

Construction of 2-dimensional Grosse-Wulkenhaar Model

Zhituo Wang*

*Laboratoire de Physique Théorique, CNRS UMR 8627,
Université Paris XI, F-91405 Orsay Cedex, France
E-mail: zhituo.wang@th.u-psud.fr*

In this talk we briefly report the recent work on the construction of the 2-dimensional Grosse-Wulkenhaar model with the method of loop vertex expansion [1]. We treat renormalization with this new tool, adapt Nelson's argument and prove Borel summability of the perturbation series. This is the first non-commutative quantum field theory model to be built in a non-perturbative sense.

*Proceedings of the Corfu Summer Institute 2011 School and Workshops on Elementary Particle Physics and Gravity
September 4-18, 2011
Corfu, Greece*

*Speaker.

1. Introduction

It is believed that in quantum field theory including gravity, space-time would become fuzzy at distances compare to the Planck scale. Noncommutative geometry is a nice way to study such fuzzy structure and noncommutative quantum field theories (NCQFT) [2, 3] are good models to study the physics in such fuzzy background. The simplest noncommutative space is the Moyal space \mathbb{R}_θ^D with the coordinates obeying the commutation relation $[x_i, x_j] = i\theta_{ij}$, and the simplest noncommutative quantum field theory model is a scale field theory with $\lambda\phi^4$ interaction defined on \mathbb{R}_θ^4 . However such model suffers from the UV/IR mixing, namely, the correlation function is still infrared divergent when integrating out the ultraviolet modes for certain non-planar graphs [4]. Several years ago H. Grosse and R. Wulkenhaar solved this problem by introducing a harmonic oscillator term in the propagator so that the theory fully obeys a new symmetry called the Langmann-Szabo duality. They have proved in a series of papers [5, 6, 7] (see also [8]) that the noncommutative ϕ_4^{*4} field theory possessing the Langmann-Szabo duality (which we call the GW_4 model hereafter) is perturbatively renormalizable to all orders.

After the work of Grosse and Wulkenhaar many other QFT models on Moyal space [9, 10, 11, 14, 12, 13, 15] or degenerate Moyal space [16, 17] have also been shown to be *perturbatively* renormalizable. More details could be found in [19, 20, 21].

The GW_4 model is not only perturbatively renormalizable but also asymptotically safe due to the vanishing of the β function in the ultraviolet regime [23], [24], [25], which means that there are neither Landau ghost which appears in commutative ϕ_4^{*4} and QED nor confinement for non-Abelian gauge theory.

So that GW_4 is a prime natural candidate for a full construction of a four dimensional just renormalizable quantum field theory [27].

In this paper we shall construct the 2 dimensional Grosse-Wulkenhaar model (GW_2 for simplicity), namely we prove that the perturbation series of the connected Schwinger's function is Borel summable, as a warm-up towards building non-perturbatively the full GW_4 model. The method we use is the Loop Vertex Expansion (LVE) invented by Vincent Rivasseau [26] (see also [1, 26, 28, 29, 30, 31] for more details), which is a combination of the intermediate field techniques with the BKAR forest formula [38, 39].

In order to simplify the reading of this paper we briefly summarize the ideas and methods of the construction procedure. We shall first obtain the intermediate field representation of the partition function for the GW_2 model by introducing the intermediate matrix field σ and integrating out the original scalar field ϕ . Then we use the BKAR tree formula to derive the perturbation series of the connected Schwinger's function. However the amplitude is divergent due to the presence of the tadpoles. So we introduce a new expansion, called the cleaning expansion, to compensate the divergences. We should stop the cleaning expansion when we have obtained enough convergent factor, as otherwise we would generate big combinatorial factors which diverge again. This is the analogue of the Nelson's argument for traditional constructive field theory [27]. After that we shall re-summing the uncompensated tadpoles. In the re-summing procedure we should analyze carefully the scales of the indices such that not all tadpoles should be re-summed. This analysis plays a role of the traditional cluster expansion. Then in the end we get the bound for the perturbation series and prove the Borel summability.

Another approaches towards the construction of GW_4 , based on the combination of a Ward identity and Schwinger-Dyson equations, is given in [32], and a numerical study of NCQFT models is given in [33].

Acknowledgments The author is very grateful to his Phd supervisor Prof. Vincent Rivasseau for many very helpful discussions, as well as the organizers of the Corfu 2011 Workshops on Elementary Particle Physics and Gravity for their invitation.

2. Moyal space and Grosse-Wulkenhaar Model

2.1 The Moyal space

The D -dimensional Moyal space \mathbb{R}_θ^D for D even is generated by the non-commutative coordinates x^μ that obey the commutation relation $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$, where Θ is a $D \times D$ non-degenerate skew-symmetric matrix. (see [18, 19] for more details).

The 2-dimensional Grosse-Wulkenhaar model in the matrix basis is defined by (see [1] for more details):

$$\begin{aligned} S[\phi] &= 2\pi\theta \text{Tr} \left[\frac{1}{2} \phi \Delta \phi + : \frac{\lambda}{4} \phi_*^4 : \right] \\ &= 2\pi\theta \sum_{m,n,k,l} \left[\frac{1}{2} \phi_{mn} \Delta_{mn,kl} \phi_{kl} + \frac{\lambda}{4} : \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} : \right]. \end{aligned} \quad (2.1)$$

where ϕ_{mn} is a Hermitian matrix and we have put the ultraviolet cutoff Λ to the matrix indices. Namely, $m, n = 1, \dots, \Lambda$. We have taken the wick ordering to the interaction term whose explicit form reads:

$$: \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} := \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} - 8\phi_{mp} \phi_{pm} T_m^\Lambda + 6\text{Tr}_m (T_m^\Lambda)^2, \quad (2.2)$$

where

$$T_m^\Lambda = \sum_{q=0}^{\Lambda} \frac{1}{q+m} = \log \frac{\Lambda+m}{m} \sim \log \Lambda, \text{ for } 1 \leq m \ll \Lambda, \quad (2.3)$$

and

$$T_\Lambda^2 = \text{Tr}_m (T_m^\Lambda)^2 = \sum_m \left(\sum_p \frac{1}{m+p} \right) \left(\sum_q \frac{1}{m+q} \right) \sim (2\ln^2 2 + \frac{\pi^2}{6}) \Lambda. \quad (2.4)$$

The kinetic term in the matrix basis reads:

$$\Delta_{mn,kl} = [\mu^2 + \frac{2}{\theta}(m+n+1)] \delta_{ml} \delta_{nk} - \frac{2}{\theta} (1 - \Omega^2) \quad (2.5)$$

$$\times [\sqrt{(m+1)(n+1)} \delta_{m+1,l} \delta_{n+1,k} + \sqrt{mn} \delta_{m-1,l} \delta_{n-1,k}]. \quad (2.6)$$

The kinetic matrix reduces to a much simpler form when $\Omega = 1$:

$$\Delta_{mn,kl} = \left[\mu^2 + \frac{4}{\theta}(m+n+1) \right] \delta_{ml} \delta_{nk}, \quad (2.7)$$

which is a diagonal matrix in the double indices and the covariance matrix $C_{mn,kl}$ reads:

$$C_{mn,kl} = \frac{1}{\mu^2 + \frac{4}{\theta}(m+n+1)} \delta_{ml} \delta_{nk}. \quad (2.8)$$

Remark that the covariance C is a $\Lambda^2 \times \Lambda^2$ dimensional diagonal matrix.

Since $\Omega = 1$ is a fixed point of this theory [24], we shall for simplicity take $\Omega = 1$ in the rest of this paper and write the covariance as $C_{mn} = C_{mn,nm} = \frac{1}{m+n+1}$ for simplicity.

3. The intermediate field representation and the Loop vertex expansion

3.1 The intermediate field representation

The partition function for the matrix model reads:

$$Z(\lambda) = \int d\mu(\phi_{mn}) e^{-S[\phi_{mn}]}, \quad (3.1)$$

where

$$d\mu(\phi_{mn}) = \pi^{-N^2/2} e^{-1/2[\sum_{mn} \phi_{mn} \Delta_{mn} \phi_{nm}]} \prod_{mn} d\text{Re}(\phi_{mn}) d\text{Im}(\phi_{nm}). \quad (3.2)$$

is the normalized Gaussian measure of the matrix fields ϕ_{mn} with covariance C given by (2.8) and $S[\phi_{mn}]$ is the Wick ordered interaction term.

We introduce the Hermitian matrix σ_{mn} as an intermediate field and the partition function can be written as:

$$Z(\lambda) = \int d\mu(\sigma) d\mu(\phi) e^{-\frac{1}{2} \sum_{mnkl} \phi_{mn} \Delta_{mn,kl} \phi_{kl} - i\sqrt{2\lambda} (\sum_{kmn} \sigma_{km} \phi_{mn} \phi_{nk} - 4 \sum_{mk} \sigma_{km} \delta_{mk} T_m^\Lambda) + \frac{5}{2} \lambda T_\Lambda^2}. \quad (3.3)$$

where

$$d\mu(\sigma_{mn}) = \pi^{-N^2/2} e^{-1/2 \sum_{mn} \sigma_{mn}^2} \prod_{mn} d\text{Re}(\sigma_{mn}) d\text{Im}(\sigma_{mn}). \quad (3.4)$$

is the normalized Gaussian measure for the hermitian matrices σ with the covariance:

$$\langle \sigma_{mn}, \sigma_{kl} \rangle = \int d\mu(\sigma) \sigma_{mn} \sigma_{kl} = \delta_{nk} \delta_{ml}. \quad (3.5)$$

Then we should integrate out the original matrix fields ϕ_{mn} , which is a $\Lambda \times \Lambda$ dimensional matrix. The result reads:

$$\begin{aligned} Z(\lambda) &= \int d\mu(\sigma) e^{2i\sqrt{2\lambda} \text{Tr} T^\Lambda \sigma - \frac{1}{2} \text{Tr} \log [1 + i\sqrt{2\lambda} C^{1/2} \hat{\sigma} C^{1/2}] + \frac{5}{2} \lambda T_\Lambda^2} \\ &= \int d\mu(\sigma) e^{2i\sqrt{2\lambda} \text{Tr} T^\Lambda \sigma - \frac{1}{2} \text{Tr} \log [1 + i\sqrt{2\lambda} C^{1/2} (I \otimes \sigma + \sigma \otimes I) C^{1/2}] + \frac{5}{2} \lambda T_\Lambda^2}, \end{aligned} \quad (3.6)$$

where $\text{Tr} T^\Lambda \sigma = \sum_m T_m^\Lambda$, I is the $\Lambda \times \Lambda$ dimensional identity matrix. $C^{1/2}$ is the square root of the covariance C with elements:

$$[C^{1/2}]_{mn,kl} = \sqrt{\frac{1}{m+n+1}} \delta_{ml} \otimes \delta_{nk}, \quad (3.7)$$

the term $e^{-1/2 \text{Tr} \log [\dots]}$ represents the $\Lambda^2 \times \Lambda^2$ determinant resulting from the Gaussian integration over ϕ_{mn} . Here we view the vector space R^{Λ^2} as $R^\Lambda \otimes R^\Lambda$. For example, the operator $C^{1/2} (I \otimes \sigma + \sigma \otimes I) C^{1/2}$ transforms the basis $e_m \otimes e_n$ of $R^\Lambda \otimes R^\Lambda$ into:

$$C^{1/2} \left(e_m \otimes \sum_k \sigma_{nk} e_k + \sum_k \sigma_{mk} e_k \otimes e_n \right) C^{1/2}. \quad (3.8)$$

However in the rest of this paper we shall simply write $\hat{\sigma}$ as σ when this doesn't make confusions.

3.2 The BKAR Tree formula and the expansion

The most interesting quantities in our model are the connected Schwinger's functions. Graphically connected functions are labeled by the spanning trees. We shall derive the connected function by the BKAR tree formula, which is a canonical way of calculating the weight of a spanning tree within an arbitrary graph.

Let us first of all expand the exponential as $\sum_n \frac{V(\hat{\sigma})^n}{n!}$. To compute the connected function while avoiding an additional factor $n!$, we give a kind of *fictitious* index v , $v = 1, \dots, n$ to each field σ in vertex $V(\sigma)$ and we could rewrite the expanded interaction term as $\sum_n \prod_{v=1}^n \frac{V(\sigma^v)}{n!}$. This means that we consider n different copies σ_v of σ with a degenerate Gaussian measure

$$d\nu(\{\sigma_v\}) = d\nu(\sigma_{v_0}) \prod_{v' \neq v_0} \delta(\sigma_{v'} - \sigma_{v_0}) d\sigma_{v'}, \quad (3.9)$$

where v_0 is an arbitrarily marked vertex.

The vacuum Schwinger's function is given by:

Theorem 3.1 (Loop Vertex Expansion [26]).

$$\begin{aligned} \log Z(\lambda, \Lambda, \mathcal{V}) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} G_{\mathcal{T}} \quad (3.10) \\ G_{\mathcal{T}} &= \left\{ \prod_{\ell \in \mathcal{T}} \sum_{m_\ell, n_\ell, p_\ell, q_\ell} \left[\int_0^1 dw_\ell \right] \right\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\}) \\ &\left\{ \prod_{\ell \in \mathcal{T}} \left[\delta_{n_\ell p_\ell} \delta_{m_\ell q_\ell} \frac{\delta}{\delta \sigma_{m_\ell, n_\ell}^{v(\ell)}} \frac{\delta}{\delta \sigma_{p_\ell, q_\ell}^{v'(\ell)}} \right] \right\} \prod_{v=1}^n V_v, \end{aligned}$$

where

- each line ℓ of the tree joins two different loop vertices $V^{v(\ell)}$ and $V^{v'(\ell)}$ which are identified through the function $\delta_{m_\ell q_\ell} \delta_{n_\ell p_\ell}$, since the propagator of σ is ultra-local.
- the sum is over spanning trees joining all n loop vertices. These trees have therefore $n-1$ lines, corresponding to σ propagators.
- the normalized Gaussian measure $d\nu_{\mathcal{T}}(\{\sigma_v\}, \{w\})$ over the fields σ_v has now a covariance

$$\langle \sigma_{mn}^v, \sigma_{kl}^{v'} \rangle = \delta_{nk} \delta_{ml} w^{\mathcal{T}}(v, v', \{w\}), \quad (3.11)$$

which depend on the "fictitious" indices. Here $w^{\mathcal{T}}(v, v', \{w\})$ equals to 1 if $v = v'$, and equals to the infimum of the w_ℓ for ℓ running over the unique path from v to v' in \mathcal{T} if $v \neq v'$.

The Feynman graphs are ribbon graphs for the GW2 model [4, 37, 36, 7]. There are three basic line structures in the LVE:

- The full resolvent $\hat{R}(\lambda, \sigma)$ is defined as:

$$\begin{aligned} \hat{R}_{mn}(\sigma, \lambda) &= -\frac{2}{i\sqrt{2\lambda}} \frac{\partial}{\partial \sigma_{mn}} \left[-\frac{1}{2} \text{Tr} \log(1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}) \right] \\ &= \left[C^{1/2} \frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}} C^{1/2} \right]_{mn} \quad (3.12) \end{aligned}$$

we define the resolvent as $R = \frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}}$ and we have $\hat{R} = [C^{1/2} R C^{1/2}]$.

- The propagators C_{mn} between the original fields ϕ_{mn} ,
- The propagators between the σ fields.

The propagators are shown in Figure 1.

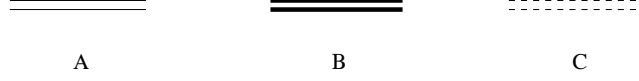


Figure 1: The propagators in LVE. A stands for the resolvent R_{mn} , B stands for the pure propagator C_{mn} , and C is the propagator of the σ fields.

There are three kinds of interaction vertices in the LVE: the counter terms, the leaf terms K with coordination number 1 and the general interaction vertices V . A leaf vertex K is generated by deriving once w.r.t the σ field on the \log_2 term:

$$\begin{aligned}
 K_{mn} &= \frac{\partial}{\partial \sigma_{mn}} \left[-\frac{1}{2} \text{Tr} \log_2 (1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}) \right] \\
 &= -\frac{1}{2} i\sqrt{2\lambda} [C^{1/2} \left(\frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}} - 1 \right) C^{1/2}]_{mn}. \tag{3.13}
 \end{aligned}$$

A general loop vertex could be obtained by deriving twice or more with respect to the σ fields:

$$\begin{aligned}
 V_{m_1 m_2 \dots m_{p-1} m_p}(\lambda, \sigma) &= \frac{\partial}{\partial \sigma_{m_1 m_2}} \dots \frac{\partial}{\partial \sigma_{m_{p-1} m_p}} \left[-\frac{1}{2} \text{Tr} \log (1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}) \right] \\
 &= -\frac{1}{2} (i\sqrt{2\lambda})^p (-1)^p \sum_{\tau} \hat{R}_{m_1 m_{\tau(1)}}(\sigma, \lambda) \dots \hat{R}_{m_{\tau(p)} m_1}(\sigma, \lambda). \tag{3.14}
 \end{aligned}$$

with $p \geq 2$ and the sum over τ is over the p cyclic permutations of the resolvents.

The basic interaction vertices are shown in Figure 2 where we didn't show explicitly the pure propagator $C^{1/2}$.

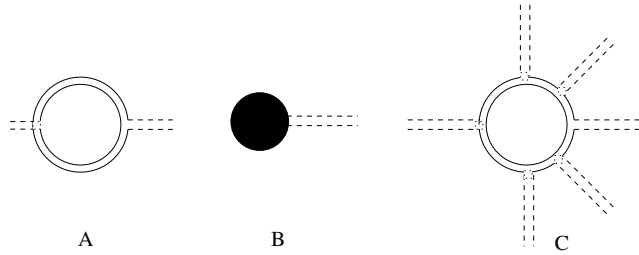


Figure 2: The basic graph elements of LVE. Graph A means the leaf K , graph B means the counter term and graph C means the most general loop vertex which has several σ fields attached.

3.3 The dual representation

Since a LVE graph in the direct representation is *planar*, the notion of duality is globally well-defined. In the dual representation we have a canonical (up to an orientation choice) and more explicit cyclic ordering of all ingredients occurring in the expansion (namely the resolvents, the pure propagators and the counterterms) [1, 31]. We shall work in the dual representation for

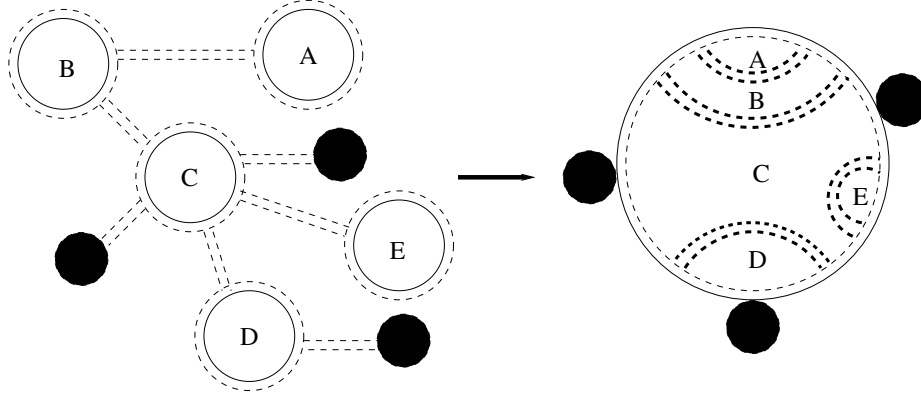


Figure 3: The dual graph of a LVE. The area enclosed by the bold dash ribbons correspond to the original loop vertices.

the rest of this paper. The amplitude of an arbitrary graph with n vertices and k counter terms is bounded by:

$$G_\Lambda \leq \sum_n \frac{n^{n-2}}{n!} \lambda^n (\ln \Lambda)^k \leq \sum (\lambda K)^n (\ln \Lambda)^k, \quad (3.15)$$

which is divergent when $\Lambda \rightarrow \infty$. The reason is simply that the counter terms T_m^Λ are divergent. We should introduce a renormalization process to cancel the counter terms to make the amplitude finite. The renormalization process is called the cleaning expansion which we will introduce in the next section.

4. The Cleaning Expansion

We shall use the multi-scale representation of the propagators and the resolvents. Introducing the Schwinger parameter representation the propagator as:

$$C_{mn} = \int_0^\infty d\alpha e^{-\alpha(\frac{\mu^2}{\theta} + m + n + 1)} = K \int_0^1 d\alpha e^{-\alpha(\frac{\mu^2}{\theta} + m + n + 1)}. \quad (4.1)$$

We decompose the propagator as:

$$C_{mn} = \sum_{j=0}^\infty C_{mn}^j, \quad (4.2)$$

where

$$C_{mn}^j = \int_{M^{-2i}}^{M^{-(2i-2)}} d\alpha e^{-\alpha(\mu^2 + \frac{4}{\theta}(m+n+1))}, \quad (4.3)$$

and M is an arbitrary positive constant. We could easily find that

$$|C_{mn}^j| \leq KM^{-2j} e^{-M^{-2j}|\mu^2 + \frac{4}{\theta}(m+n+1)|}. \quad (4.4)$$

We have also the sliced counter-term as:

$$T_m^\Lambda = \sum_{j=0}^\Lambda T_m^j. \quad (4.5)$$

Due to the cyclic order of the global Trace operator, we could rewrite the loop vertex in the non-symmetric form $\text{Tr} \log[1 + i\sqrt{2\lambda}\sigma C]$ and the resolvent defined in formula (3.12) is written as:

$$R_{mn} = \frac{\partial}{\partial \sigma_{mn}} \left[-\frac{1}{2} \text{Tr} \log(1 + i\sqrt{2\lambda}\sigma C) \right] = -i\frac{1}{2}\sqrt{2\lambda} \left[\frac{1}{1 + i\sqrt{2\lambda}\sigma C} \right] C. \quad (4.6)$$

The idea of the cleaning expansion is to expand and contract the σ fields hidden in the resolvent R so as to compensate the counter terms and generate the convergent terms. In this process we would generate either an inner tadpole, a crossing line (see figure 4) or a nesting line (see Figure 5). The amplitude of an inner tadpole is exactly the same as the counter term but with a minus sign, so this compensates exactly the counter term. The amplitude of crossing or nesting line of scale j is proportional to M^{-2j} , so that we should generate as many of them as possible. We impose also the stopping rule for the cleaning expansion to make sure that we don't expand forever (otherwise this would generate a big combinatorial factor and make the perturbation series divergent, for example, the number of graphs with n crossings is proportional to $n!$). We stop the expansion until we have gained enough convergent factors to compensate the divergent Nelson's factor. A typical cleaning expansion process is shown in Figure 4. The interested readers could look at [31, 1] for more details.

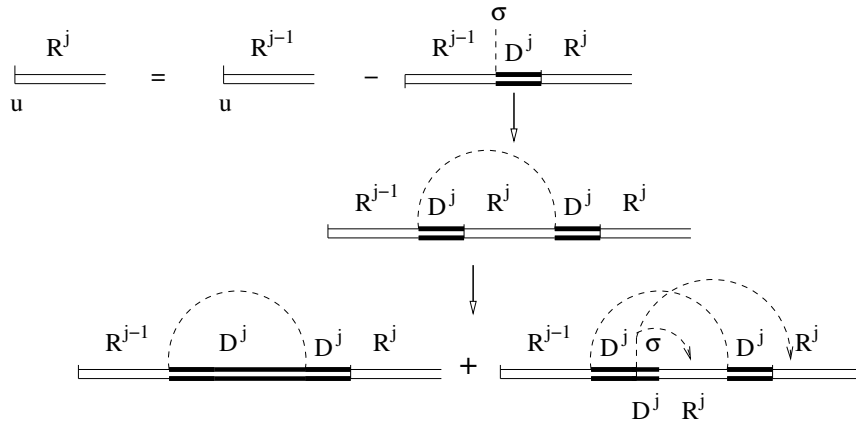


Figure 4: The cleaning expansion. The ordinary ribbon stands for the resolvents and the bold double line stands for the pure propagator. The dashed lines should be envisioned also as double lines and these stand for the σ fields. The L.H.S. of the last line means an inner tadpole while the R.H.S. stands for a crossing line.

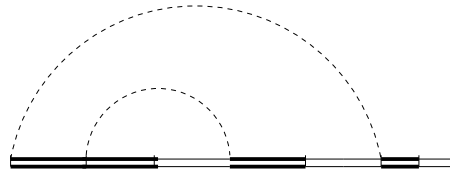


Figure 5: A graph with 2 nesting lines.

5. Nelson's argument and the bound of the connected function

In the last section we introduced the cleaning expansion, after which all inner tadpoles should

be canceled by the counter-terms in the cleaned part of the dual graph. But there might still be arbitrarily many counter terms in the uncleaned part and they are divergent. Instead of canceling all of them, we re-sum them by using the inverse formula of the Gaussian integral and integration by parts.

We write more explicitly, but loosely, the amplitude of the connected function after the LVE and the cleaning expansion as:

$$A_T^N = \prod_{\bar{\ell}, \bar{\ell}_l, \bar{\ell}_r \in \mathcal{F}} \int d\mu(w, \sigma) \int_0^1 dw_l \text{Tr} \overrightarrow{\prod} [K_{mn}^{\bar{\ell}}(\sigma) R_{mn}^{\bar{\ell}_l, \bar{\ell}_r}(\sigma) (D^{\bar{\ell}_c} T_\Lambda)], \quad (5.1)$$

where $R_{mn}(\sigma)$ is the full resolvent that contains the pure propagator C , Tr needs to follow the cyclic order according to the real positions of the leaf terms K , the resolvents R and the counter terms $T_\Lambda D$. We have used the fact that all weakening factors for the counter terms equal to one, as they are leaves in the graph. There are only weakening factors for the σ propagators that cross different regions in the dual graph.

Now we consider the function G for which we haven't expanded the counter-term:

$$G = \int d\nu(\sigma, w) \prod_{l \in \mathcal{F}} \text{Tr} [K_{mn} R_{mn}(\sigma)] e^{\text{Tri}\sqrt{2\lambda}\sigma T_\Lambda}. \quad (5.2)$$

We use the formula

$$\int d\nu(w, \sigma) f(\sigma) g(\sigma) = e^{\frac{1}{2} \frac{\partial}{\partial \sigma} C(\sigma, \sigma', w) \frac{\partial}{\partial \sigma}} f(\sigma) g(\sigma) \Big|_{\sigma=0}, \quad (5.3)$$

where $C(\sigma, \sigma', w)$ is the covariance that might depend on the weakening factor w or not. Hence

$$\begin{aligned} G &= \int d\nu(\sigma, w) \sum_{N=0}^{\infty} \frac{1}{N!} \left[\frac{1}{2} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \sigma'} \right]^N \left\{ \prod_{l \in \mathcal{F}} \int_0^1 dw_l \text{Tr} [K^{\bar{\ell}}(\sigma) R^{\bar{\ell}_l, \bar{\ell}_r}(\sigma)] e^{\text{Tri}\sqrt{2\lambda}\sigma T_\Lambda} \right\} \\ &= \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \sum_{N_3=0}^{\infty} \prod_{l \in \mathcal{F}} \int_0^1 dw_l \frac{1}{N_1! N_2!} \left(\frac{1}{2} \right)^{N_1+N_2} \left[\left(\frac{\partial}{\partial \sigma} \right)^{N_2} \left\{ \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \sigma'} \right\}^{N_1} \right] [K(\sigma) R(\sigma)] \\ &\quad \left\{ \left(\frac{\partial}{\partial \sigma'} \right)^{N_2} \frac{1}{N_3!} \left[\frac{1}{2} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \sigma'} \right]^{N_3} e^{\sum_q i\sqrt{2\lambda}\sigma_{mm} T_m^\Lambda} \right\} \Big|_{\sigma=0}. \end{aligned} \quad (5.4)$$

where we have written down explicitly the trace term in the exponential, and the term T_m^Λ is defined by formula (4.5).

While the N_1 and N_2 derivations generate connected terms, the last derivatives generate N_3 disconnected terms, see graph B in Figure 2.

If we sum the the indices m for the counter terms directly, we would have:

$$\begin{aligned} \sum_{N_3=0}^{\infty} \frac{1}{N_3!} \left[\frac{1}{2} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \sigma'} \right]^{N_3} e^{\text{Tri}\sqrt{2\lambda}\sigma T_\Lambda} \Big|_{\sigma=0} &= \sum_{N_3=0}^{\infty} \frac{1}{N_3!} \left[\frac{1}{2} \sum_{m=0}^{\Lambda} (i\sqrt{2\lambda} T_m^\Lambda)^2 \right]^{N_3} \\ &= e^{-\lambda T_\Lambda^2}, \end{aligned} \quad (5.5)$$

And the resummed amplitude reads:

$$A_T^N = \int d\nu(\sigma, w) \prod_{\bar{\ell} \in \mathcal{F}} \text{Tr} \overrightarrow{\prod} [K_{mn}^{\bar{\ell}}(\sigma) R_{mn}^{\bar{\ell}_l, \bar{\ell}_r}(\sigma)] e^{\text{Tri}\sqrt{2\lambda}\sigma T_\Lambda} e^{\lambda T_\Lambda^2}, \quad (5.6)$$

The bound in the last formula is dangerous in that $T_\Lambda^2 \sim \Lambda$. The reason for the bad factor Λ is due to that we sum all indices in $\sum_{m=0}^\Lambda (i\sqrt{2\lambda}T_m^\Lambda)^2$ in formula (5.5), which means we detach all counter terms in the uncleaned part of the dual graph even if they are convergent. But this is not optimal. For example, the tadpole of amplitude $\ln(\Lambda/m)$, where m is the scale of the border of the ribbon where the tadpole is attached, is not divergent if $m \sim c\Lambda$, where c is a number that is not much smaller than 1, say 1/2. What's more, we need to bear in mind that each counter term is attached to a pure propagator in the form of $\lambda C_{nm}T_m^\Lambda$ and C_{nm} decays as M^{-2j} , where $2j \sim \ln \max(m, n)$. Here we have ignore all the inessential factors.

So instead of considering only T_m^Λ , we need to take the whole object $C_{nm}T_m^\Lambda$, which we also call it the full counter term, into account. We first consider the case $m > n$, so we have $m \sim M^{2j}$ where $M > 1$ is an arbitrary constant. Then we have

$$C_{nm}T_m^\Lambda \sim \frac{1}{m} \ln(\Lambda/m) \leq \frac{\ln \Lambda}{m}. \quad (5.7)$$

So that the counter term doesn't cause any divergence as long as $m > \ln \Lambda$ and we could just bound these counter terms by 1 and need not to detach them from the connected graph. The counter terms become dangerous only when $m \leq \ln \Lambda$ and we need to re-sum them as introduced before.

So that we only need to sum over the indices m for $m \leq \ln \Lambda$ in formula (5.5), which reads:

$$-2\lambda \sum_{m=1}^{\ln \Lambda} \ln^2(\Lambda/m) < -2\lambda \sum_{m=1}^{\ln \Lambda} \ln^2 \Lambda = -2\lambda (\ln \Lambda)^3, \quad (5.8)$$

and the amplitude of the connected function after the resummation reads:

$$A_T^N = \int d\nu(\sigma, w) \prod_{\vec{\ell} \in \mathcal{F}} \text{Tr} \prod [K_{mn}^{\vec{\ell}}(\sigma) R_{mn}^{\vec{\ell}_l, \vec{\ell}_r}(\sigma)] e^{\text{Tri}\sqrt{2\lambda}\sigma T_\Lambda} e^{\lambda(\ln \Lambda)^3}, \quad (5.9)$$

This resummation process is shown in Figure 6.

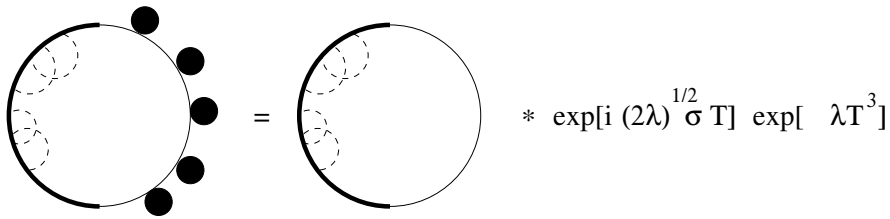


Figure 6: A sketch of the resummation of the counter terms. Where the term T means $\ln \Lambda$. The counterterms with the scale index of outer border larger than $\ln \ln \Lambda$ are not summed.

The divergent factor $e^{\lambda(\ln \Lambda)^3}$ is not dangerous as we could bound it by the following formula:

$$e^{-aj_{\max}^3} \cdot (j_{\max}^2)! \cdot e^{\lambda \ln^3 \Lambda} \sim e^{-aj_{\max}^3 + 2j_{\max}^2 \ln j_{\max} + \lambda j_{\max}^3} < 1 \quad (5.10)$$

as long as a is chosen properly, for example $a = 1.4\lambda$. Here we have use the fact that fact that $\ln \Lambda \sim j_{\max}$ and $(aj_{\max}^2)! \sim e^{2j_{\max}^2 \ln j_{\max}}$.

6. The Borel summability

In this section we consider the Borel summability of the perturbation series [27].

Theorem 6.1. *The perturbation series of the connected function for ϕ_2^4 theory is Borel summable.*

Proof For the perturbation series $\sum_{n=0} a_n \lambda^n$ to be Borel summable to the function G , we need to have

$$G(\lambda) = \sum_{n=0} a_n \lambda^n + R^{(n+1)}(\lambda), \quad (6.1)$$

where $R^{(n+1)}(\lambda)$ is the Taylor remainder. The analyticity domain C_λ for G should be at least $|\lambda| < \frac{1}{K}$ and $\text{Re}\lambda > 0$ [27], which means

$$-\frac{\pi}{4} \leq \text{Arg}\sqrt{\lambda} \leq \frac{\pi}{4}. \quad (6.2)$$

We rewrite the resolvent as

$$R = \frac{1}{1 + i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}}. \quad (6.3)$$

Since the matrix $C^{1/2}\sigma C^{1/2}$ is Hermitian, its eigenvalues are real, so that there are no poles in the denominator. In the analytic domain of G we have

$$\|R\| = \left| \frac{1}{1 + i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}} \right| \leq \sqrt{2}, \quad (6.4)$$

and

$$\|K\| = \|R - 1\| \leq 1 + \sqrt{2}. \quad (6.5)$$

However in the analytic domain C_λ the linear counter term becomes:

$$e^{\text{Tr}i\sqrt{2\lambda}\sigma T_\Lambda} = e^{\text{Tr}i|\sqrt{2\lambda}|\cos\theta\sigma T_\Lambda} e^{-\text{Tr}|\sqrt{2\lambda}|\sin\theta\sigma T_\Lambda}, \quad (6.6)$$

where $\theta = \text{Arg}\sqrt{\lambda}$. We could bound the first term in (6.6) by 1, but the second term would diverge for negative σ .

We rewrite this term as:

$$\int d\mu(\sigma) e^{-1/2\text{Tr}(\sigma + \sqrt{2\lambda}\sin\theta T_\Lambda)^2} e^{\sin^2\theta T_\Lambda^2}. \quad (6.7)$$

The term $e^{\sin^2\theta T_\Lambda^2}$ could diverge at worst as $e^{1/2T_\Lambda^2}$ for $\theta = \pm\pi/4$. But this is not dangerous, since we could still bound it with the convergent factor $e^{-aj_{max}^3} \sim e^{-aT_\Lambda^3}$ that we gained from the crossings and nesting lines.

We use simply the Taylor expansion with remainder for the connected function (6.1):

$$G(t\lambda)|_{t=1} = \frac{G^{(n)}(\lambda)}{n!}|_{t=0} + \int_0^1 dt \frac{(1-t)^n}{n!} G^{(n+1)}(t\lambda), \quad (6.8)$$

followed by explicit Wick contractions. We have for the remainder

$$\|R^{n+1}\| < |\lambda|^{n+1} K^n (2n)!! \leq |\lambda|^{n+1} [K']^n (n!), \quad (6.9)$$

where K and K' are positive numbers including the possible factors $\sqrt{2}$ or $\sqrt{2} + 1$ from the bound of the resolvent R_{mn} and of the leaf K_{mn} respectively. Hence we have proved the Borel summability of the perturbation series. \square

7. Conclusions and prospectives

In this paper we have constructed the 2-dimensional Grosse-Wulkenhaar model with the method of loop vertex expansion (LVE). The next step should be constructing the 4 dimensional case, which is the real interesting one. In this case we need also to consider the 4-point function and the abstract cluster expansion would play a more important role. This work is still in progress.

References

- [1] Zhituo Wang, “Construction of 2-dimensional Grosse-Wulkenhaar Model,” arXiv:1104.3750 [math-ph]. submitted.
- [2] M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” Rev. Mod. Phys. **73** (2001) 977 [hep-th/0106048].
- [3] R. J. Szabo, “Quantum field theory on noncommutative spaces,” Phys. Rept. **378** (2003) 207 [hep-th/0109162].
- [4] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].
- [5] H. Grosse and R. Wulkenhaar, “Power-counting theorem for non-local matrix models and renormalisation,” Commun. Math. Phys. **254** (2005) 91 [arXiv:hep-th/0305066].
- [6] H. Grosse and R. Wulkenhaar, “Renormalisation of ϕ^{*4} theory on noncommutative \mathbb{R}^{*2} in the matrix JHEP **0312**, 019 (2003) [arXiv:hep-th/0307017].
- [7] H. Grosse and R. Wulkenhaar, “Renormalisation of ϕ^{*4} theory on noncommutative \mathbb{R}^{*4} in the matrix base,” Commun. Math. Phys. **256**, 305 (2005) [arXiv:hep-th/0401128].
- [8] V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, “Renormalization of noncommutative ϕ^{*4} -theory by multi-scale analysis,” Commun. Math. Phys. **262**, 565 (2006) [arXiv:hep-th/0501036].
- [9] E. Langmann, R. J. Szabo, and K. Zarembo, Exact solution of quantum field theory on noncommutative phase spaces, JHEP **01** (2004) 017, [hep-th/0308043](#).
- [10] E. Langmann, R. J. Szabo, and K. Zarembo, Exact solution of noncommutative field theory in background magnetic fields, Phys. Lett. **B569** (2003) 95–101, [hep-th/0303082](#).
- [11] F. Vignes-Tourneret, Renormalization of the orientable non-commutative Gross-Neveu model. To appear in Ann. H. Poincaré, [math-ph/0606069](#).
- [12] H. Grosse and H. Steinacker, Renormalization of the noncommutative ϕ^3 model through the Kontsevich model. Nucl.Phys. B746 (2006) 202-226 [hep-th/0512203](#).
- [13] H. Grosse and H. Steinacker, A nontrivial solvable noncommutative ϕ^3 model in 4 dimensions, JHEP 0608 (2006) 008 [hep-th/0603052](#).
- [14] H. Grosse, H. Steinacker, Exact renormalization of a noncommutative ϕ^3 model in 6 dimensions, [hep-th/0607235](#).
- [15] Axel de Goursac, J.C. Wallet and R. Wulkenhaar, Non Commutative Induced Gauge Theory [hep-th/0703075](#).
- [16] Zhituo Wang and ShaoLong Wan, “Renormalization of Orientable Non-Commutative Complex Φ_3^6 Model”, Ann. Henri Poincaré **9** 65-90 (2008), arXiv:0710.2652 [hep-th].

- [17] H. Grosse and F. Vignes-Tourneret, “Quantum field theory on the degenerate Moyal space,” *J. Noncommut. Geom.* **4** (2010) 555 [arXiv:0803.1035 [math-ph]].
- [18] J. M. Gracia-Bondia and J. C. Varilly, “Algebras of distributions suitable for phase space quantum mechanics. 1,” *J. Math. Phys.* **29** (1988) 869.
- [19] V. Rivasseau, “Non-commutative Renormalization,” arXiv:0705.0705 [hep-th]. Séminaire Bourbaphy.
- [20] R. Wulkenhaar, “Renormalization of noncommutative ϕ_4^4 -theory to all orders”, Habilitationsschrift.
- [21] F. Vignes-Tourneret, “Renormalisation des theories de champs non commutatives,” arXiv:math-ph/0612014, PhD thesis, in french.
- [22] E. Langmann and R. J. Szabo, Duality in scalar field theory on noncommutative phase spaces, *Phys. Lett.* **B533** (2002) 168–177, hep-th/0202039.
- [23] H. Grosse and R. Wulkenhaar, “The beta-function in duality-covariant noncommutative ϕ^4 theory,” *Eur. Phys. J. C* **35** (2004) 277 [arXiv:hep-th/0402093].
- [24] M. Disertori and V. Rivasseau, “Two and three loops beta function of non commutative ϕ^4 theory,” *Eur. Phys. J. C* **50** (2007) 661 [arXiv:hep-th/0610224].
- [25] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, “Vanishing of beta function of non commutative ϕ^4 theory to all orders,” *Phys. Lett. B* **649** (2007) 95 [arXiv:hep-th/0612251].
- [26] V. Rivasseau, “Constructive Matrix Theory,” *JHEP* **0709** (2007) 008 [arXiv:0706.1224 [hep-th]].
- [27] V. Rivasseau, *From Perturbative to Constructive Renormalization*, Princeton University Press, 1991.
- [28] J. Magnen and V. Rivasseau, Constructive field theory without tears, *Ann. Henri Poincaré* **9** 403-424 (2008), arXiv:0706.2457[math-ph].
- [29] V. Rivasseau and Zhituo Wang, “Loop Vertex Expansion for Φ^{2k} Theory in Zero Dimension,” *Journal of Mathematical Physics*, **51** 092304 (2010), arXiv:1003.1037 [math-ph].
- [30] V. Rivasseau and Zhituo Wang, “How are Feynman graphs resumed by the Loop Vertex Expansion?”, arXiv:1006.4617 [math-ph].
- [31] V. Rivasseau and Zhituo Wang, “Constructive Renormalization for Φ_2^4 Theory with Loop Vertex Expansion”, arXiv:1104.3443 [math-ph].
- [32] H. Grosse and R. Wulkenhaar, “Progress in solving a noncommutative quantum field theory in four dimensions,” arXiv:0909.1389 [hep-th].
- [33] M. Panero, “Numerical simulations of a non-commutative theory: The Scalar model on the fuzzy sphere,” *JHEP* **0705** (2007) 082 [hep-th/0608202].
- [34] R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret, “Renormalization of non-commutative $\phi^4(4)$ field theory in x space,” *Commun. Math. Phys.* **267** (2006) 515 [arXiv:hep-th/0512271].
- [35] R. Gurau and V. Rivasseau, “Parametric representation of noncommutative field theory,” *Commun. Math. Phys.* **272** (2007) 811 [arXiv:math-ph/0606030].
- [36] G. 't Hooft, “A Planar Diagram Theory for Strong Interactions,” *Nucl. Phys. B* **72** (1974) 461.
- [37] T. Krajewski, V. Rivasseau, A. Tanasa and Zhituo Wang, “Topological Graph Polynomials and Quantum Field Theory, Part I: Heat Kernel Theories,” *J. Noncommut. Geom.* **4** (2010) 29 [arXiv:0811.0186 [math-ph]].

- [38] D. Brydges and T. Kennedy, Mayer expansions and the Hamilton-Jacobi equation, *Journal of Statistical Physics*, **48**, 19 (1987).
- [39] A. Abdesselam and V. Rivasseau, “Trees, forests and jungles: A botanical garden for cluster expansions,” in *Constructive Physics*, Lecture Notes in Physics 446, Springer Verlag, 1995, arXiv:hep-th/9409094.

POS(CORFU2011)057