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Nahm Transformation and Boundary State on T^2

Tsuguhiko Asakawa
E-mail: asakawa@tuhep.phys.tohoku.ac.jp

Ursula Carow-Watamura
E-mail: ursula@tuhep.phys.tohoku.ac.jp

Yoshiro Teshima
E-mail: asakawa@tuhep.phys.tohoku.ac.jp

Satoshi Watamura*
E-mail: watamura@tuhep.phys.tohoku.ac.jp

Department of Physics
Graduate School of Science
Tohoku University
Aoba-ku, Sendai 980-8578, Japan

The T-duality for a bound state of a $D2$-$D0$ brane system and its equivalence with the Nahm transformation of the corresponding gauge theory on a 2-dimensional torus were investigated, using the boundary state analysis in superstring. Contrary to the case of a 4-dimensional torus, a nontrivial change in sign in a topological charge is obtained (which seems puzzling when regarded as a D-brane charge). It is shown that both approaches agree when a minus sign is included. The boundary state in the RR-sector is reformulated using a new representation of zeromodes, we show that the RR-coupling is invariant under T-duality. Finally, we show the T-duality invariance at the level of the Chern-Simon coupling by deriving the Buscher rule for the RR potentials (also known as the 'Hori formula'), including the correct sign.

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*Speaker.

1. Introduction

The geometry which we learn at school can be described by a set of axioms deduced from the observation of the motion of particles in spacetime. The field theory of particles is based on that geometry. In this sense, one can say that the problem of the stringy geometry is to find the geometry for an observer who only sees the motion of strings, and on which the string theory can be based on. Such a geometry will be characterized by the properties of the strings which are not the properties of particles. In this context, the T-duality [2, 3] which is a one of the fundamental properties in string theory, will play an important role to understand the stringy geometry. It is also an important concept when we consider alternative theories of the geometry such as generalized geometry. In the present analysis, we compare the T-duality transformation of the boundary state in superstring with the rules in the effective theory, in particular with the Nahm-Fourier-Mukai duality.

One aim of our investigation is to analyze the compatibility among the results on T-duality found in various approaches, i.e. worldsheet, supergravity and gauge theory. The boundary state description of D-branes is a framework which fits well for such a task. A boundary state can be defined for D-branes with non-trivial gauge bundle on it, and its coupling to various closed string states is also easily estimated.

Concretely speaking, on the string side, we will use a boundary state and take the T-dual with respect to T^2 wrapped by a D2-D0 brane. Then we compare the result with the expression obtained by taking the Nahm transformation [4] of the corresponding gauge configuration on T^2 and look at the Buscher rules in the RR sector (which will lead us to the Hori formula).

The Nahm transformation was first formulated for the case of T^4 [5, 6], where $k\ SU(N)$ instantons are mapped to $N\ SU(k)$ instantons. Unfortunately, we do not have a concrete instanton solution for T^4, which is a problem since there is no single-instanton configuration. And, without an explicit solution there is no explicit boundary state construction.

Contrary to T^4, on T^2 the solution of the gauge field which corresponds to D2-D0 configuration is rather simple and thus we can construct corresponding boundary states. The advantage of a boundary states is that, once it is constructed explicitly, the comparison of T-duality and Nahm transformation may be directly done at the string level.

Of course, there were technical complications in the beginning - even on T^2 - to construct the boundary states on tori, due to the Wilson loop factor. For the bosonic string sector, Duo et al. [7] (and DiVecchia et al. [8]) proposed a systematic way to construct the gauge invariant boundary state including an appropriate cocycle factor. Using their method we constructed the boundary state on T^2 and established the precise agreement of the T-duality transformation of the boundary state with the Nahm transformation in the bosonic string theory. The expressions agree up to a nontrivial minus sign, which we shall discuss later in detail.

The organization of this article is the following. First, we briefly give the formulation of the Nahm transformation, particularly for the case of the T^2 torus, which contains some subtlety compared to the T^4. Then, we recall the formulation of the boundary state, complete it with respect to the fermionic part and give the T-dual of this superstring boundary state. Finally, we show the connection between both formulations, followed by discussion and conclusion.

This talk is based on the paper entitled "Boundary state analysis on the equivalence of T-duality and Nahm transformation in superstring theory" by T. Asakawa, U. Carow-Watamura, Y. Teshima and S. Watamura(to appear in Prog. Theor. Phys. 127, no. 4) [1].

2. Nahm transformation of T^2

We can define the Nahm transformation of $U(N)$ gauge fields with first Chern number $C_1 = k$ on T^2 [9, 1]. This works as follows: Consider a bundle $E \to T^2$ with a positive first Chern number, $C_1(E) = k > 0$ and look for the zero modes of the Dirac operator,which is parametrized by the coordinate $\tilde{x}$ of the dual torus $\tilde{T}^2$

$$\begin{aligned} D\!\!\!\!/_{\tilde{x}} &= \gamma^\mu D_{\tilde{x}\mu} \\ &= \begin{pmatrix} 0 & \mathscr{D}^+_{\tilde{x}} \\ \mathscr{D}^-_{\tilde{x}} & 0 \end{pmatrix} , \end{aligned} \tag{2.1}$$

where $\gamma_\mu = -i\sigma_\mu$ with the Pauli matrices σ_μ, $\mu = 1,2$, and

$$\begin{aligned} \mathscr{D}^+_{\tilde{x}} &= -i(\partial_1 - iA_1 - i\frac{\tilde{x}_1}{2\pi}) - (\partial_2 - iA_2 - i\frac{\tilde{x}_2}{2\pi}) , \\ \mathscr{D}^-_{\tilde{x}} &= -i(\partial_1 - iA_1 - i\frac{\tilde{x}_1}{2\pi}) + (\partial_2 - iA_2 - i\frac{\tilde{x}_2}{2\pi}) . \end{aligned} \tag{2.2}$$

If $\mathscr{D}^-_{\tilde{x}}\varphi = 0$ has no solutions for the left-handed component $\varphi \in \Gamma(T^2, \pi^* S^+ \otimes \mathscr{E}_{\tilde{x}})$, with $\mathscr{E} = \pi^* E \otimes \mathscr{P}$ over $T^2 \times \tilde{T}^2$, $\pi : T^2 \times \tilde{T}^2 \to T^2$ and $\mathscr{P}$ is the Poincare bundle, it follows from the index theorem that the right-handed spinor $\varphi \in \Gamma(T^2, \pi^* S^- \otimes \mathscr{E}_{\tilde{x}})$,

$$\mathscr{D}^+_{\tilde{x}}\varphi = 0 \tag{2.3}$$

possesses k nomalized solutions $\varphi^p (p = 1 \cdots k)$, which can be collected into an $N \times k$ matrix of zero-modes. The relation of the various bundles are given in the Figure 1.

The solutions of eq.(2.3) also give the projection to the space of zero modes $H_{\tilde{x}}$

$$\mathscr{P}_{\tilde{x}} = |\varphi\rangle\langle\varphi| : \mathscr{H} \to H_{\tilde{x}} . \tag{2.4}$$

This defines the projection to a projective module over $C(T^2)$ which is a set of sections of a k-dimensional vector bundle $\tilde{E}$ over $\tilde{T}^2$ with connection

$$\tilde{A}_\mu(\tilde{x}) = i\langle\varphi|\tilde{\partial}_\mu\varphi\rangle = i\int_{T^2} d^2x\, \varphi^\dagger \tilde{\partial}_\mu \varphi. \tag{2.5}$$

Applying the family index theorem to the parametrized Dirac operator we obtain a relation between E and $\tilde{E}$ as:

$$\mathrm{ch}(\tilde{E}) = \int_{T^2} \mathrm{ch}(\mathscr{P})\mathrm{ch}(E). \tag{2.6}$$

For the torus T^2, this gives

$$\begin{aligned} (\text{L.H.S.}) &= \mathrm{rank}(\tilde{E}) + c_1(\tilde{E}), \\ (\text{R.H.S.}) &= C_1(E) - \mathrm{rank}(E)\frac{d\tilde{x}^1 \wedge d\tilde{x}^2}{vol(\tilde{T}^2)}. \end{aligned} \tag{2.7}$$

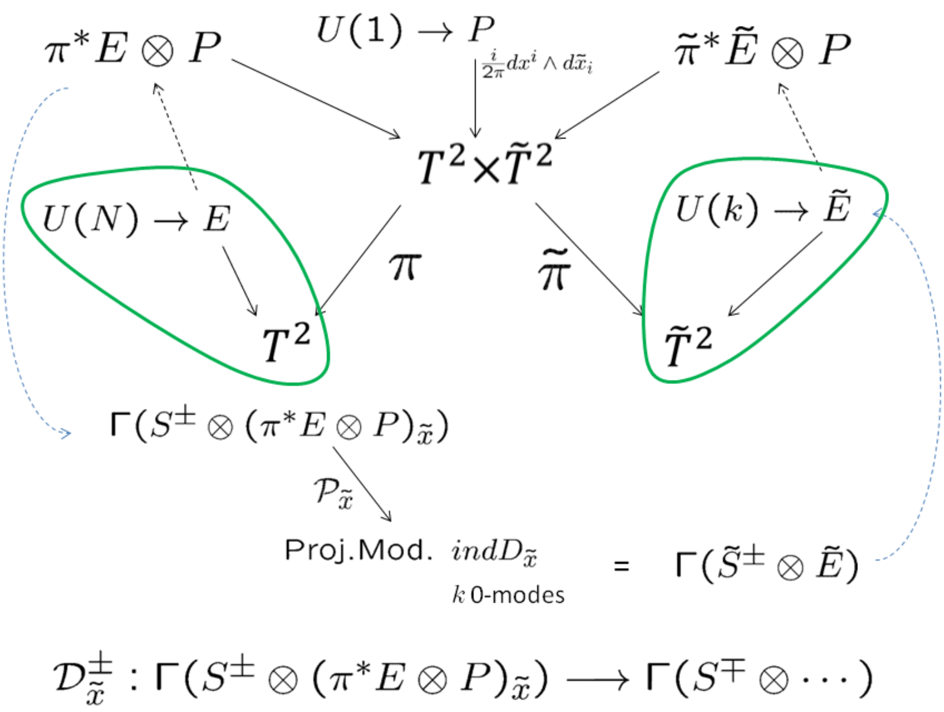

Figure 1: The schematical picture of the various bundles and maps appearing in the Nahm transformation of T^2.

$C_1(E) = \int c_1(E)$ is the first Chern number and $c_1(E)$ is the first Chern class. The above equations lead to the relations between the bundles E and $\tilde{E}$ as

$$\begin{aligned} \mathrm{rank}(\tilde{E}) &= C_1(E)\,, \\ C_1(\tilde{E}) &= -\mathrm{rank}(E) \end{aligned} \tag{2.8}$$

This tells us that for T^2 there is a relative sign under Nahm transformation and that this transformation exchanges the rank of the gauge group and first Chern number. Symbolically we can write: $(N,k) \to (k,-N)$.

Note that we obtain the same result when we construct A_μ by constructing the zero modes of the Dirac equation [1].

3. T-duality in the boundary state

The boundary state of the $D2-D0$ system on T^2 is given by

$$\begin{aligned} |B_F\rangle &= \mathscr{O}_A\,|B\rangle \\ &= \mathscr{O}_A \sqrt{\det(G+B)} \sum_{s\in\mathbb{Z}} \prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_n^\dagger GR\tilde{\alpha}_n^\dagger}\, |0;\omega m\rangle \end{aligned} \tag{3.1}$$

where where $R = (G+B)^{-1}(G-B)$,

$$\mathscr{O}_A = \mathrm{Tr}_N \prod_{\alpha=1,2} \prod_{\ell=0}^{\mathbf{m}^\alpha-1} \Omega_\alpha(\mathbf{x} + 2\pi\sqrt{\alpha'} \sum_{\beta=1}^{\alpha-1} \mathbf{m}^\beta a_\beta + 2\pi\sqrt{\alpha'}\ell a_\alpha)\exp(-S_A), \tag{3.2}$$

and the Wilson loop factor

$$\exp(-S_A) = P\exp\left(\frac{i}{2\pi\alpha'}\int_0^{2\pi} A_\alpha \partial_\sigma X^\alpha d\sigma\right) , \tag{3.3}$$

In order to obtain a gauge invariant expression we have to close the path of the Wilson loop factor in the covering space by using the transtion functions Ω_i[7, 8], as schematically shown in Figure 2.

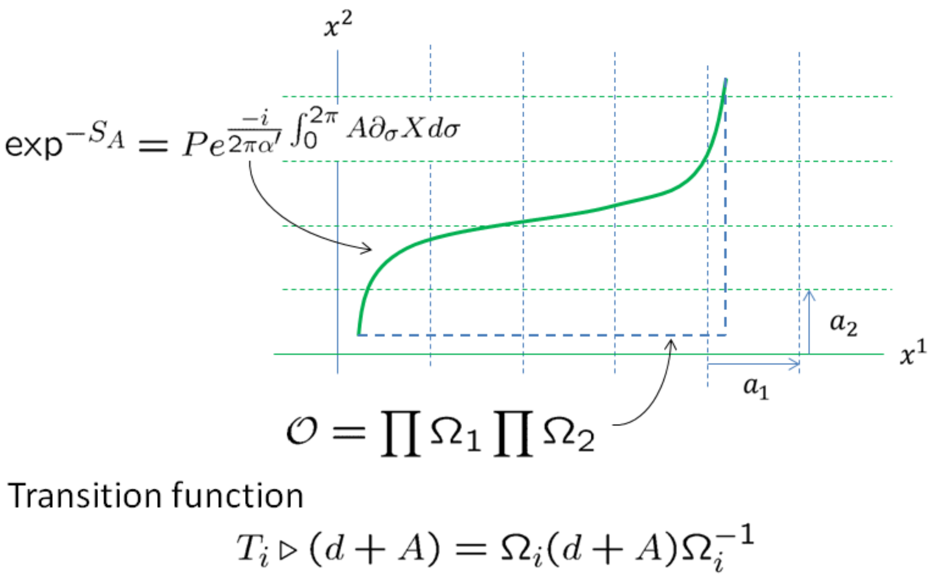

Figure 2: The path of the Wilson line is indicated by the solid line and the part of the cocycle factors is given by the dashed line.

Using the explicit form of transition functions, we can construct the boundary state of k $D0$ branes on N $D2$ branes. It is given with a Wilson loop factor of a $U(N)$ gauge field A_μ with constant flux $F_{\mu\nu}$ satisfying the following gauge field configuration

$$A_1 = 0\ ,\ \ A_2 = \frac{k}{2\pi\alpha' N}x_1\ ,\quad F_{12} = \frac{k}{2\pi\alpha' N}\ ,\quad C_1 = k\ . \tag{3.4}$$

To compare with the Nahm transformation, we construct the boundary state with the corresponding flux and then take the T-dual [10, 11, 12, 13, 14, 15]. The boundary state on T^2 with $B = 0$ and $G = \mathrm{diag}(a_1^2, a_2^2)$ is

$$\begin{aligned}|B_F\rangle = &\sqrt{(a_1a_2N)^2 + k^2} \sum_{(m_1,m_2)\in\mathbb{Z}^2} e^{-i\pi N k m_1 m_2} \\ &\times \left[\prod_{n=1}^{\infty} \exp\left\{\frac{1}{n}\frac{1}{a_1^2a_2^2N^2+k^2}(\alpha_n^{1\dagger}\alpha_n^{2\dagger})\begin{pmatrix} a_1^2a_2^2N^2 - k^2 & 2a_1a_2Nk \\ -2a_1a_2Nk & a_1^2a_2^2N^2 - k^2\end{pmatrix}\begin{pmatrix}\tilde{\alpha}_n^{1\dagger} \\ \tilde{\alpha}_n^{2\dagger}\end{pmatrix}\right\}\right] \\ &\times \left|\begin{pmatrix} -km_2 \\ km_1\end{pmatrix}, \begin{pmatrix} Nm_1 \\ Nm_2\end{pmatrix}\right\rangle .\end{aligned} \tag{3.5}$$

Taking the T-dual we have to exchange momenta and winding modes and change the sign of the oscillators $\tilde{\alpha}_\mu$. The result is

$$
\begin{aligned}
|\tilde{B}_F\rangle = & \sqrt{N^2+(\tilde{a}_1\tilde{a}_2 k)^2} \sum_{(m_1,m_2)\in\mathbb{Z}^2} e^{i\pi N k m_1 m_2} \\
& \times \left[\prod_{n=1}^{\infty} \exp\left\{ \frac{1}{n}\frac{1}{\tilde{a}_1^2\tilde{a}_2^2k^2+N^2} (\alpha_n^{1\dagger}\alpha_n^{2\dagger}) \begin{pmatrix} \tilde{a}_1^2\tilde{a}_2^2k^2-N^2 & -2\tilde{a}_1\tilde{a}_2 Nk \\ 2\tilde{a}_1\tilde{a}_2 Nk & \tilde{a}_1^2\tilde{a}_2^2k^2-N^2 \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_n^{1\dagger} \\ \tilde{\alpha}_n^{2\dagger} \end{pmatrix} \right\} \right] \\
& \times \left| \begin{pmatrix} Nm_2 \\ -Nm_1 \end{pmatrix}, \begin{pmatrix} km_1 \\ km_2 \end{pmatrix} \right\rangle, \qquad (3.6)
\end{aligned}
$$

where $\tilde{a}_i = 1/a_i$.

Comparing with the result from the Nahm transformation, we find that the T-dual of the boundary state can be also obtained by the rule:

$$
\begin{aligned}
(N,k) &\Rightarrow (k,-N) \quad \text{Nahm transformation} \\
a_i &\Rightarrow \tilde{a}_i = \frac{1}{a_i} \quad \text{Buscher rule}
\end{aligned} \qquad (3.7)
$$

4. RR sector of the boundary state

The boundary state in NSNS sector in superstring is analogous to the bosonic case. A special care is necessary for the states in RR sector due to the existence of the picture [16, 17, 18, 19]. It is known that to construct the boundary state which couples to the potential rather than the field strength, it is necessary to construct the state on the asymmetric picture. Including the Wilson line factor, the boundary state can be written as

$$
|\mathscr{B}\rangle = e^{-\int F_{\mu\nu}\theta^\mu\theta^\nu} \theta_0^\dagger \cdots \theta_p^\dagger |W\rangle \qquad (4.1)
$$

$$
|W\rangle = e^{i\beta_0\tilde{\gamma}_0} \mathscr{O}|A,-1/2\rangle C_{AB} |\tilde{B},-3/2\rangle \equiv \mathscr{O}|[C]\rangle \qquad (4.2)
$$

where $\mathscr{O}$ is the oscillator part. Here we wrote only the zeromode part of the fermionic boundary state explicitly.

The meaning of the symbols in the above expression is the following: $|A,-1/2\rangle$ means the left mover ground state in spinor representation labeled by A in the picture $-1/2$ and $|\tilde{B},-3/2\rangle$ is the corresponding right mover state. C_{AB} is the charge conjugation matrix. The objects in the right mover carry a "tilde". $\beta_0, \tilde{\gamma}_0$ are the zeromodes of the bosonic ghosts. The θ^μ denotes the fermionic creation-annihilation operator defined by a conbination of the fermionic oscilators ψ_n^μ in the RR sector :

$$
\theta^\mu = \psi_0^\mu + i\tilde{\psi}_0^\mu \;\; , \;\; \{\theta^\mu, \theta_\nu^\dagger\} = \delta_\nu^\mu .
$$

Our convention of the representation of the zeromodes is

$$
\begin{aligned}
\psi_0^\mu |A\rangle \otimes |\tilde{B}\rangle &= (\frac{1}{\sqrt{2}}\Gamma^\mu \otimes \mathbf{1})|A\rangle \otimes |\tilde{B}\rangle \;\; , \\
\tilde{\psi}_0^\mu |A\rangle \otimes |\tilde{B}\rangle &= (\Gamma_{11} \otimes \frac{i}{\sqrt{2}}\Gamma_{11}\Gamma^\mu)|A\rangle \otimes |\tilde{B}\rangle \;\; .
\end{aligned} \qquad (4.3)
$$

where we surpressed the ghost number. This representation is convenient for the following analysis and different from the one given in [18, 19] (see also Ref. [20].). The action of the θ^{μ} on the state $|[C]\rangle$ is then

$$\theta^{\dagger}_{\mu}|[C]\rangle = |[C\Gamma_{\mu}]\rangle \ . \tag{4.4}$$

and for the Dp-brane we have

$$|Dp\rangle = \theta^{\dagger}_{p}\theta^{\dagger}_{p-1}\cdots\theta^{\dagger}_{0}|[C]\rangle \tag{4.5}$$

which satisfies the boundary condition for a Dp brane

$$\begin{aligned}
&\theta^{\dagger}_{\alpha}|Dp\rangle = 0, \ \ (\alpha = 0,\cdots,p), \\
&\theta_{i}|Dp\rangle = 0, \ \ (i = p+1,\cdots,9).
\end{aligned} \tag{4.6}$$

We focus on the D2-D0 brane system here. The corresponding boundary state is thus

$$\begin{aligned}
|D2D0\rangle &= Ne^{2\pi\alpha' F_{12}\theta^1\theta^2}|D2\rangle \\
&= N(1+2\pi\alpha' F_{12}\theta^1\theta^2)|D2\rangle \\
&= N|D2\rangle + k|D0\rangle \ .
\end{aligned} \tag{4.7}$$

The T-duality of the fermion is obtained by changing the sign of the right mover for the oscillator mode: $\tilde{\psi}_n \to -\tilde{\psi}_n$. For the zeromode, T-duality transformation for the α-direction can be represented by an operator $\mathscr{T}_{\alpha}$ as

$$\mathscr{T}_{\alpha} = \theta^{\alpha} - \theta^{\alpha\dagger}, \tag{4.8}$$

which maps $\mathscr{T}_{\alpha}\theta^{\beta}\mathscr{T}^{\dagger}_{\alpha} = -\theta^{\dagger\beta}$ and $\mathscr{T}_{\alpha}\theta^{\dagger\beta}\mathscr{T}^{\dagger}_{\alpha} = -\theta^{\beta}$, and satisfies $\mathscr{T}^{\dagger}_{\alpha}\mathscr{T}_{\alpha} = 1$ and $\mathscr{T}_{\alpha}\mathscr{T}_{\alpha} = -1$. $\mathscr{T}_{\alpha}$ maps a boundary state (4.5) of a Dp-brane to that of a $D(p+1)$ or a $D(p-1)$ brane, depending on whether the direction of the T-duality map is perpendicular or parallel to the Dp brane, respectively. Using the above operator, we can take a T-duality of the state (4.7) and get

$$|D2D0\rangle' = \mathscr{T}_2\mathscr{T}_1|D2D0\rangle = k|D2\rangle - N|D0\rangle \ . \tag{4.9}$$

To calculate the overlap we need conjugate states, which can be defined on the state

$$\langle[C]| = \langle\tilde{A},-1/2|[C]_{AB}\langle B,-3/2|e^{-i\beta_0\tilde{\gamma}_0} \ , \tag{4.10}$$

as the boundary state. The action of the fermionic zeromodes on the tensor state is

$$\langle\tilde{A}|\otimes\langle B|\psi_{\mu} = \frac{-1}{\sqrt{2}}\langle\tilde{A}|\otimes\Gamma^{B}_{\mu,C}\langle C| \ , \tag{4.11}$$

$$\langle\tilde{A}|\otimes\langle B|\tilde{\psi}_{\mu} = \frac{i}{\sqrt{2}}(\Gamma_{11}\Gamma_{\mu})^{A}{}_{C}\langle\tilde{C}|\otimes\Gamma^{B}_{11,D}\langle D| \ . \tag{4.12}$$

The RR-state for antisymmetric tensor potentials can be defined as

$$\langle\mathscr{A}| = \langle[C]|\mathscr{A} = \langle[C]|\sum A_{\mu_1\cdots\mu_q}\theta^{\mu_1}\cdots\theta^{\mu_q} \ . \tag{4.13}$$

Note that the T-duality of these RR-states can also be taken by applying the operator $\mathscr{T}_{\alpha}$.

The physical state is given by applying the GSO projection

$$[GSO] = \frac{(1-(-1)^{F_0+G_0})(1+(-1)^{p+\tilde{F}_0+\tilde{G}_0})}{4} , \tag{4.14}$$

which selects the rank of the antisymmetric tensor field according to the type of the superstring. F_0 and G_0 in the GSO projection are fermion number and ghost number, respectively. Using these states, we can obtain the Chern-Simons coupling of the RR-potentials to the D-brane by

$$I_{CS} = \langle \mathscr{A} | [GSO] c_0 \tilde{c}_0 | \mathscr{B} \rangle \ . \tag{4.15}$$

The T-duality transformation of the Chern-Simons coupling follows from the relation

$$I_{CS} = \langle \mathscr{A} | [GSO] \mathscr{T}^\dagger \mathscr{T} c_0 \tilde{c}_0 | \mathscr{B} \rangle = \langle \mathscr{A} | \mathscr{T}^\dagger [GSO] c_0 \tilde{c}_0 \mathscr{T} | \mathscr{B} \rangle = \langle \mathscr{A}' | [GSO] c_0 \tilde{c}_0 | \mathscr{B}' \rangle \ . \tag{4.16}$$

where $\mathscr{T}$ is an appropriate conbination of T-dual operator $\mathscr{T}_\alpha$ and $\langle \mathscr{A}'|$ and $|\mathscr{B}'\rangle$ are the T-dual state of the RR-potentials and the boundary state, respectively. Here the T-dual operator is $\mathscr{T} = \mathscr{T}_2 \mathscr{T}_1$

Now we can discuss the consistency of the Nahm transformations and T-duality of the string for the D2-D0 system on the torus. It is straightforward to evaluate:

$$\begin{aligned}\langle \mathscr{A}' | &= \langle \mathscr{A} | \mathscr{T}_1^\dagger \mathscr{T}_2^\dagger = \langle [C] | (\mathscr{A}^{(0)} + \mathscr{A}_1^{(1)} \theta^1 + \mathscr{A}_2^{(1)} \theta^2 + \mathscr{A}_{12}^{(2)} \theta^1 \theta^2) \mathscr{T}_1^\dagger \mathscr{T}_2^\dagger \\ &= \langle [C] | (\mathscr{A}^{(0)} \theta^1 \theta^2 - \mathscr{A}_1^{(1)} \theta^2 + \mathscr{A}_2^{(1)} \theta^1 - \mathscr{A}_{12}^{(2)}) ,\end{aligned} \tag{4.17}$$

where we have expanded the sum of RR-potentials $\mathscr{A}$ in terms of θ^1 and θ^2. Note that the coefficients $\mathscr{A}^{(k)}$ do not contain θ^1 or θ^2. From this we get the T-duality rule for the RR antisymmetric field as

$$\mathscr{A}'^{(0)} = -\mathscr{A}_{12}^{(2)} \ , \ \mathscr{A}_1'^{(1)} = \mathscr{A}_2^{(1)} \ , \ \mathscr{A}_2'^{(1)} = -\mathscr{A}_1^{(1)} \ , \ \mathscr{A}_{12}'^{(2)} = \mathscr{A}^{(0)} \ . \tag{4.18}$$

This is essentially the Buscher rule for RR-potentials [21] as argued in Refs. [22, 23, 24, 25]. The transformation rule (4.18) can be represented in terms of differential forms in a compact form as

$$\mathscr{A}' = -\int_{T^2} \mathscr{A} e^{dx^i \wedge dy_i} \ , \tag{4.19}$$

where $\mathscr{A}'$ is the T-dual RR antisymmetric field with θ^1, θ^2 being replaced by dy^1, dy^2 and θ^k $(k \neq 1,2)$ replaced by dx^k, respectively.

As we have seen in the previous section, the T-duality of the boundary state is given by the Nahm transformation. For the following discussion, we use the representation of the Nahm transformation in the similar form to the family index formula (2.6) as

$$\begin{aligned}\mathrm{Tr}_k(e^{2\pi\alpha'\tilde{F}}) &= \frac{1}{(2\pi)^2\alpha'} \int_{T^2} e^{dx^i \wedge dy_i} \wedge \mathrm{Tr}_N(e^{2\pi\alpha' F}) \\ &= k - N dy^1 \wedge dy^2,\end{aligned} \tag{4.20}$$

where $\tilde{F}$ is the dual curvature 2-form.

The Chern-Simons coupling (4.15) can be represented in terms of differential forms [27, 28, 29] as

$$I_{CS} = \mu_2 \int_M \mathscr{A} \wedge \mathrm{Tr}_N(e^{2\pi\alpha' F}) \ , \tag{4.21}$$

where $\mu_2 = T_2$ is the unit of $D2$-brane charge, $M = \mathbb{R}\times T^2$ is the worldvolume, $\mathscr{A}$ is a sum of RR-potentials and F is curvature 2-form. In our gauge configuration (3.4), this reduces to

$$\begin{aligned} I_{CS} &= \mu_2 \int_M (\mathscr{A}^{(0)} + \cdots + \mathscr{A}^{(2)}_{12} dx^1\wedge dx^2)(N + k dx^1\wedge dx^2) \\ &= N\mu_2 \int_M \mathscr{A}^{(2)}_{12} dx^1 \wedge dx^2 + k\mu_0 \int_{\mathbb{R}} \mathscr{A}^{(0)}, \end{aligned} \tag{4.22}$$

where $\mu_0 = (2\pi\sqrt{\alpha'})^2\mu_2$ is the unit of $D0$-brane charge, and we have used the same notation for the RR q-form field as (4.17).

By using these two transformations (4.19) and (4.20), we obtain

$$\begin{aligned} \mu_2 \int_{\tilde{M}} \mathscr{A}' \wedge \mathrm{Tr}_k(e^{2\pi\alpha'\tilde{F}}) &= \mu_2 \int_{\tilde{M}} (\mathscr{A}'^{(0)} + \cdots + \mathscr{A}'^{(2)}_{12} dy^1\wedge dy^2)(k - N dy^1\wedge dy^2) \\ &= -\mu_2 \int (-N\mathscr{A}^{(2)}_{12} dy^1\wedge dy^2 - k\mathscr{A}^{(0)} dy^1 \wedge dy^2) \\ &= N\mu_2 \int_{\tilde{M}} \mathscr{A}^{(2)}_{12} dy^1\wedge dy^2 + k\mu_0 \int_{\mathbb{R}} \mathscr{A}^{(0)}, \end{aligned} \tag{4.23}$$

where $\tilde{M} = \mathbb{R}\times\tilde{T}^2$. This shows the invariance of the Chern-Simons term, as required. The transformation rule of RR-potentials (4.19) or the gauge flux (4.20) is ambiguous in its overall sign. It is necessary in our convention that (4.19) has an overall minus sign to obtain a consistent and T-duality invariant Chern-Simons coupling of the brane and q-form field.

5. Conclusion and Discussion

We have proved the equivalence of the Nahm transformation with the T-duality in string theory on the level of the boundary state for the case of a $ND2/kD0$ bound state on a torus T^2.

For this we gave the two dimensional version of the Nahm transformation, which interchanges the rank N of the gauge group and the flux k according to the rule $(N,k) \rightarrow (k,-N)$ together with the map T^2 to $\tilde{T}^2$, including a nontrivial relative sign.

Then, we proved the equivalence of the Nahm transformation with the T-duality transformation in superstring theory. A consistent extension of the boundary state description of magnetized D-branes on tori to the superstring was derived.

In the superstring case, the T-duality transformation of the RR-zeromode sector has to be handled carefully. Using the method for constructing the boundary state given in [19][18], we introduced a new representation of the zeromodes, which has the advantage that the boundary state and the RR-states are treated in a seamless way. Also the T-duality invariance of the Chern-Simons term follows rather naturally. Introducing the T-duality operator for the zeromode part which acts on both the boundary state and the RR q-form state, as was first introduced in [23][24] to describe the Buscher rule of RR-potentials, the relationship between T-duality rule at the superstring level and that at the low energy effective theory is clarified. The T-duality invariance of the Chern-Simons term in the effective theory requires an extra sign for the Hori formula. As a final result, we showed the compatibility among the T-duality, Buscher rule (Hori formula) and the Nahm tranform (family index formula).

It is a well known fact that when performing two Fourier transformations in sequence, this does not give back the original function but a "parity transformed function", i.e. its variable x is replaced by $-x$. This means $(\vec{x},\vec{p}) \to (\vec{p},-\vec{x}) \to (-\vec{x},-\vec{p})$. One must perform the Fourier transform 4 times to get back to the original function, which is called $\mathbb{Z}_4$-duality.

The Nahm transformation, which is a special case of Fourier-Mukai transformation, shows the same feature, symbolically written as $(N,k) \to (k,-N) \to -(N,k)$. Similarly, the RR-potentials get an overall minus sign when we transform them twice by the Hori formula. These are simply the consequence of the square of the T-duality. In fact, for $\mathscr{T} = \mathscr{T}_2\mathscr{T}_2$ we have $\mathscr{T}^2 = -\mathscr{T}_2^2\mathscr{T}_1^2 = -1$. Thus, the overall sign appearing in these formulae indicates a $\mathbb{Z}_4$-duality nature of the T-duality.

On the other hand, T-duality is usually designed to act as an $\mathbb{Z}_2$-duality [24]. This works since we can redefine both RR-potentials and the boundary state by a minus sign using the sign ambiguity. However, one has to be aware that this redefinition has to be done simultaneously in order to leave their overlap unchanged.

It would be interesting to extend our analysis to a higher even-dimensional torus T^d (d: even), where the Nahm transformation interchanges the rank and higher Chern numbers. In this way, one may find a connection to the so-called toron solutions on T^4 corresponding to $D4/D2/D0$-bound states [30].

The discrete T-duality considered here is a subgroup of the group $O(2,2;\mathbb{Z})$ for T^2 and $O(d,d;\mathbb{Z})$ for T^d. Corresponding T-duality rules for bosonic boundary states [7][8], and for RR-potentials [22][23][24] have been found. We expect that there is family of $O(d,d;\mathbb{Z})$ Nahm transformations. However, besides the $D2/D0$-bound states for $d = 2$, it also has to include tilted D-strings and a state of $D0$-branes only.

Restricting to type IIA theory and the subgroup $SO(d,d;\mathbb{Z})$, these Nahm transformations should relate gauge theories of various even dimensions, which leads us to the derived category viwpoint on D-brane bound states. Then, the corrsponding Nahm transformations would be the Fourier-Mukai transformations.

References

[1] T. Asakawa, U. Carow-Watamura, Y. Teshima and S. Watamura, Prog. Theor. Phys. 127, no.4 (2012).

[2] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984) 357.

[3] N. Sakai and I. Senda, Prog. Theor. Phys. 75(1986) 692.

[4] W. Nahm, Phys. Lett. 90B (1980) 413.

[5] P.J. Braam and P. van Baal, Commun. Math. Phys. 122 (1989) 267.

[6] H. Schenk, Commun. Math. Phys. 116 (1988) 177.

[7] D. Duo, R. Russo and S. Sciuto, JHEP12 (2007) 042.

[8] P. DiVecchia, A. Liccardo, R. Marotta, F. Pezella and I. Pesando, JHEP 11 (2007) 100.

[9] M. Hamanaka and H. Kajiura, Phys. Lett. B551 (2003) 360.

[10] T.H. Buscher, Phys. Lett. B194 (1987) 54.

[11] A. Giveon, M. Porrati and E. Ravinovich, Physics Reports 244 (1994) 77-202.

[12] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, Phys. Lett. B336 (1994) 183.

[13] E. Alvarez, J.L.F. Barbon and J. Borlaf, Nucl. Phys. B479 (1996) 218-242.

[14] T. Kugo and B. Zwiebach, Prog. Theor. Phys. 87 (1992) 801.

[15] J. Dai, R.G. Leigh and J. Polchinski, Mod. Phys. Lett. A4 (1989) 2073.

[16] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B308 (1988) 221.

[17] J. Polchinski and Y. Cai, Nucl. Phys. B296 (1988) 91-128.

[18] M. Billó, P. DiVecchia, M. Frau, A. Lerda, I. Pesando, R. Russo, and S. Sciuto, Nucl. Phys. B526 (1998) 199[hep-th/9802088].

[19] P. DiVecchia, M. Frau, I. Pesando, S. Sciuto, and A. Lerda, Nucl.Phys. B507(1997)259.

[20] V.A. Kostelecký, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, Nucl. Phys. B288 (1987) 173.

[21] E. Bergshoeff, C.M. Hull and T. Ortin, hep-th/9504081. E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, Nucl. Phys. B470 (1996) 113.

[22] D. Brace, B. Morariu and B. Zumino, Nucl. Phys. B545 (1999) 192-216; Nucl. Phys. B549 (1999) 181-193.

[23] M. Fukuma, T. Oota and H. Tanaka, Prog. Theor. Phys.103 (2000)425.

[24] S.F. Hassan, Nucl. Phys. B568 (2000) 145; Nucl. Phys. B583 (2000) 431.

[25] K. Hori, Adv. Theor. Math. Phys. 3 (1999) 281.

[26] P. Bouwknegt, 'Lectures on Cohomology, T-Duality and Generalized Geometry', Lecture Notes in Physics, vol. 807 (2010) 261-311.

[27] E. Bergshoeff and M. de Roo, Phys. Lett. B380 (1996) 265. M.B. Green, C.M. Hull and P.K. Townsend, Phys. Lett. B382(1996) 65, J. Simon, hep-th9812095.

[28] S. F. Hassan and R. Minasian, arXiv:hep-th/0008149.

[29] P. Sundell, Int. J. Mod. Phys. A **16** (2001) 3025.

[30] Z. Guralnik and S. Ramgoolam, Nucl. Phys. B **499** (1997) 241, Nucl. Phys. B **521** (1998) 129.