with equality only when $m=n=0$ and $\kappa=\tilde{\kappa}$, that is

$$d_L(\alpha_\kappa\omega_0,\alpha_\kappa\omega_0) = 2\sqrt{E_0} = \sqrt{d_{L^2}(\alpha_\kappa\omega_0,\alpha_\kappa\omega_0)}. \tag{5.5}$$

5.2 Spectral distance in the Moyal plane

The known results on the spectral distance in the Moyal plane are summarized in the following proposition.

Proposition 5.2. *1.[28] The spectral distance between any state $\varphi \in \mathscr{S}(\mathscr{A})$ of the Moyal algebra and any of its κ-translated, $\kappa \in \mathbb{C}$, is precisely the amplitude of translation*

$$d_D(\varphi, \alpha_\kappa \varphi) = |\kappa|.$$

2. [8, 28] The spectral distance on the Moyal plane takes all possible value in $[0, \infty]$.

3. [8] The distance between eigenstates ω_m is additive:

$$d_D(\omega_m, \omega_n) = \frac{\lambda_P}{\sqrt{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

We stress that there is no misprint at at point 2: the closing right bracket indicates that there exist states at infinite distance from one another. A first example of such states have been exhibited in [8]; other classes of such states have been worked out in [9, Prop. 7]. This is a crucial difference with the $\lambda_P = 0$ commutative limit: both the Moyal plane and $\mathbb{R}^2$ have infinite diameter (i.e. one can find points/pure states at arbitrarily large distance from one another), but on $\mathbb{R}^2$ any two points are at finite distance from one another.

This also has interesting consequence on the topology of the state space: the latter is not connected for the metric topology, while it is connected in the weak* topology (that coincides with the topology induced by the trace-norm, see [8]). In other terms, the topology induced by the spectral distance is not the weak* topology, meaning that the (minimal unitization) of the Moyal plane is not a (compact) *quantum metric space* in the sense of Rieffel [34].

6. Minimal length and spectral doubling

Propositions 5.1 and 5.2 stress the obvious discrepancy between the quantum length and the spectral distance:

$$\begin{aligned} d_D(\omega_m, \omega_n) = \frac{\lambda_P}{\sqrt{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} \quad &\neq \quad \sqrt{d_{L^2}(\omega_m, \omega_n)} = \sqrt{2E_m + 2E_n}, \\ d_D(\omega_m, \alpha_\kappa \omega_m) = |\kappa| \quad &\neq \quad \sqrt{d_{L^2}(\omega_m, \alpha_\kappa \omega_m)} = \sqrt{4E_m + |\kappa|^2}. \end{aligned}$$

So it seems at first sight that the two quantities do not capture the same metric information on a quantum space, and that it makes little sense to compare them. There is indeed a fundamental difference: the quantum square-length d_{L^2} is not a distance, for it does not vanish on the diagonal (i.e. when the two arguments are equal); unlike the spectral distance which is a true distance function in the mathematical sense. We show below how to solve this obvious discrepancy by

turning the quantum length into a true distance,
or by
giving a "quantum mechanics" flavor to the spectral distance.

The two point of view turn out to be equivalent thanks to the occurrence of a Pythagoras theorem in noncommutative geometry. The result is obtained thanks to a standard procedure in noncommutative geometry, consisting in doubling the spectral triple.

6.1 Spectral doubling

Doubling a spectral triple $(\mathscr{A},\mathscr{H},D)$ consists in taking its product with the standard spectral triple on $\mathbb{C}^2$ in order to obtain the new spectral triple

$$\mathscr{A}' \doteq \mathscr{A}\otimes\mathbb{C}^2, \quad \mathscr{H}' \doteq \mathscr{H}\otimes\mathbb{C}^2, \quad D' \doteq D\otimes\mathbb{I}+\Gamma\otimes D_I$$

where Γ is a grading of $\mathscr{H}$ and

$$D_I \doteq \begin{pmatrix} 0 & \bar{\Lambda} \\ \Lambda & 0 \end{pmatrix} \text{ with } \Lambda = \text{ const.} \tag{6.1}$$

Pure states of $\mathscr{A}'$ are pairs $\omega^i \doteq (\omega,\delta_i)$ where ω is a pure state of $\mathscr{A}$ and $\delta_{i=1,2}$ are the pure states of $\mathbb{C}^2$

$$\delta_1(z_1,z_2)=z_1, \quad \delta_2(z_1,z_2)=z_2 \quad \forall (z_1,z_2)\in\mathbb{C}^2. \tag{6.2}$$

Hence

$$\mathscr{P}(\mathscr{A}')\simeq\mathscr{P}(\mathscr{A})\times\mathscr{P}(\mathscr{A}) \tag{6.3}$$

and the geometry described by the doubled spectral triple $(\mathscr{A}',\mathscr{H}',D')$ is a *two-sheet model*, with associated distance $d_{D'}$.

The projection of $d_{D'}$ on each sheet gives back the distance d_D on a single sheet,

$$d_{D'}(\omega^i,\tilde{\omega}^i)=d_D(\omega,\tilde{\omega}), \tag{6.4}$$

while the distance between the sheets is constant and non-zero [29, 15],

$$d_{D'}(\omega^i,\omega^j)=d_{D_I}(\delta^i,\delta^j)=\frac{1}{|\Lambda|}. \tag{6.5}$$

The idea is to use this constant $|\Lambda|^{-1}$ to implement the notion of minimal length within the spectral distance framework. Namely, rather than comparing the quantum length with the spectral distance on a single sheet, one postulates that the quantum square-length has to be compared with the spectral distance in the double-sheeted model of quantum space-time. In other terms one aims at identifying

$$d_{L^2}(\omega,\tilde{\omega}) \quad \text{with} \quad d^2_{D'}(\omega^1,\tilde{\omega}^2). \tag{6.6}$$

To do so, the free parameter Λ is fixed as

$$|\Lambda|^{-2}=d_{L^2}(\omega,\omega)$$

for some reference state ω, so that $d^2_{D'}(\omega^1,\omega^2)=d_{L^2}(\omega,\omega)$. The point is then to check whether the identification

$$d^2_{D'} \longleftrightarrow d_{L^2} \tag{6.7}$$

holds true for other states. Obviously this has chance to be true only for those states $\tilde{\omega}$ belonging to

$$\mathscr{P}(\omega) \doteq \{\omega \in \mathscr{P}(\mathscr{A}),\, d_{L^2}(\tilde{\omega},\tilde{\omega}) = d_{L^2}(\omega,\omega)\}\,. \tag{6.8}$$

Luckily, by proposition 5.1 one has that the translated of any eigenstates ω_m satisfy the required conditions

$$\mathscr{C}(\omega_m) \subset \mathscr{P}(\omega_m). \tag{6.9}$$

Furthermore, the distance $d_{D'}$ between two states $\omega, \tilde{\omega} \in \mathscr{C}(\omega)$ localized on different sheets is known, and given by Pythagoras theorem.

Proposition 6.1. *[28] The product of the Moyal plane by $\mathbb{C}^2$ is orthogonal in the sense of Pythagoras theorem, restricted to a set of generalized coherent states $\mathscr{C}(\omega) = \{\alpha_\kappa\omega, \kappa \in \mathbb{C}\}$:*

$$d^2_{D'}(\omega^1,\tilde{\omega}^2) = d^2_D(\omega,\tilde{\omega}) + d^2_{D_I}(\delta^1,\delta^2) \tag{6.10}$$

for any $\omega, \tilde{\omega} \in \mathscr{C}(\omega)$.

Therefore, identifying in the double Moyal space $d^2_{D'}(\omega^1_m,\tilde{\omega}^2_n)$ with $d_{L^2}(\omega_m,\tilde{\omega}_n)$ amounts to identifying on a single sheet $d_D(\omega_m,\tilde{\omega}_n)$ with

$$d'_L(\omega_m,\tilde{\omega}_n) = \sqrt{d_{L^2}(\omega_m,\tilde{\omega}_n) - |\Lambda|^{-2}}. \tag{6.11}$$

Remark 6.2. *The generalization of Pythagoras theorem to the product of arbitrary spectral triples has been investigated in [15]. One finds that for the product of arbitrary unital spectral triples $(\mathscr{A}_1,\mathscr{H}_1,D_1)$, $(\mathscr{A}_2,\mathscr{H}_2,D_2)$, the following Pythagoras inequalities hold:*

$$d^2_D(\varphi_1,\tilde{\varphi}_1) + d^2_{D_I}(\varphi_2,\tilde{\varphi}_2) \leq d^2_{D'}(\varphi_1\otimes\varphi_2,\tilde{\varphi}_1\otimes\tilde{\varphi}_2) \leq 2\left(d^2_D(\varphi_1,\tilde{\varphi}_1) + d^2_{D_I}(\varphi_2,\tilde{\varphi}_2)\right). \tag{6.12}$$

for any $\varphi_1,\tilde{\varphi}_1 \in \mathscr{S}(\mathscr{A}_1), \varphi_2,\tilde{\varphi}_2 \in \mathscr{S}(\mathscr{A}_2)$.

6.2 Modified quantum length

In the analysis above, the free parameter Λ has been fixed once for all by the choice of the reference state ω. If one had started with a reference state $\omega_0 \notin \mathscr{P}(\omega)$, one would have obtained a result similar as eq. (6.11) for any state $\tilde{\omega}_0 \in \mathscr{C}(\omega_0)$. In order to collect the results for all possible choices of reference states into a single formula, it is convenient to introduce the *modified quantum length*

$$d'_L(\omega,\tilde{\omega}) \doteq \sqrt{|d_{L^2}(\omega,\tilde{\omega}) - \Lambda^{-2}(\omega,\tilde{\omega})|} \tag{6.13}$$

where

$$\Lambda^{-2}(\omega,\tilde{\omega}) = \sqrt{d_{L^2}(\omega,\omega)d_{L^2}(\tilde{\omega},\tilde{\omega})}. \tag{6.14}$$

In case $\tilde{\omega} \in \mathscr{P}(\omega)$, eq. (6.13) gives back (6.11).

The modified quantum length is the correct quantity, build from the length operator L, that should be compared with the spectral distance.

Proposition 6.3. *[27] On a set of generalized coherent states, $\mathscr{C}(\omega_m) = \{\alpha_\kappa \omega_m, \kappa \in \mathbb{C}\}$, for instance the states of optimal localization $\mathscr{C}(\omega_0)$, the identification between the spectral distance and the quantum length holds true both in the two-sheet model,*

$$d_{L^2}(\omega, \tilde{\omega}) = d^2_{D'}(\omega^1, \tilde{\omega}^2) \qquad \forall \omega, \tilde{\omega} \in \mathscr{C}(\omega_m), \tag{6.15}$$

and on a single sheet (see figure 1)

$$d_D(\omega, \tilde{\omega}) = d'_L(\omega, \tilde{\omega}) \qquad \forall \omega, \tilde{\omega} \in \mathscr{C}(\omega_m). \tag{6.16}$$

On the set of all generalized coherent states, d_D coincides with d'_L asymptotically, both in the limit of large translation

$$\lim_{\kappa \to \infty} \frac{d_D(\alpha_\kappa \omega_m, \alpha_{\tilde{\kappa}} \omega_n) - d'_L(\alpha_\kappa \omega_m, \alpha_{\tilde{\kappa}} \omega_n)}{d'_L(\alpha_\kappa \omega_m, \alpha_{\tilde{\kappa}} \omega_n)} = 0, \quad \forall m, n \in \mathbb{N},\ \tilde{\kappa} \in \mathbb{C}, \tag{6.17}$$

and for large difference of energy

$$\lim_{n \to 0} \frac{d_D(\alpha_\kappa \omega_m, \alpha_{\tilde{\kappa}} \omega_n) - d'_L(\alpha_\kappa \omega_m, \alpha_{\tilde{\kappa}} \omega_n)}{d'_L(\alpha_\kappa \omega_m, \alpha_{\tilde{\kappa}} \omega_n)} = 0, \quad \forall m \in \mathbb{N},\ \kappa, \tilde{\kappa} \in \mathbb{C}. \tag{6.18}$$

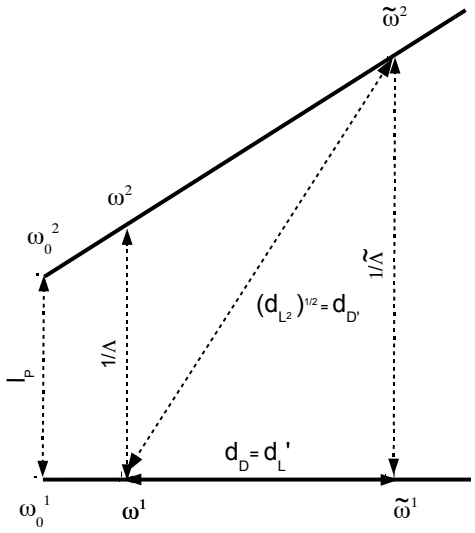

Figure 1:
The spectral distance in the double Moyal space and the (modified) quantum length: here ω is in $\mathscr{P}(\omega_m)$ and $\tilde{\omega}$ in $\mathscr{P}(\omega_n)$ for non-zero distinct integers m, n.

Eq. (6.18, 6.17) are analogous to (6.16), in that they indicate a relation between two quantities on a single copy of the Moyal plane. There is no analogous to (6.15), namely the identification of the spectral distance with the modified quantum length has no equivalent in the double Moyal space. This is because for $m \neq n$, one has $\omega_m \notin \mathscr{P}(\omega_n)$ so that to any two sets of generalized coherent states $\mathscr{C}(\omega_m), \mathscr{C}(\omega_n)$ correspond two Dirac operators with distinct free parameters

$$\Lambda = d_{L^2}^{-\frac{1}{2}}(\omega_m, \omega_m) \quad \text{or} \quad \tilde{\Lambda} = d_{L^2}^{-\frac{1}{2}}(\omega_n, \omega_n). \tag{6.19}$$

In the almost-commutative case, that is the doubling of the standard spectral triple associated to a manifold, this various Dirac operators are collected into a single operator with non-constant free parameter $\Lambda(x)$. The latter is interpreted as a Higgs field [13, 29]. At the moment it is not clear whether a similar procedure can be performed in the Moyal plane.

6.3 Turning the quantum length into a true distance

The spectral doubling, consisting in viewing a pair of states $(\tilde{\omega},\omega)$ in $\mathscr{P}(\mathbb{K})$ as living on two distinct sheets, gives a sense to the notion of minimal spectral distance. From this perspective, it furnishes the "quantum taste" to the spectral distance that we mention at the beginning of this section. One could also start from the other assumption, namely "turning the quantum length into a true distance". The natural way to do so is to make the minimum of the spectrum of the length operator zero, by defining a modified length operator

$$L' \doteq L - l_p \mathbb{I}\otimes\mathbb{I} \tag{6.20}$$

with $l_P = d_{L^2}(\omega_0,\omega_0)$ is defined in (1.5). But this guarantees that

$$d_{L'}(\omega,\omega) \doteq (\omega\otimes\omega)(L') = 0 \tag{6.21}$$

only for $\omega \in \mathscr{P}(\omega_0)$. In order to make (6.21) true for all $\omega\in\mathscr{P}(\mathbb{K})$, one is led quite naturally to the definition (6.13) of the modified quantum length d'_L.

However there is no selfadjoint operator L'^2 such that

$$d'^2_L(\tilde{\omega},\omega) \quad \text{would equal} \quad (\tilde{\omega}\otimes\omega)(L'^2). \tag{6.22}$$

For non pure states $\varphi,\tilde{\varphi}$ this is obvious since $(\tilde{\varphi}\otimes\varphi)(L'^2)$ is linear in φ and $\tilde{\varphi}$ whereas $d'^2_L(\tilde{\varphi},\varphi)$ is not. For pure states, one can check that L'^2 does not exist by writing a condition, relying on the linearity of $(\omega\otimes\tilde{\omega})(L'^2)$, that cannot be satisfied by our definition (6.13) of d'_L. Let us do it explicitly. For any $i,j\in\mathbb{N}$, we write $\omega_{ij} \doteq \omega_{\frac{1}{\sqrt{2}}(|i\rangle+|j\rangle)}$ and $|ij\rangle \doteq |i\rangle\otimes|j\rangle$, $i,j\in\mathbb{N}$. By easy computations one gets

$$(\omega_{ij}\otimes\omega_l)(L'^2) = \mathrm{Re}\,\langle jl|L'^2|il\rangle + \frac{1}{2}(\omega_i\otimes\omega_l)(L'^2) + \frac{1}{2}(\omega_j\otimes\omega_l)(L'^2) \tag{6.23}$$

so that, for $\omega_{ijk} \doteq \omega_{\frac{1}{\sqrt{3}}(|i\rangle+|j\rangle+|k\rangle)}$,

$$(\omega_{ijk}\otimes\omega_l)(L'^2) = \frac{1}{3}\left(2\omega_{ij}\otimes\omega_l + 2\omega_{ik}\otimes\omega_l + 2\omega_{jk}\otimes\omega_l - \omega_i\otimes\omega_l - \omega_j\otimes\omega_l - \omega_k\otimes\omega_l\right)(L'^2).$$

Therefore, condition (6.22) would imply

$$\begin{aligned} 3d'^2_L(\omega_{ijk},\omega_l) = {} & 2d'^2_L(\omega_{ij},\omega_l) + 2d'^2_L(\omega_{ik},\omega_l) + 2d'^2_L(\omega_{jk},\omega_l) \\ & - d'^2_L(\omega_i,\omega_l) - d'^2_L(\omega_j,\omega_l) - d'^2_L(\omega_k,\omega_l) \end{aligned} \tag{6.24}$$

But, noticing that

$$L^2 = 2(H\otimes\mathbb{I} + \mathbb{I}\otimes H - a\otimes a^* - a^*\otimes a) \tag{6.25}$$

where a has been defined in section (3.27) and $H = \frac{1}{2}(q_1^2 + q_2^2)$ is the Hamiltonian of the harmonic oscillator, one gets for i, j, k, l four integers whose differences are greater than one in absolute value,

$$d_{L^2}(\omega_i, \omega_j) = 2E_i + 2E_j, \qquad d_{L^2}(\omega_{ij}, \omega_{kl}) = E_i + E_j + E_k + E_l, \tag{6.26}$$

$$d_{L^2}(\omega_{ijk}, \omega_l) = 2E_l + \frac{2}{3}(E_i + E_j + E_k), \qquad d_{L^2}(\omega_{ijk}, \omega_{ijk}) = \frac{4}{3}(E_i + E_j + E_k) \tag{6.27}$$

with E_m defined in (5.3). Then by (6.13)

$$d'^2_L(\omega_i, \omega_l) = (\sqrt{2E_i} - \sqrt{2E_l})^2, \quad d'^2_L(\omega_{ij}, \omega_l) = (\sqrt{E_i + E_j} - \sqrt{2E_l})^2, \tag{6.28}$$

$$d'^2_L(\omega_{ijk}, \omega_l) = (\sqrt{\frac{2}{3}(E_i + E_j + E_k)} - \sqrt{2E_l})^2. \tag{6.29}$$

One then easily checks that for this choice of i, j, k, l eq.(6.24) does not hold.

Consequently there is no modified length operator L' corresponding to the modified quantum length d'_L. It is quite remarkable that the spectral distance d_D on a single copy of the Moyal plane coincides (exactly on the set of translated of a states, asymptotically on the set of generalized coherent states) with the "natural" quantity d'_L, vanishing on the diagonal, that one can build from the quantum length d_L. The two options "quantizing the spectral distance" by allowing the emergence of a non-zero minimal spectral distance, or "geometrizing the quantum length" by turning it into a true distance are two equivalent procedures.

7. Geodesics in the Moyal plane

In a quantum space, there is no natural notion of geodesics. However we may find a substitute in the notion of *optimal element* that we introduced in section in section 2.3. Recall that given a spectral triple $(\mathscr{A}, \mathscr{H}, D)$, by optimal element between two states we intend an element of the algebra that attains the supremum in the spectral distance formula, or a sequence of elements in case the supremum is not attained. Noticing that the commutator norm condition can be equivalently written as an equality instead of an inequality [23], an optimal element between $\tilde{\varphi}, \varphi \in \mathscr{S}(\mathscr{A})$ is thus either an element of $\mathscr{A}$ such that

$$|\tilde{\varphi}(a) - \varphi(a)| = d_D(\tilde{\varphi}, \varphi) \quad \text{and} \quad \|[D, a]\| = 1, \tag{7.1}$$

or a sequence of element $a_n \in \mathscr{A}$ such that

$$\lim_{n\to\infty} |\tilde{\varphi}(a_n) - \varphi(a_n)| = d_D(\tilde{\varphi}, \varphi) \quad \text{and} \quad \|[D, a_n]\| \leq 1 \quad \forall n \in \mathbb{N}. \tag{7.2}$$

For non-unital spectral triples (e.g. non-compact manifolds), one usually finds first an element that satisfies (7.1) but which is not in the algebra and needs to be "regularize at infinity", like the function (2.8) is approximated by the sequence (2.9). In this case, as explained in section 2.3, we talk about an optimal element up to regularization. In the following we are not interested in the regularization procedure, and optimal element always means "up to regularization".

In the commutative case, the commutator norm condition

$$\|[\not{\partial}, f]\| = \sup_{x \in \mathscr{M}} \left\| \nabla f_{|x} \right\|_{T_x \mathscr{M}} = 1 \tag{7.3}$$

characterizes the optimal element between δ_x and δ_y locally, in the sense that the constraint is carried by the gradient of f. The geodesics through x are retrieved as the curves tangent to the optimal element $f = d_{\text{geo}}(x, .)$. In this sense, computing the spectral distance amounts to solving the equation of the geodesics:

- eq. (7.3) plays the role of the geodesic equation;
- the optimal element $f = d_{\text{geo}}(x, .)$ fully characterizes the geodesics through x;
- the valuation of the optimal element on $\delta_x - \delta_y$ gives the integration of the line element on a minimal geodesic between x and y.

For these reasons, as a proposal for a "geodesic" between two quantum points $\varphi, \tilde{\varphi}$, we shall draw our attention on the optimal elements.

7.1 Discrete versus continuous geodesics

For eigenstates of the quantum harmonic oscillator whose difference of energy $E_n - E_m$ ($m \leq n$ to fix notation) is small, the spectral distance

$$d_D(\omega_m, \omega_n) = \lambda_P \sum_{k=m+1}^{n} \frac{1}{\sqrt{2k}} \tag{7.4}$$

appears as a middle Riemann sum approximation of the modified quantum length

$$d'_L(\omega_m, \omega_n) = \sqrt{2E_n} - \sqrt{2E_m} = \lambda_P \left(\sqrt{2n+1} - \sqrt{2m+1} \right) = \lambda_P \int_{m+\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{\sqrt{2k}} dk. \tag{7.5}$$

From a geometrical point of view, one may interpret this result saying that the spectral distance and the quantum length are the integration of the same quantum line element

$$\lambda_P \frac{1}{\sqrt{2k}} dk \tag{7.6}$$

but along two distinct geodesics: a continuous one for the quantum length, a discrete one for the spectral distance. In a word, *both the spectral distance and the length operator quantize the line element; with the spectral distance one also quantizes the geodesics.*

Let us develop this idea from the point of view of the optimal element. Recall that on the Euclidean plane, the function $l(x_\mu)$ in (2.10) yields both the length operator $L = l(dq_\mu)$ and the optimal element $l(q_\mu)$ between any two pure states $\delta_x, \delta_{\lambda x}$, $\lambda \in \mathbb{R}^+$. This is no longer true in the Moyal plane. To see it, it is convenient to work with the complex coordinates introduced in (3.23), as well as with their universal differential

$$da = \frac{1}{\sqrt{2}}(dq_1 + i dq_2), \quad da^* = \frac{1}{\sqrt{2}}(dq_1 - i dq_2). \tag{7.7}$$

Proposition 7.1. *[27][10] On the Moyal quantum plane, the length operator can be equivalently defined as $L = l_i(da)$, with*

$$l_1(z) \doteq \sqrt{z\bar{z} + z\bar{z}} \ \text{ or } \ l_2(z) \doteq \sqrt{2(z\bar{z} - \lambda_P^2)} \ \text{ or } \ l_3(z) \doteq \sqrt{2(\bar{z}z + \lambda_P^2)}. \tag{7.8}$$

[10] Notice some change of notations with respect to [27]: there we assumed that $\mathscr{L}(l_0)$ were $l_0(a)$, but there is no guaranty that this should be true. Also we used indistinctly l_i for $l_i(a)$, which might have been confusing.

The optimal element between any two eigenstates of the Hamiltonian of the quantum harmonic oscillator is - up to regularization at infinity - $\mathscr{L}(l_0)$ where l_0 is a solution of

$$(\partial_z l_0 \star z) \star (\partial_z l_0 \star z)^* = \frac{1}{2} z^* \star z. \tag{7.9}$$

Neither $l_1(a)$ nor $l_2(a)$ or $l_3(a)$ are optimal elements between eigenstates.

If $l_1(a)$ were the optimal element, then the identification between the modified quantum length d'_L and the spectral distance d_D on the set eigenstates of the harmonic oscillator, discussed in proposition 6.3, would hold true exactly and not only asymptotically. Indeed one checks that [27]

$$|\omega_m(l_1(a)) - \omega_n(l_1(a))| = \lambda_P |\sqrt{2m+1} - \sqrt{2n+1}| = d'_L(\omega_m, \omega_n). \tag{7.10}$$

We may interpret this equation as a definition of an "optimal element for the modified quantum length", namely we assume that (7.10) is the supremum of $\omega_m - \omega_n$ on the unit ball of $\mathbb{K}$ for some (still to determine) semi-norm, distinct from $\|[D,.]\|$. Having in minds that optimal elements provide a notion of geodesics in a quantum space, $\mathscr{L}(l_0)$ and $l_1(a)$ thus appear as two proposals for a geodesic on the quantum space, with associated geodesic distance $d_D(\omega_m, \omega_n), d'_L(\omega_m, \omega_n)$.

7.2 Shift vs. identity

Let us now consider translated states. The function

$$l_\kappa(z) = \frac{z e^{-i\Xi} + \bar{z} e^{i\Xi}}{\sqrt{2}}, \qquad \text{with} \qquad \Xi \doteq \operatorname{Arg} \kappa, \tag{7.11}$$

yields the optimal element (up to regularization at infinity) between any state φ and its κ-translated both on the Euclidean plane (through the pointwise action of l_κ) and the Moyal plane (through its $\star$-action). For the latter, this has been shown in [28, Theo. III.9], for the former in [14, Prop. 3.2]). In particular $l_\kappa(z)$ is an optimal element between δ_x and δ_y viewed as the $\kappa = \frac{y_1 - x_1 + i(y_2 - x_2)}{\sqrt{2}}$-translated of δ_x: one the one hand,

$$|\delta_x(l_\kappa) - \delta_{x+\kappa}(l_\kappa)| = l_\kappa(\kappa) = \sqrt{2}|\kappa| = |x - y|, \tag{7.12}$$

on the other hand

$$[\not{\partial}, l_\kappa] = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} l_\kappa \\ \partial l_\kappa & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & e^{i\Xi} \\ e^{-i\Xi} & 0 \end{pmatrix} \tag{7.13}$$

has obviously norm 1. Notice that

$$[\not{\partial}, l_\kappa]^* [\not{\partial}, l_\kappa] = \mathbb{I}. \tag{7.14}$$

It is quite remarkable that the same function l_κ gives an optimal element between translated states, regardless of the commutativity of the algebra. In a sense, (7.13) indicates that in both the Euclidean and the quantum planes, the derivatives of the optimal element between translated states is proportional to the identity, meaning that the "geodesic" is smooth. Quantum versus classical is not relevant.

Let us now re-examine the optimal element $\mathscr{L}(l_0)$ between eigenstates of the quantum harmonic oscillator. Modulo regularization at infinity, it can be characterized [8, Prop. 3.7] as a solution of

$$[\partial\!\!\!/, \mathscr{L}(l_0)] = -i \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}, \tag{7.15}$$

where S is the shift operator (eq. (7.9) actually follows from it). One has

$$\mathbb{I} - [\partial\!\!\!/, \mathscr{L}(l_0)]^* [\partial\!\!\!/, \mathscr{L}(l_0)] = e_0 \tag{7.16}$$

where e_0 is the projection on h_0. Eq. (7.15) indicates that the derivative of the optimal element for the spectral distance between eigenstates is the shift S, meaning that the "geodesic" is discrete.

Notice that what prevents $\mathscr{L}(l_0)$ to satisfy (7.14) is that the set of eigenstates of the harmonic oscillator - identified to $\mathbb{N}$ - is not a group (unlike the set of translated states). The shift acting on $l^2(\mathbb{N})$ is not a unitary operator, so that the optimal element between eigenstates verifies (7.16) instead of (7.14). The latter would be verified if one could take into account states with negative energy (the shift on $l^2(\mathbb{Z})$ is unitary).

8. Conclusions and outlook

There is no *quantum* standard meter: the DFR and θ-Minkowski length operator L make a minimal length emerge from the Moyal plane, on the contrary Connes's spectral distance provides the same Moyal plane with a metric structure that does not imply any minimal distance.

Because of this discrepancy, stemming from the non-zero minimum l_P of the spectrum of L opposed to the continuum of value $[0,\infty]$ taken by the spectral distance d_D on the Moyal plane, there is no obvious way to compare these two approaches. However, one can extract from the length operator a quantity d'_L - the modified quantum length - that coincides exactly with the spectral distance d_D on any set $\mathscr{C}(\omega_m)$ of generalized coherent states, and asymptotically on their union $\mathscr{C} = \underset{m\in\mathbb{N}}{\cup} \mathscr{C}(\omega_m)$.

Thanks to Pythagoras theorem for the product of spectral triple, this way of turning the quantum length d_L into a true distance d'_L is equivalent to implement a minimal non-zero length into the spectral distance framework by doubling the spectral triple.

As a tentative physical interpretation, we stress that a pair of states $(\varphi, \tilde{\varphi})$ can be viewed either as two states of a single system, or as one state $\varphi \otimes \tilde{\varphi}$ of a two-point system. The spectral distance d_D measures the distance between the two states of the same system, hence $d_D(\varphi,\varphi) = 0$ (no difference between a system in a state φ, and the same system in the same state φ). On the contrary, two copies of the same system can be in the same quantum state φ, yet, they are two distinct copies. Hence $d_L(\varphi,\varphi) = (\varphi\otimes\varphi)(L) \neq 0$. By doubling the spectral triple, one reconciles the two points of view: a pair of quantum points $(\omega, \tilde{\omega})$ in $\mathscr{P}(\mathbb{K})$ can be equivalently seen as

- a state $\omega \otimes \tilde{\omega}$ of $\mathscr{P}(\mathbb{K}) \otimes \mathscr{P}(\mathbb{K})$, on which one evaluates the length operator;
- a pair of states $\left(\omega^1, \tilde{\omega}^2\right)$ in $\mathscr{P}(\mathbb{K}) \otimes \mathscr{P}(\mathbb{C}^2)$, between which one computes $d_{D'}$.

For this to make sense, the correct objects to compare are either

- the double-sheet spectral distance $d_{D'}$ with the quantum square-length d_{L^2},

or, equivalently thanks to Pythagoras theorem

- the single-sheet spectral distance d_D with the modified quantum length d'_L.

The discrepancy between the corrected quantum length and the spectral distance that remains between eigenstates of the harmonic oscillator with a small difference of energy has a natural interpretation in terms of integrations of the same noncommutative line element along two distinct geodesics: a discrete geodesic for spectral distance, a continuous one for the corrected quantum length. .

As outlook, let us mention the following points:

- there is a recent result of Wallet [39] on homothetic transformation of the Moyal plane. To study the renormalizability of quantum field theory on noncommutative spacetimes, Grosse and Wulkenhaar had added an harmonic term Ωx^2 to the Lagrangian. The spectral distance computed with the corresponding Dirac operator D_Ω is

$$d_{D_\Omega} = \frac{1}{\sqrt{1+\Omega^2}} d_D,$$

  where D is the Dirac operator corresponding to the theory without harmonic term. This indicates an intriguing link between renormalizability and the metric structure of space-time.

- besides the length operator and the spectral distance, there exists (at least) a third proposal for a quantized version of the distance in the physics literature, namely the length operator in loop quantum gravity [5]. Its definition relies on a crucial way on the holonomy of a suitable connection (Wilson loops). Interestingly, the holonomy of a connection also appears in the spectral distance formula when one considers the so-called *fluctuation of the metric* for a spectral triple based on a algebra of matrix valued functions $C_0(\mathscr{M}) \otimes M_n(\mathbb{C})$. The space of pure state $\mathscr{P}$ is a $U(n)$- trivial bundle over $\mathscr{M}$, and the connection associated to a covariant Dirac operator defines on $\mathscr{P}$ an horizontal distribution in the sense of sub-Riemannian geometry. The difference between the spectral distance and the horizontal distance associated to the connection heavily depends on the size of the holonomy group [25, 26]. So it would be interesting to compare these two metric interpretations of the holonomy: loop quantum gravity and noncommutative geometry.

- the Heisenberg group, which comes out naturally in our context as the exponential of the quantum coordinates q_μ, can also be seen as a sub-Riemannian geometry [30]. By defining an appropriate covariant Dirac operator as in the preceding remark, one could provide the Heisenberg group with a spectral distance. It would be interesting to understand whether this distance is similar as the one coming from the spectral triple of the Moyal plane.

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A. Appendix

A.1 Exact Poincaré covariance vs. deformed Poincaré invariance

In the construction of section 3.1 leading to the Moyal algebra, there is still some freedom in the choice of the symplectic form σ. The latest is constrained by the transformation law of the commutator relation (1.3) under Poincaré transformations. Explicitly, identifying operators with their irreducible representation so that (1.3) reads

$$[q_\mu, q_\nu] = i\lambda_P^2 Q_{\mu\nu} = i\lambda_P^2\, \theta_{\mu\nu}\mathbb{I}, \tag{A.1}$$

we assume there is a unitary representation of the Poincaré group $\mathbb{R}^d \ltimes O(d-1,1)$ such that

$$q_\mu \mapsto q'_\mu \doteq \Lambda_\mu^\alpha q_\alpha + a_\mu \mathbb{I} \qquad \Lambda \in SO(d-1,1), a \in \mathbb{R}^d. \tag{A.2}$$

The commutation relation (A.1) is obviously not Poincaré invariant since

$$[q'_\mu, q'_\nu] = [a_\mu, a_\nu]\mathbb{I} + q_\alpha([a_\mu\mathbb{I}, \Lambda_\nu^\alpha] - [a_\nu\mathbb{I}, \Lambda_\mu^\alpha]) + \Lambda_\mu^\alpha [q_\alpha, q_\beta]\Lambda_\nu^\beta \neq [q_\mu, q_\nu]. \tag{A.3}$$

However Poincaré covariance can be restored observing that the generators of the classical Poincaré group commute with each other, in which case (A.3) reduces to

$$[q'_\mu, q'_\nu] = i\lambda_P^2 \Lambda_\mu^\alpha \Lambda_\nu^\beta \theta_{\alpha\beta}\mathbb{I} = i\lambda_P^2\, (\mathrm{Ad}_\Lambda \Theta)_{\mu\nu}\mathbb{I}. \tag{A.4}$$

In other term the commutator relations (A.1) are covariant under Poincaré transformations as soon as one requires the matrix $\Theta = \{\theta_{\mu\nu}\}$ to transform under the adjoint action of the Poincaré group. This requirement is the building block of the DFR model of *Poincaré covariant* quantum spacetime.

Alternatively, one may impose the commutators to be invariant under the action of the symmetry group of the quantum space. This forces to deform the Poincaré group into the quantum group θ-Poincaré [2]. The latest is characterized by a non-trivial commutation relation between the generators of translations,

$$[a_\mu, a_\nu] = i\theta_{\mu\nu} - i\theta_{\alpha\beta}\Lambda_\mu^\alpha \Lambda_\nu^\beta, \tag{A.5}$$

so that (A.3) yields

$$[q'_\mu, q'_\nu] = i\lambda_P^2 \theta_{\mu\nu}\mathbb{I}. \tag{A.6}$$

This is a model of *deformed-Poincaré invariant* spacetime, called *canonical noncommutative spacetime* (NCS) or θ-Minkowski.

Note that in both DFR and θ-Minkowski the numerical value of the Planck length is an invariant under symmetry transformations: either because (A.6) is indeed invariant in θ-Minkowski; or because in the (4 dimensional) DFR model λ_P is retrieved as the norm of the tensor $Q_{\mu\nu}$, which is a Poincaré invariant quantity.

To some extent, the invariant deformed-Poincaré NCS can be viewed as the covariant-Poincaré DFR restricted to one point on the orbit $\Sigma = \{\mathrm{Ad}_\Lambda \Theta, \Lambda \in SO(d-1,1)\}$. This is discussed at length in [33]. From our purposes, once fixed the matrix Θ then the relevant C^*-algebra is $\mathbb{K}$. For the full DFR model one should take into account the action (A.2) of the Poincaré group. The relevant algebra is then (see [19] for the original argument, [32] for a recent presentation)

$$\mathscr{E} = C_0(\Sigma, \mathbb{K}) = C_0(\Sigma) \otimes \mathbb{K}, \tag{A.7}$$

namely the C^*-algebra of $\mathbb{K}$-valued smooth functions vanishing at infinity.

A.2 Pair of quantum points

In the commutative case, a point x of $\mathbb{R}^d$ is a pure state δ_x of $C_0(\mathbb{R}^d)$. Similarly, we take as a "quantum point" a pure state of the algebra $\mathbb{K}$ (in θ-Minkowski) or $\mathscr{E}$ (in the DFR model). Pure states of $\mathscr{E}$ are couples

$$\omega_S \doteq (\delta_S, \omega) \quad \text{with} \quad \delta_S \in \mathscr{P}(C_0(\Sigma)) \simeq \Sigma,\ \omega \in \mathscr{P}(\mathbb{K}). \tag{A.8}$$

A pair of quantum points $(\omega_S, \tilde{\omega}_{\tilde{S}})$ defines a two-"quantum point" state $\omega_S \otimes \tilde{\omega}_{\tilde{S}}$. The latter is a pure state of the tensor product of complex algebras $\mathscr{E} \otimes \mathscr{E}$. However, to guarantee that

$$[q_\mu \otimes \mathbb{I}, q_\nu \otimes \mathbb{I}] = [\mathbb{I} \otimes q_\mu, \mathbb{I} \otimes q_\nu] = i\lambda_P^2 Q_{\mu\nu}(\mathbb{I} \otimes \mathbb{I}) \tag{A.9}$$

(that is, the commutators of the coordinates of two independent quantum points are equal), it has been proposed in [3] that the tensor product $\mathscr{E} \otimes_{C_0(\Sigma)} \mathscr{E}$ over the center $C_0(\Sigma)$ of $\mathscr{E}$ should be used instead. This has the following importance consequence:

Proposition A.1. [see e.g. [27]] *Pure states of $\mathscr{E} \otimes_{C_0(\Sigma)} \mathscr{E}$ are pairs $(\omega_S, \tilde{\omega}_S)$ composed of two pure states of $\mathscr{E}$ corresponding to the same point $S \in \Sigma$.*

Consequently, for the DFR model, θ-Minkowski and the Moyal plane, a pair of quantum points is a pair of pure-states $(\omega, \tilde{\omega})$ of $\mathbb{K}$. So from our length/distance perspective, these three models of quantum spaces are equivalent.

Let us mention that in loop quantum gravity, the behaviour under Lorentz transformations of the minimum of a quantum observable (specifially: the area) has been investigated in [36]. It could be interesting to see whether the analysis developed would make sense in this context.

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